An improvement of dimension-free Sobolev imbeddings in r.i. spaces

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Abstract

We prove a dimension-invariant imbedding estimate for Sobolev spaces of first order, and we establish a kind of optimality.

Keywords: Sobolev space, imbedding theorem, small Lebesgue space, logarithmic Lebesgue space, rearrangement-invariant Banach function space

2000 Mathematics Subject Classification: Primary 46E35; Secondary 46E30

1 Introduction

The topic of the dimension free imbedding has attracted interest in couple of last years. The first paper based on the classical Gross logarithmic inequality apparently appeared in [14]. Tuned and extended versions based either on a

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†This paper had to be finished during the visit of the first author to the Institute of Mathematics in Praha, from June, 25 until July, 5 2012, but the second author suddenly passed away on June, 17. The present version contains the work made by the coautors during such visit. The first author wishes to thank Šárka Nečasová and the Institute for the kind ospitality, partially supported by the Grant Agency of the Czech Republic under no. 201/10/1920 and RVO 67985840

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generalized version of the Gross inequality [12] or on extrapolation procedure for Sobolev spaces are in [15] and [16]. A deep analysis of the problem, using the isoperimetric theorem, appeared in [18] and another interesting approach can be found in [23]. Let us explain the core of the problem. Recall the well-known Sobolev imbedding theorem, according to which $W_0^{1,p}(Q_n)$ (here $p < n$ and Q_n is the unit cube in $\mathbb{R}^n$, for simplicity) is imbedded into the Lebesgue space $L_{p_n^*}(Q_n)$, where $p_n^* = \frac{np}{n-p}$. Clearly $p_n^* \searrow p$ as $n \to \infty$ and the question is whether there exists some residual improvement of the integrability independent of n, that is, the question about existence of some space X such that $X \hookrightarrow L_p$, $X \subsetneqq L_p$ and $\|f\|_X \le c\|\nabla f\|_p$ with c independent of f and n. This identical result of the above cited paper is that $L_p(\log L)^{p/2}(Q_n)$ is a suitable target space in this sense.

Here we shall show that the target space $L_p(\log L)^{p/2}(Q_n)$ *can be improved.* Rather surprisingly, it turns out that the better target space is $L_{(p,p'/2}(Q_n)$, a certain *small Lebesgue space* $L_{(p,\theta}(Q_n)$, introduced in [7]. Moreover, we show that the established imbedding is in a good sense optimal in the framework of rearrangement invariant spaces.

In Section 2 we recall basic properties of $L_{(p,\theta}$, Sections 3 and 4 contain the main results and their proofs.

2 The small Lebesgue spaces

Let us recall the definition of the small Lebesgue spaces $L_{(p,\theta}$. Note that it is rather complicated, however, quite natural in the light of work on duality properties of grand Lebesgue spaces and extrapolation of Lebesgue spaces. Again we shall consider functions defined on the unit cube in $\mathbb{R}^n$, that is, with their non-increasing rearrangements supported in $[0,1]$. The space $L_{(p,\theta}$ $(p > 1,\ \theta > 0)$ consists of functions f with the finite Banach function norm (see [7])

$$\|f\|_{L_{(p,\theta}} = \inf_{f=\sum f_j} \left(\sum_j \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{\theta}{p'-\varepsilon}} \|f_j\|_{(p'-\varepsilon)'} \right) < \infty, \qquad (2.1)$$

where $(p'-\varepsilon)'$ denotes the index conjugate to $(p'-\varepsilon)$, that is,

$$(p'-\varepsilon)' = \frac{p'-\varepsilon}{p'-\varepsilon-1} = \frac{p-\varepsilon(p-1)}{1-\varepsilon(p-1)}.$$

In the formula (2.1) one can consider $0 < \varepsilon < \varepsilon_0$ for any $0 < \varepsilon_0 < p' - 1$ to arrive at the same space (up to an equivalence of the corresponding functionals), see [10].

The small Lebesgue spaces have been introduced in [7] as associate spaces of the grand Lebesgue spaces defined by Iwaniec and Sbordone in [13], therefore they are Banach function spaces (see [3, Theorem 2.2 p.8]). In [9] and in [8] a new, equivalent, expression for the norm appeared in the form of an integral, which is a certain gauge for measuring speed of the blowup of f^* (the nonincreasing rearrangement of f) at zero, namely,

$$\|f\|_{L_{(p,\theta}} \sim \int_0^1 \left(\int_0^t f^*(s)^p \, ds \right)^{1/p} \frac{dt}{t\,(\log(1/t))^{1-\theta/p'}}. \tag{2.2}$$

Note that the small Lebesgue spaces have turned out to have relevant applications in Analysis, particularly in Partial Differential Equations (see references in [4]) or in extrapolation of (sub)linear operators (in [5]).

For reader's convenience we state the following claim on comparison of the small Lebesgue spaces and the logarithmic Lebesgue (Zygmund) spaces.

Proposition 2.1 ([4])**.** *Let $1 < p < \infty$ and $\theta > 0$. Then*

$$\bigcup_{\beta > 1} L_p(\log L)^{\frac{\beta\theta}{p'-1}} \subset L_{(p,\theta} \subset L_p(\log L)^{\frac{\theta}{p'-1}} \tag{2.3}$$

with continuous imbeddings. Moreover, both inclusions in (2.3) *are proper.*

The key to the dimension-free imbedding of a Sobolev space into a suitable small Lebesgue space is comparison of parameters in the definition of the small Lebesgue space and those in the Sobolev imbedding theorem and the claim on the best constant. Recall that in $\mathbb{R}^n$ (see [22]),

$$n^{1/2}\|f\|_{\frac{np}{n-p}} \le c\|\nabla f\|_p, \tag{2.4}$$

where c is independent of n (with p fixed). Consider sufficiently small ε and put

$$(p' - \varepsilon)' = \frac{np}{n-p}.$$

Then (after some elementary calculation)

$$n = \frac{p(p'-\varepsilon)}{(p'-\varepsilon)' - p} = \frac{1}{\varepsilon} \cdot \frac{p(p'-\varepsilon)}{p-1}.$$

Hence one of the possible terms on the right hand side of (2.1) is (one can neglect ε in the exponent provided ε is small enough) is const. $n^{\theta/p'}\|f\|_{\frac{np}{n-p}}$ and we conclude that

$$\|f\|_{(p,\theta} \le \text{const.}\|\nabla f\|_p,$$

for

$$\frac{\theta}{p'} = \frac{1}{2}, \quad \text{that is,} \quad \theta = \frac{p'}{2}.$$

Summarizing, we have

$$\|f\|_{(p,p'/2} \le c\|\nabla f\|_p \tag{2.5}$$

for functions supported in Q_n.

The same result can be obtained using the integral formula for the norm in $L_{(p,p'/2}$. Indeed, according to (2.2), we have

$$\begin{aligned}
&\int_0^1 \left(\int_0^t f^*(s)^p\,ds\right)^{1/p} \frac{dt}{t\,(\log(1/t))^{1/2}} \\
&\qquad \le \int_0^1 \left(\int_0^t f^*(s)^{np/(n-p)\,ds}\right)^{(n-p)/n} \left(\int_0^t ds\right)^{p/n} \frac{dt}{t\,(\log(1/t))^{1/2}} \\
&\qquad \le \frac{c}{\sqrt{n}}\|\nabla f\|_p \int_0^1 \frac{dt}{t^{1-p/n}\,(\log(1/t))^{1/2}} = \frac{c}{\sqrt{n}}\|\nabla f\|_p I_n.
\end{aligned}$$

We estimate I_n. After substitutions,

$$\begin{aligned}
I_n &= \int_0^\infty \frac{e^{-\xi}}{e^{-\xi(1-p/n)}\xi^{1/2}}\,d\xi = \int_0^\infty \frac{e^{-\xi p/n}}{\xi^{1/2}}\,d\xi \\
&= \int_0^\infty \frac{e^{-y}}{\sqrt{ny}}\,\frac{n}{\sqrt{p}}\,dy = c\cdot\sqrt{n},
\end{aligned}$$

where c is independent of n.

Combining the above estimates we again get the desired imbedding.

The best space for the dimension-free imbeddings achieved in [14], [15], [16], [18], [23], is the logarithmic Lebesgue space $L_p(\log L)^{p/2}$. According to definition,

$$\|f\|_{(p,\theta} \sim \int_0^1 \left(\int_0^t f^*(s)^p\,ds\right)^{1/p} \frac{dt}{t\,(\log(1/t))^{1-\theta/p'}},$$

hence for $\theta = p'/2$,

$$\begin{aligned}
\|f\|_{(p,\,p'/2} &\sim \int_0^1 \left(\int_0^t f^*(s)^p\,ds\right)^{1/p} \frac{dt}{t\,(\log(1/t))^{1/2}} \\
&= \int_0^1 \left(\int_0^t \left[f^*(s)\,\frac{1}{t\,(\log(1/t))^{1/2}}\right]^p ds\right)^{1/p} dt \\
&\geq \left(\int_0^1 f^*(s)^p \left(\left[\frac{(\log(1/t))^{1/2}}{1/2}\right]_s^1\right)^p ds\right)^{1/p} \\
&= 2^{1/p}\left(\int_0^1 f^*(s)^p \left(\log\frac{1}{s}\right)^{p/2} ds\right)^{1/p}.
\end{aligned}$$

Note that the same follows from the formula (2.3) (the right wing imbedding). Hence the conclusion is that we have found a better target space for the dimension-free imbedding. Let us point out that the imbedding $L_{(p,\,p'/2} \hookrightarrow L_p(\log L)^{p/2}$ is proper (see [4]). Hence we have improved all the results from the earlier papers. We state it in the following

Theorem 2.2. *There exists a constant c independent of n such that*

$$\|f\|_{(p,\,p'/2} \leq c\|\nabla f\|_{L_p(Q_n)}, \text{ for all } f \in W_0^{1,p}(Q_n),\ Q_n \text{ the unit cube in } \mathbb{R}^n.$$

3 The optimality property of the small Lebesgue space

First we check what happens if we replace Lebesgue spaces by Lorentz spaces in the definition of the small Lebesgue space. For our purposes, we will need next Lemma in the case of functions over the interval $(0,1) \subset \mathbb{R}$. Note that the case $\theta = 1$ can be found in [9, Lemma 3.1].

Lemma 3.1. *Fix $p > 1$. Then for all measurable functions f*

$$\inf_{m>p} m^{1/2}\|f^*\|_{\frac{mp}{m-p}} \sim \inf_{m>p} m^{\frac{1}{2}}\|f^*\|_{\frac{mp}{m-p},p}.$$

Proof. Consider f in the space $L_{mp/(m-p),p}$, for some $m > p$, otherwise there is nothing to prove. We have

$$\left(\int_0^1 f^*(t)^{\frac{mp}{m-p}}dt\right)^{\frac{m-p}{mp}} = \left(\int_0^1 [t^{\frac{m-p}{mp}} f^*(t)]^p \cdot f^*(t)^{\frac{mp}{m-p}-p} \cdot t^{1-\frac{m-p}{m}}\frac{dt}{t}\right)^{\frac{m-p}{mp}}$$
$$= \left(\int_0^1 [t^{\frac{m-p}{mp}} f^*(t)]^p \cdot f^*(t)^{\frac{p^2}{m-p}} \cdot t^{\frac{p}{m}} \cdot \frac{dt}{t}\right)^{\frac{m-p}{mp}}$$
$$\leq \left(\int_0^1 [t^{\frac{m-p}{mp}} f^*(t)]^p \cdot t^{\frac{p}{m}} \cdot t^{-\frac{p}{m}} \cdot \|f\|_{\frac{mp}{m-p}}^{\frac{p^2}{m-p}}\frac{dt}{t}\right)^{\frac{m-p}{mp}}$$
$$= \|f\|_{\frac{mp}{m-p}}^{\frac{p}{m}} \cdot \left(\int_0^1 [t^{\frac{m-p}{mp}} f^*(t)]^p \frac{dt}{t}\right)^{\frac{m-p}{mp}}.$$

From this

$$\|f\|_{\frac{mp}{m-p}}^{1-\frac{p}{m}} \leq \|f\|_{\frac{mp}{m-p},p}^{p\cdot\frac{m-p}{mp}},$$

i.e.

$$\|f\|_{\frac{mp}{m-p}} \leq \|f\|_{\frac{mp}{m-p},p}.$$

Now we prove the converse estimate. We have

$$\left(\int_0^1 [t^{\frac{2m-p}{2mp}} f^*(t)]^p \frac{dt}{t}\right)^{1/p} = \left(\int_0^1 f^*(t)^p t^{\frac{2m-p}{2m}-1}dt\right)^{1/p} = \left(\int_0^1 f^*(t)^p t^{-\frac{p}{2m}}dt\right)^{1/p}$$
$$\leq \left(\int_0^1 f^*(t)^{\frac{pm}{m-p}}dt\right)^{\frac{1}{p}\cdot\frac{m-p}{m}} \cdot \left(\int_0^1 t^{-\frac{p}{2m}\cdot\frac{m}{p}}dt\right)^{\frac{p}{m}}$$
$$= \|f\|_{\frac{mp}{m-p}} \cdot \left(\int_0^1 t^{-\frac{1}{2}}dt\right)^{\frac{p}{m}} = \|f\|_{\frac{mp}{m-p}} \cdot ([2\sqrt{t}]_0^1)^{\frac{p}{m}}$$
$$= 2^{\frac{p}{m}}\|f\|_{\frac{mp}{m-p}} \leq 2\|f\|_{\frac{mp}{m-p}}.$$

Both the above estimate are of course true also for non-integer $m > p$. $\square$

Our next result is a first optimality property of the norm of small Lebesgue spaces. Namely, from (2.1), considering the trivial decomposition $(f_j) = (f, 0, 0, ..., 0, ...)$, we get that for all f

$$\|f\|_{L_{(p,\theta}} \leq \inf_{m>p} m^{\theta/p'}\|f\|_{\frac{mp}{m-p}} \tag{3.1}$$

and a natural problem occurs, namely, to know whether $\|\cdot\|_{L_{(p,\theta}}$ is the greatest norm satisfying (3.1). The answer is positive as shown in the following lemma, where, again, we consider the case of functions over the interval $(0,1) \subset \mathbb{R}$.

Lemma 3.2. *Let $p > 1$ and let X be a normed space, whose elements are measurable functions, endowed with the norm $\|\cdot\|_X$. If for all f*

$$\|f\|_{L_{(p,\theta}} \leq \|f\|_X \leq \inf_{m>p} m^{\theta/p'} \|f\|_{\frac{mp}{m-p}} \tag{3.2}$$

then X is a Banach function space, $\|f\|_X \sim \|f\|_{L_{(p,\theta}}$, and therefore $X = L_{(p,\theta}$.

Proof. By assumption we already have

$$\|f\|_{L_{(p,\theta}} \leq \|f\|_X, \tag{3.3}$$

therefore we have to show the opposite inequality. Let f be in $L_{(p,\theta}$, and let $f = \sum f_j$ be any decomposition of f. Applying the right wing inequality of (3.2) to each f_j, we get

$$\begin{aligned} \|f\|_X = \|\sum f_j\|_X \leq \sum \|f_j\|_X \\ \leq \sum \inf_{m>p} m^{\theta/p'} \|f_j\|_{\frac{mp}{m-p}} \sim \sum \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{\theta}{p'-\varepsilon}} \|f_j\|_{(p'-\varepsilon)'} \end{aligned} \tag{3.4}$$

Passing to the infimum over all decompositions on the right hand side, we get the opposite inequality. The other claims are an immediate consequence of the equivalence proved. □

Now let us go back to the problem of the dimension-free imbeddings of $W_0^{1,p}$.

At first it will be convenient to fix an appropriate notation for the following argument. Note that in Lemma 3.1 the underlying space is one dimensional, and m has the role of an integer parameter. Now this parameter will be linked to the dimension of the unit cube under consideration, therefore we will use m to denote the dimension, and, accordingly, the symbol Q_m will mean the unit cube in $\mathbb{R}^m$

$$Q_m = \left(-\frac{1}{2}, \frac{1}{2}\right)^m \subset \mathbb{R}^m \tag{3.5}$$

Moreover, we will need to consider several times spaces with underlying measure space Q_m or the interval $(0,1) \subset \mathbb{R}$: for the sake of simplicity, in this latter case we omit the set in the symbol and we will write the corresponding spaces by $L_{mp/(m-p)}$, $L_{mp/(m-p),p}$ $L_{(p,\theta}$. The same convention will be applied to the subscripts of the symbol $\|\cdot\|$. As in the previous sections, p will denote a fixed real number greater than 1.

The following result, which goes back to the estimates by O' Neil ([19], see also Peetre [20]), is the refinement of the classical Sobolev embedding theorem in the context of Lorentz spaces.

Theorem 3.3. *Let $1 < p < m$. Then*

$$\|f\|_{L_{mp/(m-p),\,p}(Q_m)} \leq c_{m,p} \|\nabla f\|_{L_p(Q_m)} \qquad \text{for all } f \in W_0^{1,p}(Q_m) \tag{3.6}$$

The Lorentz space on the left hand side of (3.6) is in fact optimal in the framework of the rearrangement-invariant Banach function spaces: this is proved in [6, Theorem 5.11] (see also [17]; for a less implicit statement see [21, Theorem 4.1]). The statement of optimality can be written in our context as follows.

Theorem 3.4. *Let $1 < p < m$ and let Y_m be a rearrangement-invariant Banach function space over $(0,1) \subset \mathbb{R}$. If*

$$\|f^*\|_{Y_m} \leq c_{m,p} \|\nabla f\|_{L_p(Q_m)}, \text{ for all } f \in W_0^{1,p}(Q_m)\,,$$

then

$$\|f^*\|_{Y_m} \leq c_{m,p} \|f^*\|_{mp/(m-p),\,p}, \text{ for all } f \in W_0^{1,p}(Q_m) \tag{3.7}$$

The last preparatory result is the computation of the best constant in (3.6), due to Alvino ([1]). In the next statement we make explicit only the dipendence of the best constant with respect to the dimension.

Theorem 3.5. *Let $1 < p < m$. Then*

$$\|f\|_{L_{mp/(m-p),\,p}(Q_m)} \leq c_p m^{-1/2} \|\nabla f\|_{L_p(Q_m)} \qquad \text{for all } f \in W_0^{1,p}(Q_m) \tag{3.8}$$

Note that in [1] the function f is simply with compact support in $\mathbb{R}^m$, but a straightforward scaling argument shows that the dependence of the dimension remains unchanged.

Now we can state and prove the main result of this paper.

Theorem 3.6. *Let $p > 1$ and let Y be a rearrangement-invariant Banach function space over $(0,1) \subset \mathbb{R}$. If for all $m > p$*

$$\|f^*\|_Y \leq c_p m^{1/2} \|f^*\|_{mp/(m-p),\,p} \qquad f \in W_0^{1,p}(Q_m), \tag{3.9}$$

then $L_{(p,\,p'/2} \hookrightarrow Y$, i.e.

$$\|f^*\|_Y \leq c_p \|f^*\|_{(p,\,p'/2}. \tag{3.10}$$

Remark 3.7. Because of (3.8), the assumption (3.9) is stronger than

$$\|f^*\|_Y \leq c_p \|\nabla f\|_{L_p(Q_m)} \qquad \text{for all } f \in W_0^{1,p}(Q_m), \text{ for all } m > p$$

Remark 3.8. Comparing the optimalities in (3.7) and (3.10), we see that the estimate in Theorem 3.6 involves function spaces independent of the dimension m. The role of the space Y is to determine the possible gain of summability of a function in $W_0^{1,p}$, in the sense of (3.9), ensured regardless the dimension.

Proof of Theorem 3.6. Fix $m > p$, $f \in L_{mp/(m-p),\,p}$, and $\epsilon > 0$. Since C_0^∞ functions are dense in Lorentz spaces, let $\varphi \in L_{mp/(m-p),\,p} \cap C_0^\infty$ be such that

$$\Big| \|f^*\|_{mp/(m-p),\,p} - \|\varphi\|_{mp/(m-p),\,p} \Big| \leq \|f^* - \varphi\|_{mp/(m-p),\,p} < \epsilon \tag{3.11}$$

and therefore

$$\|\varphi^*\|_{mp/(m-p),\,p} \leq \|f^*\|_{mp/(m-p),\,p} + \epsilon \tag{3.12}$$

In [2] it is proved that the property of a function to be Lipschitz is inherited by its decreasing rearrangement: therefore φ^* is Lipschitz and therefore the function ψ defined by

$$\psi(x) = \varphi^*((2\max\{|x_i|\})^m) \quad \forall\, x = (x_1, \ldots, x_m) \in Q_m$$

is such that

$$\psi \in W_0^{1,p}(Q_m) \tag{3.13}$$

$$\psi^* = \varphi^* \tag{3.14}$$

We may apply the argument above choosing $\epsilon = 1/k$, where $k \in \mathbb{N}$, so that we get a sequence (φ_k), which is Cauchy in $L_{mp/(m-p),\,p}$, enjoying the properties (3.13) and (3.14) with ψ replaced by ψ_k. Plugging ψ_k in (3.9) we get, by (3.14), that $\varphi_k^* \in Y$, which means that $\varphi_k \in Y$ (since Y is rearrangement-invariant).

On the other hand, for any $k, h \in \mathbb{N}$, the function $(\varphi_k - \varphi_h)^*$ is again Lipschitz and by the same argument as before is the decreasing rearrangement of a function in $W_0^{1,p}(Q_m)$. By (3.9) we get

$$\|\varphi_k - \varphi_h\|_Y \leq c_p m^{1/2} \|\varphi_k - \varphi_h\|_{mp/(m-p),\, p} \tag{3.15}$$

and therefore (φ_k) is Cauchy also in Y. Since Y is complete, (φ_k) is convergent, and this shows that $f^* \in Y$. Letting $k \to \infty$ in (3.15), we get

$$\|f^* - \varphi_h\|_Y \leq c_p m^{1/2} \|f^* - \varphi_h\|_{mp/(m-p),\, p}$$

from which, by (3.11),

$$\|f^*\|_Y \leq \|\varphi_h\|_Y + c_p m^{1/2} \frac{1}{h}. \tag{3.16}$$

By (3.12) and (3.16)

$$\frac{\|f^*\|_Y}{m^{1/2}\|f^*\|_{mp/(m-p),\, p}} \leq \frac{\|\varphi_h\|_Y + c_p m^{1/2} \frac{1}{h}}{m^{1/2}\left(\|\varphi_h^*\|_{mp/(m-p),\, p} - \frac{1}{h}\right)}$$

and the right hand side is bounded by a constant depending only on p, let's call it again c_p, because of (3.9) applied to φ_h.

Hence

$$\|f^*\|_Y \leq c_p m^{1/2} \|f^*\|_{mp/(m-p),\, p} \quad \text{for all } f \in L_{mp/(m-p),\, p} \tag{3.17}$$

Now let f be any measurable function over $(0, 1) \subset \mathbb{R}$; the interesting case is that it belongs to at least one space $L_{mp/(m-p),\, p}$, $m > p$. Since the sequence of spaces $(L_{mp/(m-p),\, p})_m$ is increasing, the same f can be considered as element of infinitely many spaces $L_{mp/(m-p),\, p}$, and the norm $\|f^*\|_Y$ can be estimated by all possible right hand sides in (3.17). The consequence is that for all measurable functions f we get

$$\|f^*\|_Y \leq c_p \inf_{m>p} m^{1/2} \|f^*\|_{mp/(m-p),\, p}$$

i.e., by Lemma 3.1,

$$\|f^*\|_Y \leq c_p \inf_{m>p} m^{1/2} \|f^*\|_{mp/(m-p)} \tag{3.18}$$

On the other hand, by (3.1), also $\|f\|_{(p,p'/2}$ is smaller than the right hand side of (3.18). We may therefore apply Lemma 3.2 with $\theta = p'/2$ and

$$\|f\|_X = \max\{\|f\|_{(p,p'/2},\ \|f\|_Y\}$$

to get

$$\max\{\|f\|_{(p,p'/2}\ \|f\|_Y\} \leq c_p\|f\|_{(p,p'/2}$$

from which

$$\|f\|_Y \leq c_p\|f\|_{(p,p'/2}$$

The theorem is proved. □

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