A global construction of homogeneous planar STIT tessellations

Joseph Mecke, Friedrich-Schiller-Universität Jena,

Werner Nagel, Friedrich-Schiller-Universität Jena,

Viola Weiß, Fachhochschule Jena

Abstract
Homogeneous (stationary) random tessellations of the Euclidean plane are constructed which have the characteristic property to be stable with respect to iteration (or nesting), STIT for short. A new approach is presented that describes the tessellation in the whole plane. So far, it was only known how to construct those tessellations within bounded windows.

Key words: stochastic geometry; random tessellation; Poisson point processes; iteration/nesting of tessellations; stability of distributions

AMS subject classification: 60D05, 52A22

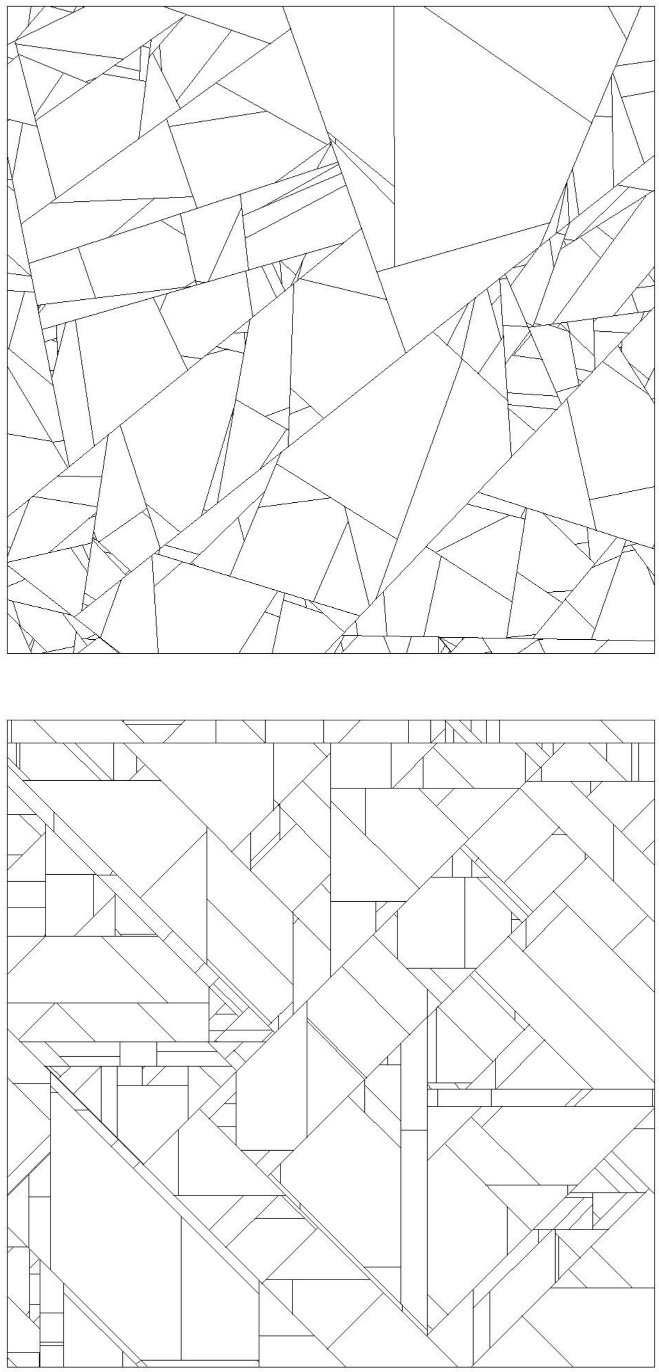

Figure 1: *Simulations of homogeneous STIT tessellations. The directional distributions are discrete uniform distributions on 8 or 4 directions, resp. (Kindly provided by Joachim Ohser, Hochschule Darmstadt.)*

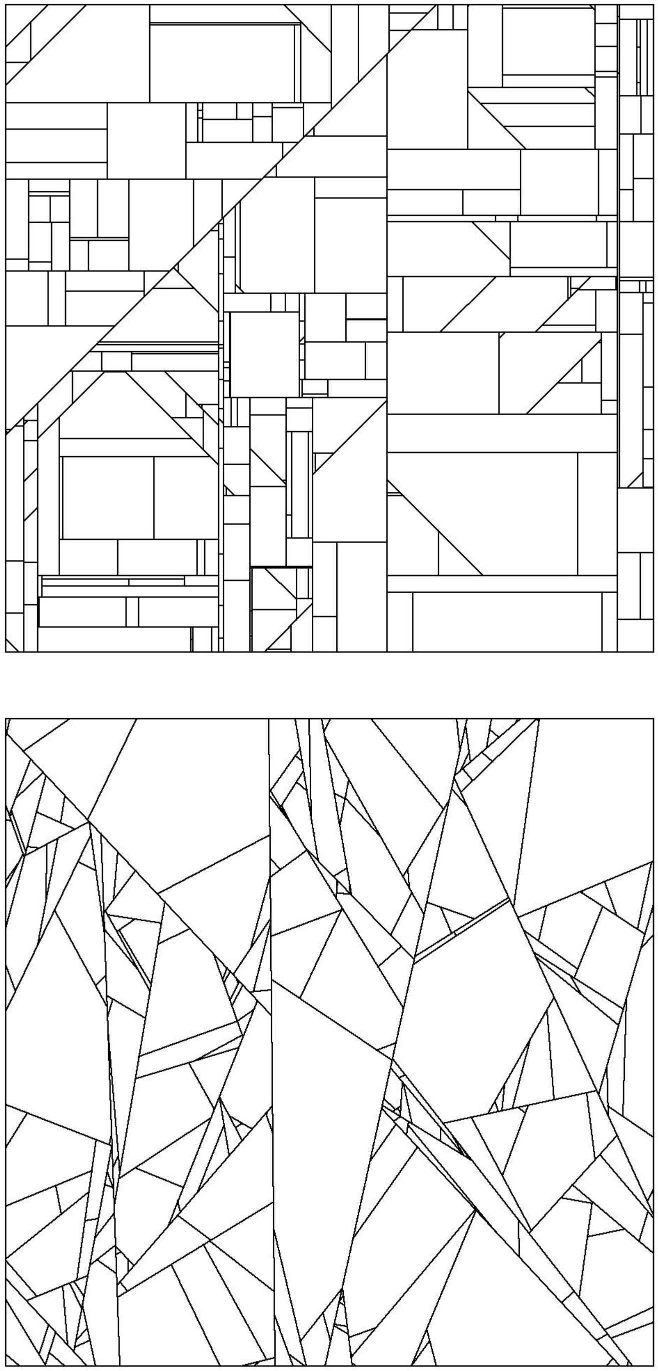

Figure 2: *Simulations of homogeneous STIT tessellations. The directional distributions are: Horizontal and vertical with 5/12 and the other two directions with 1/12 each (top), and directional distribution with density* $|\cos\alpha|/2$ *(bottom). (Kindly provided by Joachim Ohser, Hochschule Darmstadt.)*

1 Introduction

In an earlier paper [7] a new type of random tessellations was introduced, the so-called STIT tessellations. This name indicates that a characteristic feature is their stochastic stability with respect to the operation iteration (or nesting) of tessellations. The existence and a uniqueness result were already shown and a geometric construction was described that can be carried out in any bounded window. But it was not clear how to extend the construction to the whole infinite plane or space.

In the present paper an alternative construction of the planar STIT tessellations is developed, and this construction is not restricted to a window. Basically, the construction is given by a deterministic map of a Poisson point process Π on $[0,\pi)\times\mathbb{R}\times(0,\infty)$, where $\mathbb{R}$ denotes the set of real numbers. The points are interpreted as marked lines in the plane where the mark indicates the time of birth of the line. Thus, for any time $t>0$ the set of all lines that are already born, forms a tessellation. The key idea is to consider the *o*-cells, i.e. the cells containing the origin, as a process depending on the time t. This sequence of *o*-cells is monotonically decreasing. Since they are filling the whole plane they give rise to a non-homogeneous (i.e. spatially non-stationary) tessellation Ψ_t where the cells are the set differences in the descending sequence of *o*-cells. In a final step, the cells of the tessellation Ψ_t are subdivided by an iteration/nesting of a tessellation according to the above mentioned 'old' construction in [7] for bounded windows.

A random tessellation in the plane can be described as a random ensemble of non-overlapping compact convex polygons – the cells – that fill the plane. Equivalently, it can be considered as the random closed set – RACS for short – of all the boundaries of the cells. We will deliberately make use of both the definitions.

In order to prove that the new construction indeed yields a homogeneous STIT tessellation its capacity functional is calculated and it is shown that it coincides with that one which was given in [7].

In the appendix we recall the definitions of iteration and of STIT.

2 The construction

The construction starts with an auxiliary spatial-temporal line process in the plane. The generated sequences of *o*-cells is then used as the 'rough material'. Finally, this ensemble of convex polygons is divided further by random chords. This last step can also be understood as a generalized iteration of tessellations.

2.1 A birth process for lines in the plane

In the Euclidean plane $\mathbb{R}^2$ we denote a line by $g(\alpha,p)=\{(x,y)\in\mathbb{R}^2 : x\cos\alpha + y\sin\alpha = p\}$, $(\alpha,p)\in[0,\pi)\times\mathbb{R}$, i.e. the line with the signed distance $p\in\mathbb{R}$ from the origin and the normal direction $\alpha\in[0,\pi)$ w.r.t. the abscissa axis.

The starting point is a Poisson point process Π on the space $[0,\pi) \times \mathbb{R} \times (0,\infty)$ with the intensity measure

$$\tilde{\Lambda} = \vartheta \times \ell \times \ell_+ \tag{1}$$

where ϑ is a probability measure that is not concentrated on a single point, i.e. $\vartheta(\{\alpha\}) < 1$ for all $\alpha \in [0,\pi)$, ℓ is the Lebesgue measure on $\mathbb{R}$, and ℓ_+ is its restriction to $(0,\infty)$, i.e. $\ell_+(\cdot) = \ell(\cdot \cap (0,\infty))$.

To point $(\alpha, p, t) \in [0,\pi) \times \mathbb{R} \times (0,\infty)$ that belongs to Π there corresponds the birth of the line $g(\alpha, p)$ at time t.

For $t > 0$ define

$$\Gamma_t = \{g(\alpha,p) : (\alpha,p,s) \in \Pi, s < t\}. \tag{2}$$

This is a homogeneous (spatially stationary, i.e. its distribution is invariant w.r.t. translations in the plane) line process with directional distribution ϑ and length intensity (mean total line length per unit area) $L_A(\Gamma_t) = t$. Thus we have a stochastic process $(\Gamma_t)_{t>0}$ on the space of line ensembles. The definition yields

$$\Gamma_{t_1} \subseteq \Gamma_{t_2} \quad \text{for} \quad t_1 < t_2. \tag{3}$$

Furthermore, due to the properties of Poisson process Π, the line processes Γ_t and $\Gamma_{t_2} \setminus \Gamma_{t_1}$ are independent if $t \leq t_1 < t_2$.

2.2 The process of cells that contain the origin

For all $t > 0$ the Poisson line process Γ_t is a.s. (almost surely) not empty. Since it is assumed that the directional distribution ϑ is not concentrated in a single point, Γ_t a.s. generates a tessellation with a compact convex polygon Z_t that contains the origin o in its interior. This polygon will be referred to as the o-cell of Γ_t. This yields the stochastic process $(Z_t)_{t>0}$. In the following, several of the assertions hold a.s. even if we do not indicate this everywhere.

The isotony (3) implies

$$Z_{t_1} \supseteq Z_{t_2} \quad \text{for} \quad t_1 < t_2. \tag{4}$$

Let $\mathbb{Z}$ denote the set of all integers. There is a monotonic sequence $(\sigma_k)_{k\in\mathbb{Z}}$ of times where $(Z_t)_{t>0}$ changes its state. These σ_k can be defined as follows. For any $t > 0$ there is a finite random set of points $(\alpha_1, p_1, s_1), \ldots, (\alpha_\eta, p_\eta, s_\eta) \in \Pi$ with $0 < s_1 < \ldots < s_\eta \leq t$ such that all the edges of Z_t are segments on the lines $g(\alpha_1, p_1), \ldots, g(\alpha_\eta, p_\eta)$. Denote $A(Z_t) = s_\eta$, i.e. the last of the times of births of the edges of Z_t. Now we can introduce a sequence $(\sigma_k)_{k\in\mathbb{Z}}$ by

$$\begin{array}{rcl} \sigma_1 &=& A(Z_1), \\ \sigma_{k-1} &=& \sup\{A(Z_t) : t > 0, A(Z_t) < \sigma_k\}, \text{ for } k = 1, 0, -1, \ldots \\ \sigma_{k+1} &=& \inf\{A(Z_t) : t > 0, A(Z_t) > \sigma_k\}, \text{ for } k = 1, 2, 3, \ldots \end{array} \tag{5}$$

Thus the process $(Z_t)_{t>0}$ is piecewise constant. The time σ_k is the time when the interior $int\, Z_{\sigma_{k-1}}$ is the first time hit by a line:

$$\sigma_k = \inf\left\{s : \exists(\alpha,p,s) \in \Pi : g(\alpha,p) \cap \left(int\, Z_{\sigma_{k-1}}\right) \neq \emptyset\right\}.$$

Since for any $t > 0$ we have $A(Z_t) \leq t$ we obtain $\lim_{k\to-\infty} \sigma_k = 0$. On the other hand, $\lim_{k\to\infty} \sigma_k = \infty$ since for all $t > 0$

$$P\left(\exists(\alpha, p, s) \in \Pi : s \geq t,\, g(\alpha, p) \cap (int\, Z_t) \neq \emptyset\right) = 1.$$

It is crucial for the construction that the process of the o-cells fills the whole plane.

Lemma 1

$$\bigcup_{k\in\mathbb{Z}} Z_{\sigma_k} = \mathbb{R}^2 \quad a.s. \tag{6}$$

Proof: For $x \in \mathbb{R}^2$ and $t > 0$ denote by B_x the circle of radius 1 and center x and by $(\alpha_1, p_1, s_1), ..., (\alpha_\xi, p_\xi, s_\xi) \in \Pi$ the finite set with $0 < s_1 < ... < s_\xi \leq t$ such that $\{g(\alpha_1, p_1), ..., g(\alpha_\xi, p_\xi)\}$ is the set of all lines in Γ_t that intersect the set $conv(B_x \cup \{o\})$, i.e. the convex hull of B_x and the origin. Then $x \in int\, Z_{\sigma_k}$ for all $\sigma_k < s_1$. Thus

$$P\left(\exists k \in \mathbb{Z} : B_x \subset Z_{\sigma_k}\right) = 1.$$

With (4) and a coverage of $\mathbb{R}^2$ by circles B_x, x from a countable set, the proof can be completed. □

On the other hand one can show that $\bigcap_{k\in\mathbb{Z}} Z_{\sigma_k} = \{o\}$.

2.3 A preliminary tessellation of $\mathbb{R}^2$

For $u > 0$ we define a tessellation Ψ_u that is derived from the sequence $(Z_{\sigma_k})_{k\in\mathbb{Z}}$ using only those Z_{σ_k} with $\sigma_k \leq u$. The cells of Ψ_u are

$$Z_u \quad \text{and} \quad \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}},\ \sigma_k < u, \tag{7}$$

where $\overline{B}$ denotes the topological closure of a set $B \subset \mathbb{R}^2$. All these cells are compact, convex and have a pairwise disjoint interior. Since, according to (5), $Z_{\sigma_k} = Z_{\sigma_{k-1}} \cap g(\alpha, p)^{+1}$ with $(\alpha, p, \sigma_k) \in \Pi$ and $g(\alpha, p)^{+1}$ the closed half-plane that contains the origin and is generated by the line $g(\alpha, p)$. Due to (6) the cells fill the plane, i.e.

$$Z_u \cup \bigcup_{\sigma_k < u} \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}} = \mathbb{R}^2.$$

The random tessellation Ψ_u is non-homogeneous (spatially non-stationary). Intuitively, the older cells of Ψ_u with $\sigma_{k-1},\ \sigma_k$ close to the time 0 (the moment of the 'Big Bang') are very far from the origin $o \in \mathbb{R}^2$ and they are stochastically larger than the younger ones. Also $(\Psi_u)_{u>0}$ is a process of tessellations.

2.4 The final steps of the construction

For $u > 0$ the random tessellation Φ_u is now constructed on the basis of Ψ_u. Sequences of additional chords are nested into the cells of Ψ_u. In each of the cells the nested tessellation is equivalent to that one which is given in [7] for bounded windows. We recall this here briefly and adapt it to the present situation.
Assign to each cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$, $\sigma_k < u$ of Ψ_u, i.e. with the exception of Z_u, a random tessellation $Y(u-\sigma_k, \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}})$ as it is defined in [7] and also described below.

2.4.1 Construction of $Y(t,W)$ in a fixed window W

Let W be a compact convex polygon, $t > 0$ and the distribution ϑ on $[0,\pi)$ as it was introduced in 2.1. Denote by $[W]$ the set of all lines g with $g \cap W \neq \emptyset$ and

$$\Lambda([W]) = \tilde{\Lambda}\left(\{(\alpha,p,s) \in [0,\pi) \times \mathbb{R} \times (0,\infty) : g(\alpha,p) \in [W], 0 < s \leq 1\}\right).$$

Let $\{(\tau_j,\gamma_j)\},\ j = 1,2,...$ be an i.i.d. sequence of pairs of independent random variables. The τ_j are exponentially distributed with parameter $\Lambda([W])$ and the γ_j are random lines with the distribution $\Lambda([W])^{-1}\Lambda(\cdot \cap [W])$, i.e. they have the directional distribution ϑ. The idea is the following. After the lifetime τ_1 the polygon W is divided by the random chord $W \cap \gamma_1$ into the two closed polygons $W \cap \gamma^{+1}$ and $W \cap \gamma^{-1}$ where γ^{+1} and γ^{-1} denote the two closed half-planes generated by γ_1. Then these two polygons are treated separately, their lifetimes, that begin at time τ_1, are τ_2 and τ_3 respectively. At the end of their lifetimes appear $(W \cap \gamma_1^{-1}) \cap \gamma_2^{+1}$, $(W \cap \gamma_1^{-1}) \cap \gamma_2^{-1}$ and $(W \cap \gamma_1^{+1}) \cap \gamma_3^{+1}$, $(W \cap \gamma_1^{+1}) \cap \gamma_3^{-1}$. Some of these intersections can also be empty. This division procedure is repeatedly applied to all polygons that emerge. The state at time t is the random tessellation $Y(t,W)$, inside of W. The construction can be illustrated by a tree, see Figure 3. A corresponding realization of $Y(t,W)$ where ϑ is concentrated on the horizontal and the vertical directions is shown in Figure 4. For a more detailed description and explanation of the construction see [7].

2.4.2 Properties of the tessellation $Y(t,W)$ for a fixed window W

For a fixed window W a process $(Y(t,W))_{t>0}$ of tessellations is defined by the construction. Let W_1, W_2 be compact convex polygons in $\mathbb{R}^2$. Then for all $t > 0$

$$W_1 \subset W_2 \implies Y(t,W_2) \cap W_1 \stackrel{D}{=} Y(t,W_1) \tag{8}$$

where $\stackrel{D}{=}$ means the identity of the distributions; cf. the proof of Theorem 1 in [7]. The following formulas relate the time for the construction with the intensity of the resulting tessellation; cf. Lemma 5 in [7] and (8).

$$\begin{aligned} Y(t,W) &\stackrel{D}{=} W \cap \tfrac{1}{t} Y(1,W) && \text{for } 0 < t < 1, \\ W \cap t\,Y(t,W) &\stackrel{D}{=} Y(1,W) && \text{for } t > 1. \end{aligned} \tag{9}$$

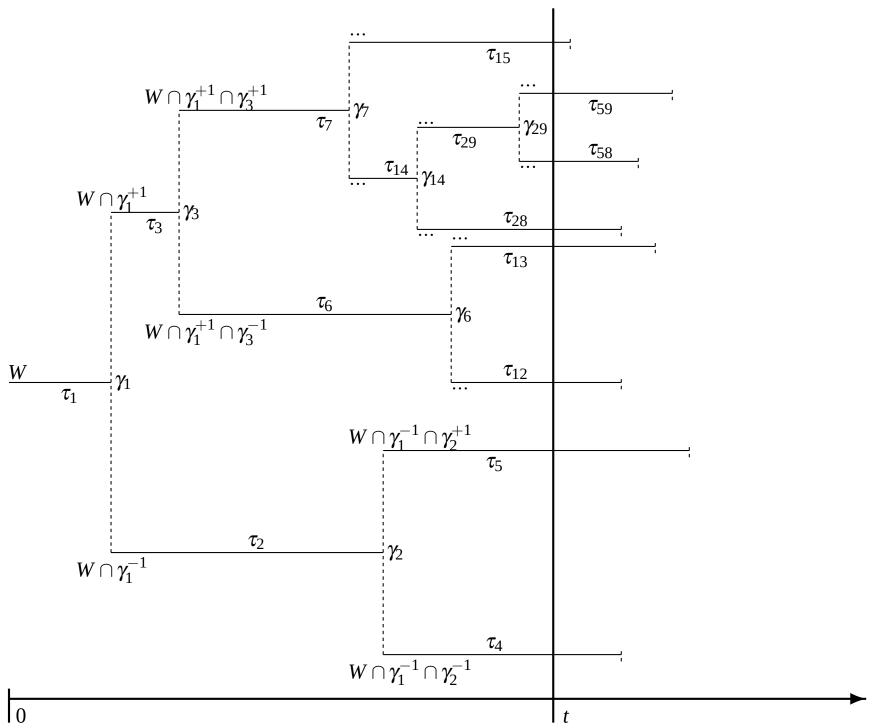

Figure 3: *Scheme of time intervals until time t. The lengths of the horizontal segments are proportional to the τ-values.*

2.4.3 Nesting of tessellations into Ψ_u

Let $W = [0,1]^2$ be fixed and $\left\{ (Y_k(t,W))_{t>0} \right\}_{k\in\mathbb{Z}}$ be a sequence of i.i.d. processes on the same probability space as Π and with the same distribution as $Y(t,W)$ that is defined in 2.4.1. Now, for a cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ of Ψ_u the window W is rescaled and shifted such that

$$\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}} \subset aW + b, \quad \text{with appropriate} \quad a \in (0,\infty),\ b \in \mathbb{R}^2.$$

Finally, for such a and b the tessellation $Y_k(a(u-\sigma_k),W)$ is rescaled and shifted and then restricted to $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$, i.e.

$$M_k = \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}} \ \cap\ (aY_k(a(u-\sigma_k),W)+b)$$

yields a partition of the cells of Ψ_u by the edges from Y_k. Due to the consistency properties that where recalled in 2.4.2 we have $aY_k(a(u-\sigma_k),W) \stackrel{D}{=} Y_k(u-\sigma_k, aW)$ and the result does not depend on the particular choice of $a,\ b$.
For $u > 0$, the random index $\kappa(u) = \max\{k : \sigma_k < u\}$ and Ψ_u as in 2.3 we define the tessellation

$$\Phi_u = \bigcup_{k=-\infty}^{\kappa(u)} \partial \left(\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}} \right) \cup M_k \tag{10}$$

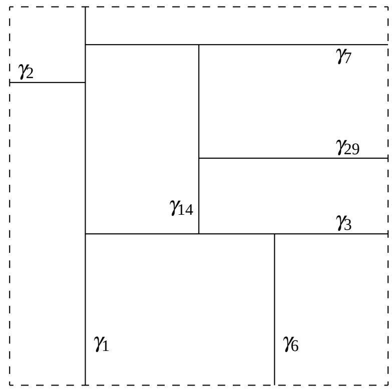

Figure 4: *A realization of $Y(t,W)$ which corresponds to the scheme that is shown in Figure 3. Notice that not necessarily all lines γ_i yield an edge: If, e.g., γ_{14} would fall to the left of γ_1 then it had to be ignored. Consequently, in this case γ_{29} would reach from γ_1 to the right hand boundary of the window W (dashed line).*

where ∂ denotes the boundary of a set. This tessellation can also be considered as the result of an iteration of Ψ_u with a sequence of independent but not identically distributed tessellations. The length intensity of the nested tessellations depends on the time of birth of the cells of the 'frame' tessellation Ψ_u. The o-cell of Φ_u is Z_u, i.e. it coincides with that one of Γ_u as well as of Ψ_u.

3 The distribution of the tessellation Φ_u

We consider the constructed tessellation Φ_u, $u > 0$, as a RACS – the set of all boundaries of cells – in $\mathbb{R}^2$. Thus its distribution is uniquely determined by the capacity functional (see [3] or [9]) and the calculation of this functional will yield the proof that the constructed tessellation has the same distribution as the homogeneous (spatially stationary) STIT tessellations of [7].

Theorem 1 *For $u > 0$ the tessellation Φ_u as described in Section 2 is a random homogeneous STIT tessellation with length intensity $L_A = u$ and directional distribution ϑ.*

Proof: Due to the results in [8] it suffices to calculate the capacity functional of the RACS Φ_u, i.e.

$$T_{\Phi_u}(C) = P(\Phi_u \cap C \neq \emptyset) \quad \text{for all} \quad C \in \left\{ \bigcup_{i=1}^{n} C_i : C_i \text{ compact convex }, n \in \mathbb{N} \right\}.$$

Thus induction over the number n of connected components is possible. Denote by $[A|B]$ the set of all lines that separate two sets A and B. The convex hull of A is

written as $conv A$. Recall that Z_u denotes the o-cell of the tessellation Ψ_u and thus, by definition, also the o-cell of Φ_u.
(i) If C is compact and convex then $\Phi_u \cap C = \emptyset$ if either $C \subset Z_u$ or if there is a time $t \in (0,u)$ when C is separated from o by an edge of Ψ_u and then in the remaining time $u-t$ it is not hit by an edge of the tessellation that is nested into that cell of Ψ_u where C is located, shortly written as $Y(u-t) := aY_k(a(u-t),W)+b$, cf. 2.4.3. Formally, denote by P_Π the distribution of the Poisson process Π and by $P^{(\alpha,p,t)}$ its Palm distribution w.r.t. the point (α,p,t). The indicator function $\mathbf{1}(\cdot)$ is defined to be 1 if the condition in brackets is fulfilled and 0 otherwise. In the following we apply standard formulas for Poisson (line) processes, the refined Campbell formula and Slivnyak's theorem, see [4] or Proposition 12.1.VI in [1]. Furthermore, we make use of the independence and of the results in [7] for the $Y(u-t)$, in particular that its typical cell has the same distribution as that one of a Poisson line tessellation. It is straightforward that $\Lambda([conv(C \cup \{o\})]) = \Lambda([C]) + \Lambda([C|\{o\}])$. This yields

$$\begin{aligned}
& P(\Phi_u \cap C = \emptyset) \\
= \; & P(C \subset Z_u) \\
& + \int P_\Pi(\mathrm{d}\varphi) \int \varphi(\mathrm{d}(\alpha,p,t))\, \mathbf{1}\left(\{g(\alpha',p') : (\alpha',p',t') \in \varphi, t' < t\} \cap [C|\{o\}] = \emptyset\right) \times \\
& \quad \times \mathbf{1}(0 < t < u) \cdot \mathbf{1}(g(\alpha,p) \in [C|\{o\}]) \cdot P(C \cap Y(u-t) = \emptyset) \\
= \; & P(C \subset Z_u) \\
& + \int \tilde{\Lambda}(\mathrm{d}(\alpha,p,t)) \int P^{(\alpha,p,t)}(\mathrm{d}\varphi)\, \mathbf{1}\left(\{g(\alpha',p') : (\alpha',p',t') \in \varphi, t' < t\} \cap [C|\{o\}] = \emptyset\right) \times \\
& \quad \times \mathbf{1}(0 < t < u) \cdot \mathbf{1}(g(\alpha,p) \in [C|\{o\}]) \cdot P(C \cap Y(u-t) = \emptyset) \\
= \; & \exp\{-u\Lambda([conv(C \cup \{o\})])\} \\
& + \int_0^u \mathrm{d}t\, \exp\{-t\Lambda([conv(C \cup \{o\})])\} \cdot \Lambda([C|\{o\}]) \cdot \exp\{-(u-t)\Lambda([C])\} \\
= \; & \exp\{-u\Lambda([conv(C \cup \{o\})])\} \\
& + \exp\{-u\Lambda([C])\} \cdot \Lambda([C|\{o\}]) \int_0^u \mathrm{d}t\, \exp\{-t\Lambda([C|\{o\}])\} \\
= \; & \exp\{-u\Lambda([conv(C \cup \{o\})])\} - \exp\{-u(\Lambda([C]) + \Lambda([C|\{o\}]))\} + \exp\{-u\Lambda([C])\} \\
= \; & \exp\{-u\Lambda([C])\}.
\end{aligned}$$

(ii) Now consider $C = \bigcup_{i=1}^n C_i$, C_i compact convex and $n > 1$ and assume that for all sets D with less than n compact connected components holds $P(\Phi_u \cap D = \emptyset) = P(Y(u) \cap D = \emptyset)$ where $Y(u)$ is a random homogeneous STIT tessellation with length intensity $L_A = u$ and directional distribution ϑ as given in [7].
The event $\Phi_u \cap C = \emptyset$ can be partitioned into the three disjoint events that either
(a) $conv C \cap \Phi_u = \emptyset$ or
(b) there is a time $t \in (0,u)$ when C is separated from o by an edge of Ψ_u and then in the remaining time $u-t$ their components are separated but not hit by an edge of the tessellation $aY_k(u-t,W)+b$ that is nested into that cell of Ψ_u where C is located or
(c) there is a time $t \in (0,u)$ when the components C_i, $i = 1,2,\dots,n$ are separated by a line into two parts, $D_1,\ D_2$ say, which thus belong to two different cells of Ψ_u and then are not hit in the further construction. Since both D_1 and D_2 consist of less than n connected components induction becomes possible.
The case (a) corresponds to (i) above for $conv C$ instead of C.
Case (b) is treated using formula Lemma 4 in [7]. The symbol $\sum_{D_1,D_2}$ denotes the sum over all partitions of C into two parts, say $D_1 = \bigcup_{i\in J} C_i$ and $D_2 = \bigcup_{i\in\{1,\dots n\}\setminus J} C_i$, with nonempty $J \neq \{1,\dots,n\}$. Analogously to (i), the refined Campbell theorem yields

$$
\begin{aligned}
&\int_0^u \mathrm{d}t\ P(C \subset Z_t)\cdot\Lambda([C|\{o\}]) \sum_{D_1,D_2} \Lambda([D_1|D_2])\times \\
&\times \int_0^{u-t} \mathrm{d}t_1\ \exp\{-t_1\Lambda([convC])\}\cdot P(Y(u-t-t_1)\cap D_1=\emptyset)\cdot P(Y(u-t-t_1)\cap D_2=\emptyset) \\
= \quad & \sum_{D_1,D_2} \Lambda([D_1|D_2])\cdot\Lambda([C|\{o\}]) \int_0^u \mathrm{d}t_1\ P(Y(u-t_1)\cap D_1=\emptyset)\cdot P(Y(u-t_1)\cap D_2=\emptyset)\times \\
&\times \int_0^{t_1} \mathrm{d}t\ \exp\{-t\Lambda([conv(C\cup\{o\})])\}\cdot\exp\{-(t_1-t)\Lambda([convC])\} \\
= \quad & \sum_{D_1,D_2} \Lambda([D_1|D_2]) \int_0^u \mathrm{d}t_1\ \left(\exp\{-t_1\Lambda([convC])\} - \exp\{-t_1\Lambda([conv(C\cup\{o\})])\}\right)\times \\
&\times P(Y(u-t_1)\cap D_1=\emptyset)\cdot P(Y(u-t_1)\cap D_2=\emptyset) \\
= \quad & \sum_{D_1,D_2} \Lambda([D_1|D_2]) \int_0^u \mathrm{d}t\ \exp\{-t\Lambda([convC])\}\cdot P(Y(u-t)\cap D_1=\emptyset)\cdot P(Y(u-t)\cap D_2=\emptyset) \\
&-\sum_{D_1,D_2} \Lambda([D_1|D_2]) \int_0^u \mathrm{d}t\ \exp\{-t\Lambda([conv(C\cup\{o\})])\}\times \\
&\qquad\quad \times P(Y(u-t)\cap D_1=\emptyset)\cdot P(Y(u-t)\cap D_2=\emptyset).
\end{aligned}
$$

In case (c), if there is a time $t \in (0,u)$ when the components of C are separated for the first time by a line into two parts, D_1, D_2, then there is a $k \in \mathbb{Z}$ such that $t = \sigma_k$ and thus, without loss of generality, $D_1 \subset \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and $D_2 \subset Z_{\sigma_k}$. According with the construction, $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ is a cell of Ψ_u that is further divided by a tessellation $aY_k(a(u-\sigma_k),W)+b$. On the other hand, for $0<t<u$ the conditional probability

$$P(\Phi_u \cap D_2 = \emptyset | D_2 \subset Z_t) = P(\Phi_{u-t} \cap D_2 = \emptyset)$$

due to the properties of the Poisson point process Π. Furthermore, since the number of connected components of D_1 is less then n, the induction assumption implies $P(\Phi_{u-t} \cap D_1 = \emptyset) = P(Y(u-t) \cap D_1 = \emptyset)$.
The summary of the calculations for (a), (b) and (c) yields

$$\begin{aligned}
& P(\Phi_u \cap C = \emptyset) \\
= \ & \exp\{-u\Lambda([conv C])\} \\
& + \sum_{D_1,D_2} \Lambda([D_1|D_2]) \int_0^u \mathrm{d}t \ \exp\{-t\Lambda([conv C])\} \times \\
& \qquad \times P(Y(u-t) \cap D_1 = \emptyset) \cdot P(Y(u-t) \cap D_2 = \emptyset) \\
& - \sum_{D_1,D_2} \Lambda([D_1|D_2]) \int_0^u \mathrm{d}t \ \exp\{-t\Lambda([conv(C \cup \{o\})])\} \times \\
& \qquad \times P(Y(u-t) \cap D_1 = \emptyset) \cdot P(Y(u-t) \cap D_2 = \emptyset) \\
& + \int_0^u \mathrm{d}t \, P(C \subset Z_t) \sum_{D_1,D_2} \Lambda([D_1|D_2]) \cdot P(Y(u-t) \cap D_1 = \emptyset) \cdot P(\Phi_{u-t} \cap D_2 = \emptyset) \\
= \ & \exp\{-u\Lambda([conv C])\} \\
& + \sum_{D_1,D_2} \Lambda([D_1|D_2]) \int_0^u \mathrm{d}t \exp\{-t\Lambda([conv C])\} \cdot P(\Phi_{u-t} \cap D_1 = \emptyset) \cdot P(\Phi_{u-t} \cap D_2 = \emptyset).
\end{aligned}$$

The coincidence of the results in (i) and (ii) with the formulas in the Lemmas 3 and 4 in [7] and their combination with Theorems 1 and 2 in [7] yield that Φ_u is indeed a homogeneous STIT tessellation with $L_A = u$ and ϑ. □

4 Outlook on a process of cell division that is homogeneous in the plane

Our construction of Φ_u suggests the following:
To every cell of Φ_u there can be assigned a time of birth at which it appears as one of the two parts in which a former cell is subdivided by a straight line generated by Π or by the algorithms of nesting. All cells which have a time of birth smaller than s, $s < u$, form a tessellation $\widetilde{\Phi}_s$. In this way, we have constructed a stochastic process $(\widetilde{\Phi}_s)_{0<s<u}$ on the time intervall $(0,u)$, the states of which are tessellations of the plane.
It can be shown that $\widetilde{\Phi}_s$ is distributed as Φ_s. This allows us to replace the notation $\widetilde{\Phi}_s$ by Φ_s. In particular, the random tessellations Φ_s are STIT tessellations. Furthermore, they are homogeneous in the plane. But much more stronger, we guess that the process as a whole is spatially homogeneous. This can be expressed in an equivalent way:

Conjecture 1 *For any $0 < s_1 < ... < s_n < u$ the distribution of the n-tuple $[\Phi_{s_1}, .., \Phi_{s_n}]$ is invariant under all shifts of the plane; $n = 1, 2, 3, ...$*

Our process has the Markov property. It should be constructed on the whole positive time axis.
Summarizing the above considerations, we formulate:

Conjecture 2 *Given a non-degenerate directional distribution ϑ, there exists a stochastic process $(\Phi_t)_{t>0}$, the states of wich are tessellations of the Euclidean plane. It has the following properties.*
1. The process is homogeneous in the plane in the sense of Conjecture 1.
2. For every $t > 0$ the random state Φ_t is a homogeneous STIT tessellation with length intensity t and directional distribution ϑ.
3. For $0 < s < t < \infty$ the net of edges of Φ_s is a subset of the net of edges of Φ_t.
4. The process has the Markov property.
5. Given $0 < s < t < \infty$, the transition from Φ_s to Φ_t can be regarded as an iteration procedure, where Φ_s is the frame, and the cells of Φ_s are filled according to the distribution of Φ_{t-s}.

From property 2 we conclude that $t\,\Phi_t$ is distributed as Φ_1, i.e. up to a scaling factor all 1-dimensional distributions are the same.
The process may be interpreted as a process of cell divisions. After a certain lifetime every cell is divided into two parts by a straight cut. The set of points in time in which a cell division happens anywhere in the plane is dense in $(0,\infty)$ but countable.
Some parts of Conjecture 2 are already proved implicitly in the preceding paragraphs.

5 Appendix: Iteration of tessellations

For tessellations, the operation of iteration (also referred to as nesting) is defined as follows. Let $Y^1, Y^2, \ldots$ be a sequence of i.i.d. homogeneous tessellations in $\mathbb{R}^2$ and denote $\mathscr{Y} = \{Y^1, Y^2, \ldots\}$. Further assume that Y^0 is a homogeneous tessellation which is independent of $\mathscr{Y}$. The set of all cells of a tessellation Y is written as $C(Y)$. Assume that the cells of Y^0 are numbered and denoted as $p_1, p_2, \ldots$. The iteration of the tessellation Y^0 and the sequence $\mathscr{Y}$ is defined as the tessellation

$$\mathscr{I}(Y^0, \mathscr{Y}) = Y^0 \cup \bigcup_{p_i \in C(Y^0)} (Y^i \cap p_i). \tag{11}$$

This formula describes the operation in terms of the boundaries of the cells. For the cells themselves it means that the cells p_i of the so called 'frame' tessellation Y^0 are independently subdivided by the cells p_{ik}, $k = 1, 2, \ldots$ (or their edges respectively) of the tessellations Y^i which intersect the interior of p_i, i.e. the new cells are of the type $p_i \cap p_{ik}$.

For a real number $r > 0$ the homothetic tessellation rY is generated by transforming all points $x \in Y$ into rx. Accordingly, $r\mathscr{Y}$ means that this transformation is applied to all tessellations of the sequence $\mathscr{Y}$. Let L_A be the mean total length per unit area, the length intensity, of any of the tessellations $Y^0, Y^1, \ldots$. Then $\mathscr{I}(Y^0, \mathscr{Y})$ has the length intensity $2L_A$, and $\mathscr{I}(2Y^0, 2\mathscr{Y})$ has the length intensity L_A, respectively.

Let Y^0 be a homogeneous tessellation and $\mathscr{Y}^1, \mathscr{Y}^2, \ldots$ a sequence of sequences of tessellations such that all the occurring tessellations (including Y^0) are i.i.d. Then the sequence $\mathscr{I}_2(Y^0), \mathscr{I}_3(Y^0), \ldots$ of rescaled iterations is defined by (see [6])

$$\begin{aligned}
\mathscr{I}_2(Y^0) &= \mathscr{I}(2Y^0, 2\mathscr{Y}^1)\,, \\
\mathscr{I}_m(Y^0) &= \mathscr{I}(mY^0, m\mathscr{Y}^1, \ldots, m\mathscr{Y}^{m-1}) \\
&= \mathscr{I}(\mathscr{I}(mY^0, m\mathscr{Y}^1, \ldots, m\mathscr{Y}^{m-2}), m\mathscr{Y}^{m-1})\,, \qquad m = 3, 4, \ldots
\end{aligned}$$

Here m is the rescaling factor which is chosen to keep the parameter L_A of the tessellation $\mathscr{I}_m(Y^0)$ constant for all m. We use the abbreviation $\mathscr{I}_m(Y^0)$ since it is assumed that all the other tessellations in the sequences $\mathscr{Y}^1, \mathscr{Y}^2, \ldots$ are independent and have the same distribution as Y^0.

Definition 1 *A homogeneous tessellation Y is said to be stable with respect to iteration (STIT) if*

$$Y \stackrel{D}{=} \mathscr{I}_m(Y) \quad \text{for all } m = 2, 3, \ldots\,,$$

i.e. if its distribution is not changed by repeated rescaled iteration with sequences of tessellations with the same distribution.

The algorithm that is presented in subsection 2.4.1 yields a STIT tessellation in a bounded window W.

A list of references concerning iteration was given in [6]. An approach to iteration which is more general than described above was developed in [2]. In the present paper the step of the construction in paragraph 2.4.3 uses the idea of nesting of independent but not identically distributed tessellations into a non-homogeneous 'frame' tessellation.

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