Families of irreptiles

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Abstract

We present several classes of polygons each admitting a dissection into finitely many smaller similar copies of itself.

1 Introduction

The question for tilings of the plane by congruent images of some given polygon A leads in a natural way to the concept of a reptile. A is called a *reptile* if it can be dissected into finitely many pairwise congruent images of A under suitable similarities of the plane. We speak of a *dissection* if the covering pieces can only have boundary points in common.

Many known examples of reptiles are polyominoes or polyiamonds. A polygon is called a *polyomino (polyiamond)* if it has a connected interior and possesses an edge-to-edge dissection into finitely many congruent squares (equilateral triangles).

A family of reptiles that are polyominoes can be obtained as follows (see [2], [3, p. 97], [5, p. 54], and the first illustration in Figure 1). Fix an integer $k \geq 1$ and dissect a square S into $(2k)^2$ congruent smaller squares $S_1, \ldots, S_{(2k)^2}$. Let δ_c denote the rotation about the centre c of S by an angle of $\frac{\pi}{2}$. Now choose a simple polygonal arc Γ contained in the union of the boundaries of the pieces S_i that connects c with a point from the boundary of S such that $\Gamma \cap \delta_c(\Gamma) = \{c\}$. Then Γ, $\delta_c(\Gamma)$, and a quarter of the boundary of S bound a reptile $A \subseteq S$ (shaded in Figure 1). Indeed, since A splits into k^2 congruent squares S_i and since S as well as any other square admits a dissection in four congruent similar copies of A, A can be dissected into $4k^2 = (2k)^2$ pairwise congruent images of A under suitable similarities. A similar procedure starting with an equilateral triangle gives a family of reptiles that are polyiamonds (see [2], [5, p. 175], and the second part of Figure 1).

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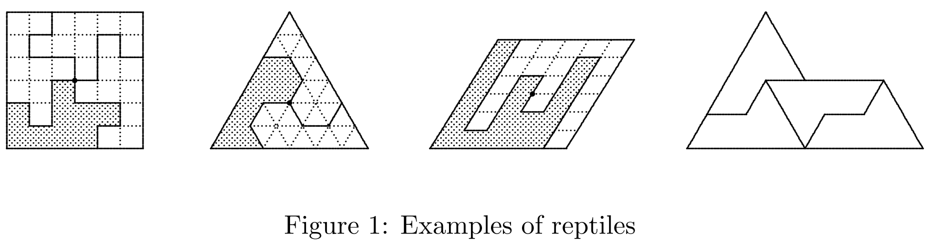


Figure 1: Examples of reptiles

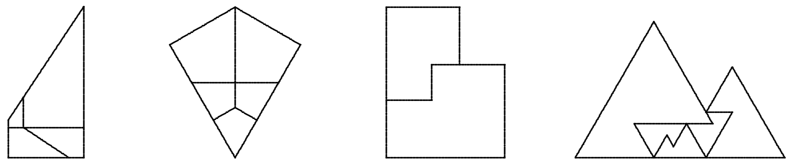


Figure 2: Examples of irreptiles (see [8])

The third family illustrated in Figure 1 contains reptiles obtained from an arbitrary parallelogram P. One splits P into $(2k)^2$ congruent smaller similar copies $P_1, \dots, P_{(2k)^2}$ and fixes a simple polygonal arc Γ connecting two points of the boundary of P and symmetric with respect to the centre of P that is contained in the union of the boundaries of the pieces P_i. Then Γ dissects P into a polygon A and a congruent image of A. Hence A is a reptile, because A is the union of $2k^2$ of the pieces P_i, which are similar to P. In the context of polyominoes this idea can be found for example in [3, p. 97] and [5, p. 52].

The last example in Figure 1 is the so-called sphinx, which is a polyiamond composed by six equilateral triangles. It is a reptile with an odd number of vertexes, whereas the number of vertexes in all previous examples is even.

If a polygon A splits into finitely many similar copies of A that are not necessarily pairwise congruent then A is called an *irreptile* (see [8]). Every irreptile A gives rise to a dissection of the plane into images of A under similarities whose similarity ratios are at least 1. Scherer's nice book [8] gives an insight into the great richness of irreptiles. Figure 2 shows four examples. Further results on irreptiles appear sporadically in the literature or on the internet, mainly in the context of recreational mathematics and often concerning polyominoes or polyiamonds (see e.g. [7]).

In the present paper we describe several rather large classes of irreptiles, that are no polyominoes and, mostly, no polyiamonds. These classes contain many examples from [8]. Our emphasis is on a large variety of shapes, but not on optimal dissections (i.e. into a minimal number of pieces). In some cases it will turn out that it is possible to obtain dissections into similar copies being based on proper similarities only.

Irreptiles with many vertexes cannot be convex. They have to have so-called *reflex vertexes* where the size of the corresponding inner angle exceeds π. Indeed, if an irreptile has a total number of v vertexes then the number v_r of reflex vertexes is bounded by

$$\tfrac{v}{2} - 3 < v_r < \tfrac{v}{2} - 1$$

(see [6], [4]). So if v is even then $v_r = \frac{v}{2} - 2$ and if v is odd then $v_r \in \{\frac{v-5}{2}, \frac{v-3}{2}\}$. As far as we know, in all known examples with odd v one has $v_r = \frac{v-3}{2}$. Is $v_r = \frac{v-5}{2}$ possible? Does there exist a convex pentagon that is an irreptile?

In the sequel we use the symbols $\mathrm{cl}(A)$, $\mathrm{int}(A)$, and $\mathrm{conv}(A)$ for denoting the closure, the interior, and the convex hull of a set $A \subseteq \mathbb{R}^2$, respectively.

2 An uncountable family based on isosceles triangles

Given a real parameter $\xi > 0$, the origin $0 = (0,0)$ together with the vectors $b_1 = (1,0)$ and $b_2 = (\frac{1}{2}, \xi)$ span an isosceles triangle. We fix moreover two integer parameters $0 < k \leq l$. Then $c = kb_1 + lb_2$ is the centre of the parallelogram $P = \mathrm{conv}\{0, 2kb_1, 2lb_2, 2kb_1 + 2lb_2\}$. We denote the reflection with respect to the point c by δ_c. Now we pick a simple polygonal arc Γ with the following properties:

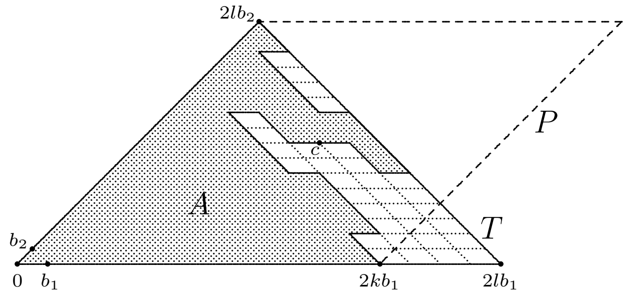


Figure 3: Proof of Proposition 1

(i) Γ connects the vertexes $2kb_1$ and $2lb_2$ of P and $\Gamma \setminus \{2kb_1, 2lb_2\} \subseteq \mathrm{int}(P)$,

(ii) all vertexes of Γ belong to the lattice $\mathbb{Z}b_1 + \mathbb{Z}b_2 = \{ib_1 + jb_2 : i, j \in \mathbb{Z}\}$ and all edges of Γ are parallel to b_1 or $b_2 - b_1 = (-\frac{1}{2}, \xi)$,

(iii) Γ is symmetric with respect to c, that is, $\Gamma = \delta_c(\Gamma)$, and

(iv) Γ is contained in the triangle $T = \mathrm{conv}\{0, 2lb_1, 2lb_2\}$.

Then Γ splits P into two polygons A and $\delta_c(A)$, where $0 \in A$ without loss of generality (see Figure 3).

Proposition 1. (a) *The polygon A defined above is an irreptile. Moreover, if the length $\|b_2\| = \sqrt{\frac{1}{4} + \xi^2}$ is rational then there exists a dissection of A into finitely many images of A under suitable proper similarities.*

(b) *The number v_A of vertexes of A is odd. If v_Γ denotes the number of inner vertexes of Γ then $v_A = v_\Gamma + 3$ and the number of reflex vertexes of A is $\frac{v_\Gamma}{2}$.*

Proof. **(a)** A splits into proper congruent images of the triangle $\mathrm{conv}\{0, b_1, b_2\}$, which itself is a proper similar copy of T. Hence it suffices to show that T can be dissected into finitely many similar images of A. Since $A \subseteq T$, it is enough to show that $\mathrm{cl}(T \setminus A)$ has a dissection of that kind.

Both A and T have only one edge parallel to b_2, namely $\mathrm{conv}\{0, 2lb_2\}$. Hence the boundary of $\mathrm{cl}(T \setminus A)$ consists of line segments parallel to b_1 and $b_2 - b_1$ only (see Figure 3). Since all vertexes of $\mathrm{cl}(T \setminus A)$ belong to the lattice $\mathbb{Z}b_1 + \mathbb{Z}b_2 = \mathbb{Z}b_1 + \mathbb{Z}(b_2 - b_1)$, $\mathrm{cl}(T \setminus A)$ splits into finitely many translates of the parallelogram $P^- = \mathrm{conv}\{0, b_1, b_2 - b_1, b_2\}$ (dotted in Figure 3). Thus it remains to find a dissection of P^- into similar copies of A, which have to be proper if $\|b_2\|$ is rational.

The parallelogram $P^+ = \mathrm{conv}\{0, b_1, b_2, b_1 + b_2\}$ splits into lk translates of

$$\mathrm{conv}\{0, \tfrac{b_1}{l}, \tfrac{b_2}{k}, \tfrac{b_1}{l} + \tfrac{b_2}{k}\} = \tfrac{1}{2kl} P = \tfrac{1}{2kl}(A \cup \delta_c(A)).$$

P^- is the image of P^+ under a reflection with respect to a vertical axis. This gives rise to a dissection of P^- into $2kl$ similar copies of A. So A is an irreptile.

If $\|b_2\| = \frac{m}{n}$, $m, n \in \{1, 2, \ldots\}$, is rational then there is a rotation γ fixing 0 such that $\gamma(b_2) = -\|b_2\| b_1 = -\frac{m}{n} b_1$ and hence $\gamma(b_1) = \frac{b_2 - b_1}{\|b_2 - b_1\|} = \frac{n}{m}(b_2 - b_1)$. Now P^- is splitted into kn^2lm^2 translates of

$$\mathrm{conv}\{0, -\tfrac{b_1}{kn^2}, \tfrac{b_2 - b_1}{lm^2}, -\tfrac{b_1}{kn^2} + \tfrac{b_2 - b_1}{lm^2}\} = \gamma(\tfrac{1}{2klmn} P) = \gamma\big(\tfrac{1}{2klmn}(A \cup \delta_c(A))\big).$$

This gives a dissection into proper similar images of A.

(b) $v_A = v_\Gamma + 3$, because 0, $2kb_1$, and $2lb_2$ are the only vertexes of A that are no inner vertexes of Γ.

By (iii), c is no vertex of Γ and the vertexes of Γ appear in pairs $(x, \delta_c(x))$. So v_Γ is even and v_A is odd.

If x is an inner vertex of Γ then x is a reflex vertex of A if and only if $\delta_c(x)$ is a convex vertex of A. Thus the number of reflex vertexes of A is $\frac{v_\Gamma}{2}$. □

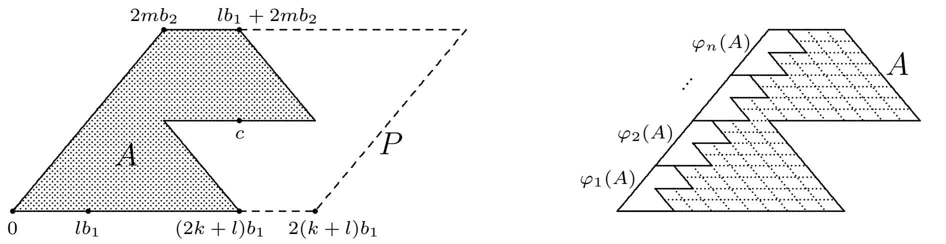


Figure 4: Proof of Proposition 2

Proposition 1 gives irreptiles with arbitrary odd numbers of vertexes $v_A \geq 3$. In the following we modify the construction for obtaining even numbers $v_A \geq 4$.

We choose ξ, b_1, and b_2 as above and fix arbitrary integer parameters $k, l, m \geq 1$. Now we consider the parallelogram $P = \text{conv}\{0, 2(k+l)b_1, 2mb_2, 2(k+l)b_1 + 2mb_2\}$. δ_c denotes the reflection with respect to the centre $c = (k+l)b_1 + mb_2$ of P. We pick a simple polygonal arc Γ such that

(i) Γ connects $(2k+l)b_1$ with $lb_1 + 2mb_2$ and $\Gamma \setminus \{(2k+l)b_1, lb_1 + 2mb_2\} \subseteq \text{int}(P)$,

(ii) all vertexes of Γ belong to the lattice $\mathbb{Z}b_1 + \mathbb{Z}b_2$ and all edges of Γ are parallel to b_1 or $b_2 - b_1$, and

(iii) Γ is symmetric with respect to c.

Again Γ splits P into two polygons A and $\delta_c(A)$, where $0 \in A$ (see Figure 4).

Proposition 2. (a) *The polygon A defined above is an irreptile. Moreover, if the length $\|b_2\| = \sqrt{\frac{1}{4} + \xi^2}$ is rational then there exists a dissection of A into finitely many images of A under suitable proper similarities.*

(b) *The number v_A of vertexes of A is even. If v_Γ denotes the number of inner vertexes of Γ then $v_A = v_\Gamma + 4$ and the number of reflex vertexes of A is $\frac{v_\Gamma}{2}$.*

Proof. For every integer $n \geq 2$, the homothetic copies $\varphi_i(A) = \frac{1}{n}A + \frac{2m(i-1)}{n}b_2$, $1 \leq i \leq n$, have pairwise disjoint interiors and cover the edge $\text{conv}\{0, 2mb_2\}$ of A (see the right-hand part of Figure 4). We assume n to be chosen large enough such that all $\varphi_i(A)$ are contained in A. Then the remaining polygon $\text{cl}(A \setminus (\varphi_1(A) \cup \ldots \cup \varphi_n(A)))$ is formed by vertexes from the lattice $\frac{1}{n}(\mathbb{Z}b_1 + \mathbb{Z}b_2) = \frac{1}{n}(\mathbb{Z}b_1 + \mathbb{Z}(b_2 - b_1))$ and by edges parallel to b_1 and $b_2 - b_1$ only. Thus we can decompose it into translates of $\frac{1}{n}P^- = \frac{1}{n}\text{conv}\{0, b_1, b_2 - b_1, b_2\}$ (dotted in Figure 4) and it suffices to prove that P^- admits a dissection into suitable similar copies of A. This remainder of part **(a)** and the verification of part **(b)** can be treated as in the proof of Proposition 1. □

3 A countable family of polyiamonds

Let $b_1 = (0, 1)$ and $b_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and fix arbitrary integers $k, l > 0$. δ_c is to denote the reflection with respect to the centre $c = kb_1 + lb_2$ of the parallelogram $P = \text{conv}\{0, 2kb_1, 2lb_2, 2kb_1 + 2lb_2\}$. Let Γ be a simple polygonal arc such that

(i) Γ connects the vertexes $2kb_1$ and $2lb_2$ of P and $\Gamma \setminus \{2kb_1, 2lb_2\} \subseteq \text{int}(P)$,

(ii) all vertexes of Γ belong to $\mathbb{Z}b_1 + \mathbb{Z}b_2$ and all edges of Γ are parallel to b_1 or $b_2 - b_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and

(iii) Γ is symmetric with respect to c.

Γ dissects P into two polygons A and $\delta_c(A)$, where $0 \in A$ (see the left-hand part of Figure 5). Note that this kind of polygons is closely related with that from the first part of the previous section. Here the parameter ξ is restricted to $\frac{\sqrt{3}}{2}$, so that

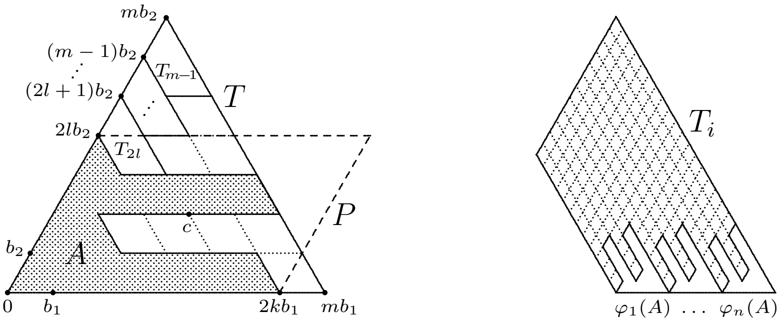


Figure 5: Proof of Proposition 3

A is a polyiamond. In contrast with that, the choice of k and l is more flexible, since $k > l$ is no longer forbidden, and the restriction (iv) from Section 2 is dropped.

Proposition 3. (a) *The polygon A defined above is an irreptile. There exists a dissection of A into finitely many images of A under suitable proper similarities.*

(b) *The number v_A of vertexes of A is odd. If v_Γ denotes the number of inner vertexes of Γ then $v_A = v_\Gamma + 3$ and the number of reflex vertexes of A is $\frac{v_\Gamma}{2}$.*

Proof. Let m be the smallest integer such that the equilateral triangle $T = \mathrm{conv}\{0, mb_1, mb_2\}$ covers A. Since A splits into finitely many equilateral triangles, it suffices to show that T admits a dissection into finitely many proper similar copies of A.

In contrast with the situation of Proposition 1, now $\mathrm{cl}(T \setminus A)$ has an edge $\mathrm{conv}\{2lb_2, mb_2\}$ parallel to b_2. (It vanishes if $m = 2l$.) For every $i \in \{2l, \dots, m-1\}$, we define a trapezoid $T_i = \mathrm{conv}\{ib_2, (i-1)b_2 + b_1, (i-1)b_2 + 2b_1, (i+1)b_2\}$. Then $\mathrm{cl}(T \setminus (A \cup T_{2l} \cup \dots \cup T_{m-1}))$ has all its vertexes in $\mathbb{Z}b_1 + \mathbb{Z}b_2 = \mathbb{Z}b_1 + \mathbb{Z}(b_2 - b_1)$ and all its edges are parallel to b_1 or $b_2 - b_1$. We split it into parallelograms (dotted in the left-hand part of Figure 5) and dissect them into proper similar copies of A as we did in the proof of Proposition 1. Now it remains to prove that every T_i, $2l \le i \le m-1$, has a dissection of the same kind.

Let γ be a rotation about the origin with angle $\frac{2\pi}{3}$. The lower edge of $\gamma(A)$ is the only one parallel to b_1 and has length $2l$. Hence, for every integer $n \ge 1$, there exist translates $\varphi_j(A)$, $1 \le j \le n$, of $\frac{1}{2ln}\gamma(A)$ such that the lower edge of T_i splits into the lower edges of $\varphi_1(A), \dots, \varphi_n(A)$. We assume n to be fixed large enough such that all these translates are subsets of T_i (see the right-hand part of Figure 5). The remaining polygon $\mathrm{cl}(T_i \setminus (\varphi_1(A) \cup \dots \cup \varphi_n(A))$ is formed by vertexes from $\frac{1}{2ln}(\mathbb{Z}b_2 + \mathbb{Z}(b_2 - b_1))$ and and edges parallel to b_2 and $b_2 - b_1$. Hence it splits into finitely many rhombs similar to $\mathrm{conv}\{0, b_2, b_2 - b_1, 2b_2 - b_1\}$ (dotted in Figure 5). Dissections of these rhombs into proper similar copies of A are obtained as in the proof of Proposition 1. This completes the verification of **(a)**.

Claim **(b)** can be proved as in Proposition 1. □

One example of an irreptile found by the last construction is the sphinx (see the last example from Figure 1). Proposition 3 says that a dissection representing the sphinx as an irreptile can be realized by the aid of proper similarities only (see Figure 6 as an example). It is worth noting that this is impossible for the sphinx as a reptile.

Proposition 4. *Let the sphinx S be dissected into $n \ge 2$ pairwise congruent similar copies $\varphi_1(S), \dots, \varphi_n(S)$ of itself. Then at least one of the similarities φ_i is an improper map.*

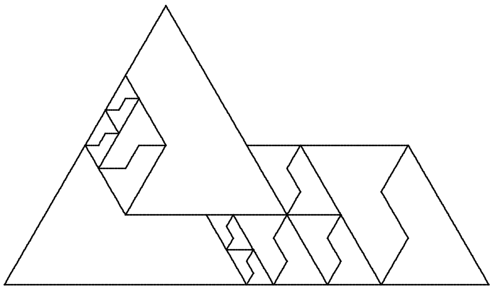
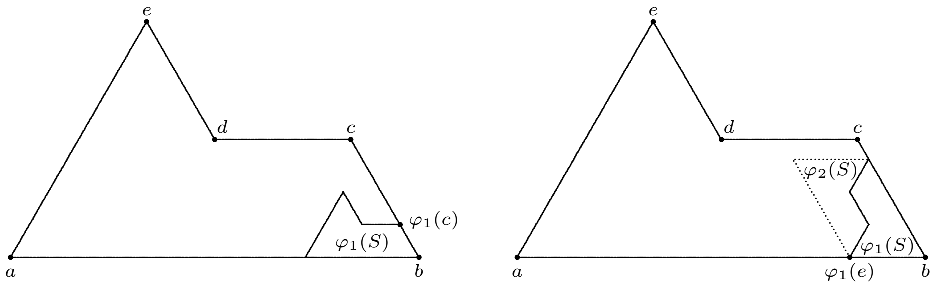


Figure 6: A dissection of the sphinx based on proper similarities only



Case 1: $b = \varphi_1(b)$. Case 2: $b = \varphi_1(a)$.

Figure 7: Proof of Proposition 4

Proof. Suppose that all φ_i, $1 \leq i \leq n$, are proper. Let the vertexes of S be denoted by $a, \ldots, e$ as in Figure 7. Then b must be a vertex of one of the $\varphi_i(S)$, say of $\varphi_1(S)$. The inclusion $\varphi_1(S) \subseteq S$ is possible only if $b = \varphi_1(a)$ or $b = \varphi_1(b)$.

In the latter case (Case 1 in Figure 7) $\varphi_1(c)$ had to be a vertex of one of the tiles $\varphi_i(S)$, $2 \leq i \leq n$. This is impossible, because all $\varphi_i(S)$ are proper congruent images of $\varphi_1(S)$.

In the remaining case $b = \varphi_1(a)$ (Case 2 in Figure 7) the edge $\operatorname{conv}\{\varphi_1(e), \varphi_1(d)\}$ of $\varphi_1(S)$ had to be an edge of another tile, say of $\varphi_2(S)$. Again using that $\varphi_2(S)$ is a proper congruent image of $\varphi_1(S)$ we conclude that the position of $\varphi_2(S)$ relative to $\varphi_1(S)$ is as it is illustrated by the dotted lines in Figure 7.

Now $\varphi_1(e)$ plays a similar role in the remaining polygon $\varphi_3(S) \cup \ldots \cup \varphi_n(S)$ as the vertex b did with respect to S. Repeated application of the above arguments shows that the horizontal strip of S over the edge $\operatorname{conv}\{a, b\}$ whose height agrees with that of the parallelogram $P = \varphi_1(S) \cup \varphi_2(S)$ had to be dissected into translates of P, a contradiction. □

We close this section with the remark that an adjacent modification of the above construction yields polyiamond irreptiles with an even number of vertexes. This coincides with the particular case of Proposition 2 where $\xi = \frac{\sqrt{3}}{2}$.

4 A countable family based on isosceles right triangles

We fix two integers $1 \leq l \leq k$ and consider the rectangle $R = \operatorname{conv}\{0, 2ke_1, 2le_2, 2ke_1 + 2le_2\}$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$. δ_c is to denote the reflection with

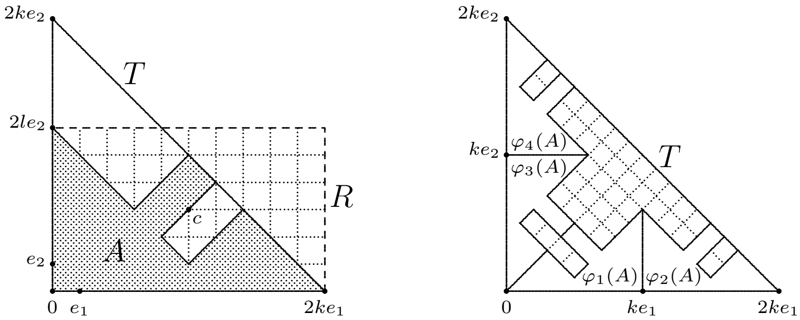


Figure 8: Proof of Proposition 5

respect to the centre $c = ke_1 + le_2$ of R. Let Γ be a simple polygonal arc such that

(i) Γ connects $2ke_1$ with $2le_2$ and $\Gamma \setminus \{2ke_1, 2le_2\} \subseteq \text{int}(R)$,

(ii) all vertexes of Γ belong to $\mathbb{Z} \times \mathbb{Z}$ and all edges of Γ are parallel to $e_1 + e_2$ or $e_1 - e_2$,

(iii) Γ is symmetric with respect to c, and

(iv) Γ is contained in the triangle $T = \text{conv}\{0, 2ke_1, 2ke_2\}$.

Γ dissects R into two polygons A and $\delta_c(A)$, where $0 \in A$ (see the left-hand part of Figure 8). In contrast with the previous examples, the present family contains polygons A whose edges have four different directions.

Proposition 5. **(a)** *The polygon A defined above is an irreptile.*

(b) *The number v_A of vertexes of A is odd. If v_Γ denotes the number of inner vertexes of Γ then $v_A = v_\Gamma + 3$ and the number of reflex vertexes of A is $\frac{v_\Gamma}{2}$.*

Proof. It suffices to show that T admits a dissection into finitely many similar copies of A, because A splits into isosceles right triangles. We choose similarities $\varphi_1, \ldots, \varphi_4$ with similarity ratio $\frac{1}{2}$ such that $T = \varphi_1(T) \cup \ldots \cup \varphi_4(T)$ and $\varphi_1(0) = \varphi_2(0) = ke_1$, $\varphi_3(0) = \varphi_4(0) = ke_2$, $\varphi_1(2ke_1) = \varphi_3(2ke_1) = 0$, $\varphi_2(2ke_1) = 2ke_1$, $\varphi_4(2ke_1) = 2ke_2$. Then the remainder $\text{cl}(T \setminus (\varphi_1(A) \cup \ldots \cup \varphi_4(A)))$ has all its vertexes in $\frac{1}{2}(\mathbb{Z} \times \mathbb{Z})$ and all its edges are parallel to $e_1 + e_2$ or $e_1 - e_2$ (see the right-hand part of Figure 8). Hence this remainder splits into squares (dotted in the illustration) which can be dissected into similar copies of $R = A \cup \delta_c(A)$. This proves **(a)**.

Claim **(b)** can be verified as in Proposition 1. □

5 An uncountable family related to rhombs

Let b_1 and b_2 be two vectors spanning a rhomb, that is, b_1 and b_2 are linearly independent and of the same length. Given an integer $k \geq 0$, we define A as the polygon bounded by the simply closed polygonal arc connecting 0, $(2k+1)b_1$, $(2k+1)b_1 + b_2$, $2kb_1 + b_2$, $2kb_1 + 2b_2$, $\ldots$, $(k+1)b_1 + kb_2$, $(k+1)b_1 + (k+1)b_2$, 0 (see the left-hand part of Figure 9).

Proposition 6. *The polygon A defined above is an irreptile. Among the $2k+3$ vertexes of A there are k reflex vertexes.*

Proof. Let γ be the reflection with respect to the axis $\mathbb{R}(b_1 + b_2) = \{\xi(b_1 + b_2) : \xi \in \mathbb{R}\}$ and let δ_c be the reflection with respect to the centre $c = (k + \frac{1}{2})b_1 + (k+1)b_2$ of the parallelogram $P = \text{conv}\{0, (2k+1)b_1, 2(k+1)b_2, (2k+1)b_1 + 2(k+1)b_2\}$. Then P is dissected into $A, \gamma(A), \delta_c(A)$, and $\delta_c\gamma(A)$ (see the left-hand part of Figure 9).

The images $\varphi_1(A) = \frac{k}{k+1}A$ and $\varphi_2(A) = -\frac{1}{k+1}\gamma(A) + (k+1)(b_1 + b_2)$ are contained in A, cover the edge $\text{conv}\{0, (k+1)(b_1 + b_2)\}$ of A, and have disjoint

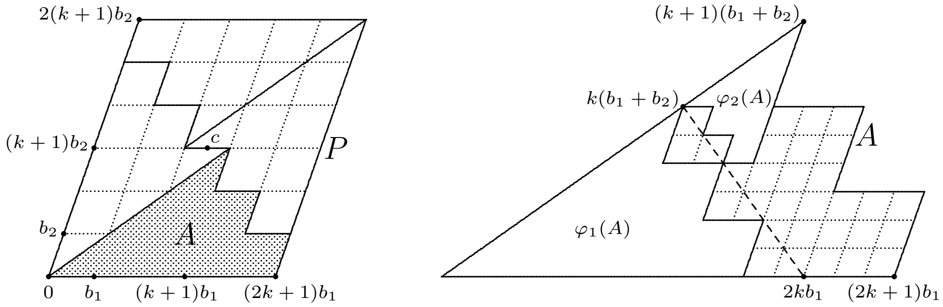


Figure 9: Proof of Proposition 6

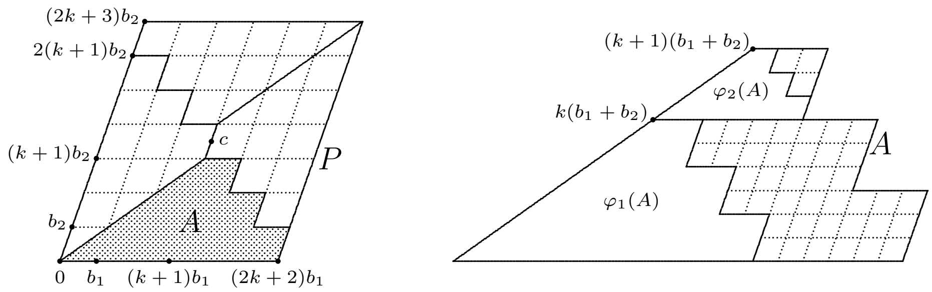


Figure 10: Proof of Proposition 7

interiors, because they are separated by the line through $2kb_1$ and $k(b_1+b_2)$ (see the right-hand part of Figure 9). Hence A splits into $\varphi_1(A)$, $\varphi_2(A)$, and finitely many rhombs similar to $\mathrm{conv}\{0, b_1, b_2, b_1+b_2\}$. Any of these rhombs can be dissected into finitely many similar copies of $P = A \cup \gamma(A) \cup \delta_c(A) \cup \delta_c\gamma(A)$. So A is an irreptile.

The vertexes of A can easily be counted. In particular, $(2k+1-i)b_1 + ib_2$, $1 \le i \le k$, are reflex vertexes. □

Proposition 6 gives irreptiles with an odd number of vertexes. Even numbers can be obtained as follows. We fix b_1, b_2, and k as above and define A as the polygon bounded by the polygonal arc 0, $(2k+2)b_1$, $(2k+2)b_1+b_2$, $(2k+1)b_1+b_2$, $(2k+1)b_1+2b_2$, $\ldots$, $(k+2)b_1+kb_2$, $(k+2)b_1+(k+1)b_2$, $(k+1)b_1+(k+1)b_2$, 0 (see the left-hand part of Figure 10).

Proposition 7. *The polygon A defined above is an irreptile. Among the $2k+4$ vertexes of A there are k reflex vertexes.*

Proof. We refer to Figure 10 and leave the details to the reader. □

6 A countable family of non-lattice pentagons

Each of the previously defined irreptiles has its vertexes in some lattice $\mathbb{Z}b_1+\mathbb{Z}b_2$. In the following we describe a family of pentagons including infinitely many non-lattice members. We use the following technical tool.

Lemma. *Let P_λ and P_μ be two parallelograms with the same sizes of angles, P_λ having edges of lengths 1 and λ and P_μ having edges of lengths 1 and μ. If there are an integer $m \ge 0$ and rational numbers $r_0, \ldots, r_m$ with $r_0 \ge 0$ and $r_i > 0$ for*

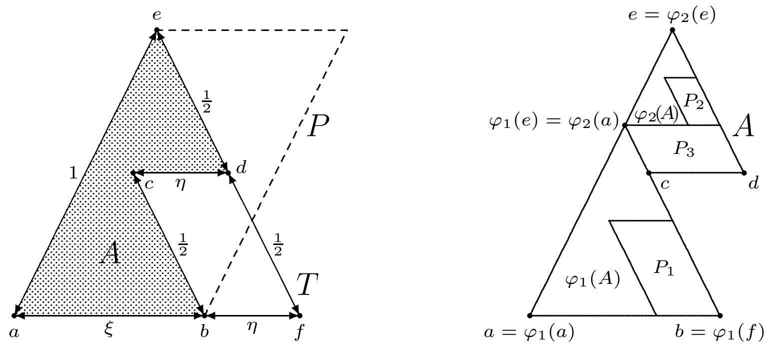


Figure 11: Proof of Proposition 8

$1 \le i \le m$ such that

$$\lambda = r_0\mu + \cfrac{1}{r_1\mu + \cfrac{1}{\ddots + \cfrac{1}{r_m\mu}}}$$

then P_λ can be dissected into finitely many similar copies of P_μ.

Proof. Theorem 5 from [1] includes the above claim for rectangles. The generalization to parallelograms is obvious. □

Proposition 8. *Let ξ, η be real numbers with $0 < \eta < \xi$ and $\xi + \eta < 2$ such that there are integers $m, n \ge 0$ and rational numbers $p_0, \dots, p_m$, $q_0, \dots, q_n$ with $p_0, q_0 \ge 0$ and $p_1, \dots, p_m, q_1, \dots, q_n > 0$ satisfying*

$$2\eta = p_0\xi + \cfrac{1}{p_1\xi + \cfrac{1}{\ddots + \cfrac{1}{p_m\xi}}} \quad \text{and} \quad \frac{2\eta(\xi+\eta)}{\xi-\eta} = q_0\xi + \cfrac{1}{q_1\xi + \cfrac{1}{\ddots + \cfrac{1}{q_n\xi}}}.$$

Moreover, let the pentagon A be obtained by cutting off a parallelogram of edge lengths $\frac{1}{2}$ and η from an isosceles triangle T with edges of lengths 1, 1, and $\xi + \eta$ as illustrated in Figure 11. Then A is an irreptile.

Proof. We use a, b, c, d, e, f for denoting the vertexes of A and T as in Figure 11. δ is to denote the reflection with respect to the centre of the segment from c to d. Then A and $\delta(A)$ form a dissection of a parallelogram P with edges of lengths 1 and ξ.

Let φ_1 and φ_2 be dilatations with fixed points a and e, respectively, such that $\varphi_1(f) = b$ and $\varphi_2(a) = \varphi_1(e)$. The similarity ratio of φ_1 is $\frac{\|b-a\|}{\|f-a\|} = \frac{\xi}{\xi+\eta}$. Now A splits into $\varphi_1(A)$, $\varphi_2(A)$, and three parallelograms P_1, P_2, P_3 with angles of the same sizes as those of P (see the right-hand part of Figure 11).

Both P_1 and P_2 are similar to the parallelogram $\operatorname{conv}\{b, c, d, f\}$. Hence in both cases the ratio of the edge lengths is $1 : 2\eta$. By the lemma, the first technical assumption above guarantees that P_1 and P_2 can be dissected into finitely many similar copies of $P = A \cup \delta(A)$.

The lengths of the edges of P_3 are $\|d - c\| = \eta$ and

$$\begin{aligned}\|\varphi_1(e) - c\| &= \|\varphi_1(e) - \varphi_1(f)\| - \|c - b\| \\ &= \tfrac{\xi}{\xi+\eta}\|e - f\| - \|c - b\| = \tfrac{\xi}{\xi+\eta} - \tfrac{1}{2} = \tfrac{\xi-\eta}{2(\xi+\eta)}.\end{aligned}$$

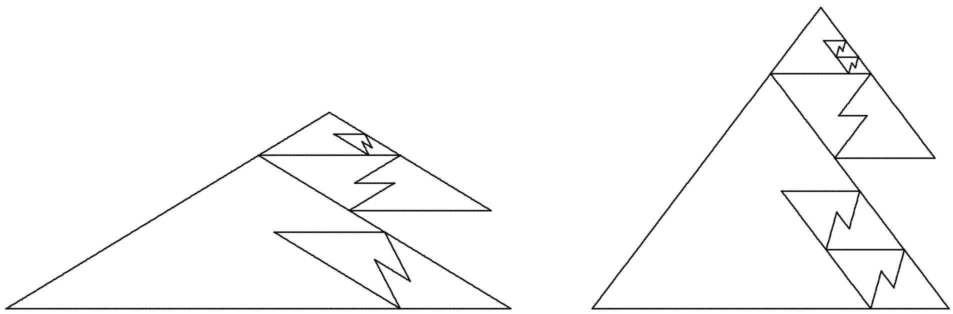


Figure 12: Two non-lattice examples

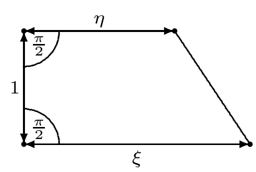


Figure 13: The trapezoid $H(\xi, \eta)$

Hence P_3 is similar to a parallelogram with edges of lengths 1 and $\frac{\|d-c\|}{\|\varphi_1(e)-c\|} = \frac{2\eta(\xi+\eta)}{\xi-\eta}$. The second technical assumption and the lemma show that P_3 admits a finite dissection into similar copies of $P = A \cup \delta(A)$, too. This completes the proof. $\square$

For obtaining a class of particular examples, we consider the parameters $m = 1$, $p_0 = 0$, $p_1 = k \in \{1, 2, \ldots\}$, $n = 0$, and $q_0 = 1$. This gives $\xi = \frac{2}{\sqrt{2k(\sqrt{17}-3)}}$ and $\eta = \frac{\sqrt{17}-3}{4}\xi$, in particular $0 < \eta < \xi$ and $\xi + \eta < 2$ for all $k \in \{1, 2, \ldots\}$. All these examples are non-lattice polygons, because $\frac{\|d-c\|}{\|b-a\|} = \frac{\eta}{\xi} = \frac{\sqrt{17}-3}{4}$ is irrational. Figure 12 shows the cases $k = 1$ and $k = 2$.

7 An uncountable family of trapezoids

In [8] Scherer introduces trapezoids $H(\xi, \eta)$ whose parallel edges of lengths ξ and η are perpendicular to a third edge of length 1 (see Figure 13). He shows that, for every $\xi > 0$, $H(\xi, \frac{1}{\xi})$ splits into four smaller similar copies of $H(\xi, \frac{1}{\xi})$. This gives another family of non-lattice irreptiles. The example with $\xi = 2$ is illustrated in Figure 2.

Moreover, Scherer shows that $H(\xi, \eta)$ is an irreptile if both ξ and η are rational. This admits the following generalization.

Proposition 9. *If the ratio λ of the lengths of the parallel edges of a trapezoid T satisfies $\frac{\lambda}{(\lambda+1)^2} = \frac{m}{n}$ with $m, n \in \{1, 2, \ldots\}$ then there exists a dissection of T into $2(m + n + 1)$ proper similar copies of T.*

Proof. Suppose that the the parallel edges are horizontal and have the lengths λ and 1 without loss of generality. Figure 14 illustrates the required dissection of T. P is a parallelogram formed by two congruent copies of T. The length of the horizontal edges of P is $\lambda + 1$. The similarity ratio of the maps $\gamma, \gamma_1, \ldots, \gamma_m$ is $\varrho_1 = \frac{\lambda}{\lambda+m(\lambda+1)}$.

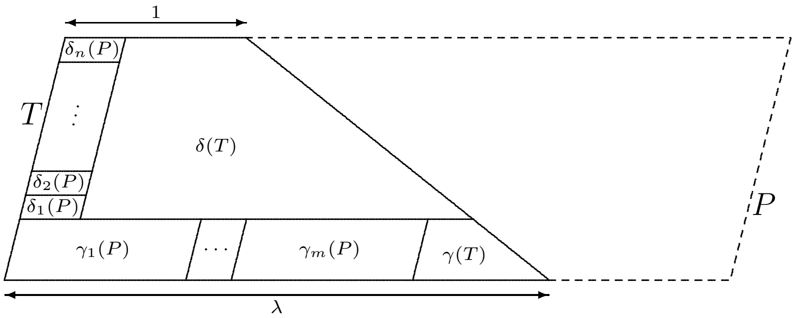


Figure 14: Proof of Proposition 9

Hence the length of the lower edge of the trapezoid $\gamma(T) \cup \gamma_1(P) \cup \ldots \cup \gamma_m(P)$ coincides with that of the lower edge of T. For δ, we choose a ratio $\varrho_2 = 1 - \varrho_1 = \frac{m(\lambda+1)}{\lambda+m(\lambda+1)}$, so that the heights of $\gamma(T) \cup \gamma_1(P) \cup \ldots \cup \gamma_m(P)$ and of $\delta(T)$ add up to that of T. Finally, $\delta_1, \ldots, \delta_n$ use the ratio $\varrho_3 = \frac{\varrho_2}{n} = \frac{m(\lambda+1)}{n(\lambda+m(\lambda+1))}$. Hence $\delta(T) \cup \delta_1(P) \cup \ldots \cup \delta_n(P)$ is a trapezoid, too. It remains to show that the length of its upper edge agrees with that of T, that is, $1\varrho_2 + (\lambda+1)\varrho_3 = 1$. One easily checks this by the aid of the assumption $\frac{\lambda}{(\lambda+1)^2} = \frac{m}{n}$. □

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