Two-Step Shortlisting by Imperfect Experts

Stephan Kolassa Stefan Schwarz
{kolassa,delgado}@minet.uni-jena.de
Institut für Angewandte Mathematik
Friedrich-Schiller-Universität Jena
Germany

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Abstract

We explore various models of decision making using an intermediate shortlisting step. The shortlisting and the final decision are done by two imprecise experts who may or may not be identical and/or independent.

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Introduction

Decision-making processes often consist of more than one step:

- Junior management or consultants analyze a large number of possible approaches to a problem and present a short list of alternatives. Senior management then reviews this list and makes a final choice or decision.
- Computerized Decision Support Systems such as *Recommender Systems* (Resnick & Varian, 1997) make a small set of recommendations from which the end user selects one course of action.
- Chess professionals use the k-best mode of computer chess programs and make the final choice themselves when analyzing games or researching new variants (see Bareev, 2001).

These examples are instances of "division of labor"-approaches for making decisions based on two-step algorithms. One implicit assumption of such a procedure is that the sequential application of more than one expert improves the final decision. This assumption is rarely explicitly acknowledged.

In fact, real world experts are not perfect. They cannot measure hidden costs and benefits of a course of action and need to reduce complexity to be able to handle reality.

We analyze some two-step shortlisting models with imperfect experts using Monte Carlo simulations and theoretical approaches.

1 Monte Carlo Simulations for a Continuous Model

1.1 The Basic Model

Model 1.1 (Model Cont). Consider $3n$ independently distributed random variables

$$x_1, \dots, x_n \text{ uniformly distributed in } [0,1],$$
$$\gamma_1, \dots, \gamma_n \text{ uniformly distributed in } [0,\Gamma],$$
$$\text{and } \delta_1, \dots, \delta_n \text{ uniformly distributed in } [0,\Delta].$$

We consider two experts, Alice and Bob. Alice observes $x_i + \gamma_i$ for $1 \leqslant i \leqslant n$ and constructs a shortlist $\mathcal{S} \subseteq \{1, \dots n\}$ consisting of the indices of her k largest observations $x_i + \gamma_i$.

Bob now observes $x_i + \delta_i$ for $i \in \mathcal{S}$ and selects the index $\widetilde{\imath}$ of his largest observation.

The correct index i^* Alice and Bob are looking for is the index of the maximal x_i. $\Diamond$

We note that i^* is almost surely well-defined and that the zero set on which it is not will not influence the result.

The rationale of this model is that the x_i quantify some "real" value of the ith alternative and that the γ_i and δ_i are imprecisions in Alice and Bob's observation of the data. The larger Γ or Δ are, the less precise are the associated experts.

We are interested in the probability that this process returns the index i^* of the "real" maximum $\max_{1 \leqslant i \leqslant n} x_i$: the hitting ratio

$$h_{\mathrm{Cont}}(n \xrightarrow{\Gamma} k \xrightarrow{\Delta} 1).$$

Having introduced Model 1.1 as one of the concrete models we are interested in, we now give an abstract framework for all models we will consider:

Definition. Consider some shortlisting model Mod with r experts and a parameter space Π. The j^{th} expert is described by some parameter $\pi_j \in \Pi$. We are also given $n \in \mathbb{N}$ as well as $k_1, \cdots, k_r \in \mathbb{N}$ such that $n \geqslant k_1 \geqslant k_2 \geqslant \cdots \geqslant k_r$. Finally, we are given some distinguished *correct index* i^* which may be fixed or probabilistic.

The objective of the r experts is to identify i^* among the indices $1, \dots, n$.

The first expert is given a set of alternatives with indices $1, \dots, n$. He compiles a shortlist $\mathcal{S}_1 \subseteq \{1, \dots, n\}$ such that $\#\mathcal{S}_1 = k_1$. He passes $\mathcal{S}_1$ to the second expert, who compiles $\mathcal{S}_2 \subseteq \mathcal{S}_1$ such that $\#\mathcal{S}_2 = k_2$ and so on.

The *hitting ratio* under parameters $\pi_1, \dots, \pi_r$ is the probability that in model Mod with parameters $\pi_1, \dots, \pi_r$, the sequential shortlisting yields the correct index i^* in the final list $\mathcal{S}_r$. We denote the hitting ratio by

$$h_{\mathrm{Mod}}(n \xrightarrow{\pi_1} k_1 \xrightarrow{\pi_2} k_2 \xrightarrow{\pi_3} \cdots \xrightarrow{\pi_r} k_r).$$

We will conduct Monte Carlo simulations for some models and use the *empirical hitting ratio* as an estimator for the hitting ratio:

$$h^{\mathrm{emp}}_{\mathrm{Mod}}(n \xrightarrow{\pi_1} k_1 \xrightarrow{\pi_2} k_2 \xrightarrow{\pi_3} \cdots \xrightarrow{\pi_r} k_r) := \frac{\#\{\text{Monte Carlo runs in which } i^* \text{ is returned}\}}{\#\{\text{all Monte Carlo runs}\}}.$$

$\Diamond$

We emphasize that no other information is passed along than unordered sets of indices!

The special case of the above in which we will be most interested is that of $r = 2$ and $k_2 = 1$: one expert first shortlists k indices from $\{1, \ldots, n\}$, and a second one then chooses exactly one index $\tilde{\imath}$. The hitting ratio then counts how often we have $\tilde{\imath} = i^*$.

In fact, the only place where the final list $\mathcal{S}_r$ has cardinality $\#\mathcal{S}_r > 1$ is Lemma 5.7.

For instance, the parameter space Π in Model 1.1 is given by the possible values of Γ and Δ, i. e. $\Pi = \mathbb{R}_{\geqslant 0}$.

Returning to Model 1.1, we will consider additionally the expected value of the true value returned $E(x_{\tilde{\imath}})$, the *expected result*.

If the experts differ significantly in precision, it is obvious that letting the more precise expert do all the work will yield the best performance:

Remark. 1. For all $\Gamma > 0$, there exists a $\Delta(\Gamma) > 0$ such that for all $\Delta < \Delta(\Gamma)$ and for all $k \neq n$, we have

$$h_{\text{Cont}}(n \xrightarrow{\Gamma} n \xrightarrow{\Delta} 1) > h_{\text{Cont}}(n \xrightarrow{\Gamma} k \xrightarrow{\Delta} 1).$$

2. For all $\Delta > 0$, there exists a $\Gamma(\Delta) > 0$ such that for all $\Gamma < \Gamma(\Delta)$ and for all $k \neq 1$, we have

$$h_{\text{Cont}}(n \xrightarrow{\Gamma} 1 \xrightarrow{\Delta} 1) > h_{\text{Cont}}(n \xrightarrow{\Gamma} k \xrightarrow{\Delta} 1).$$

The analogue holds for the expected result. □

We will mostly restrict our attention to the case $\Gamma = \Delta$, i. e. Alice is as "precise" as Bob.

We collected empirical data on Model 1.1 by Monte Carlo simulations. Because of the well-known regularity problems of many random number generators, we conducted all simulations twice, using two different well accepted random number generators: a generator based on a combination of four linear congruence generators as proposed by L'Ecuyer and Andres (1997) and a generator as described by Press, Teukolsky, Vetterling and Flannery (1992, p. 282). The results were quantitatively the same, and we will only give the results of the simulations that used L'Ecuyer and Andres' generator.

All computations were performed on a Compaq AlphaServer DS20E with 2 Alpha EV6.7 (21264A) processors at 667 MHz and 4 GB RAM running Compaq Tru64 UNIX V5.1.

The results of simulations of Model 1.1 with $\Gamma = \Delta = 0.5$ are shown in Figures 1 and 2.

We observe:

1. For $n \leqslant 20$, the empirical hitting ratio is unimodal in k, i. e., it first increases and then decreases.

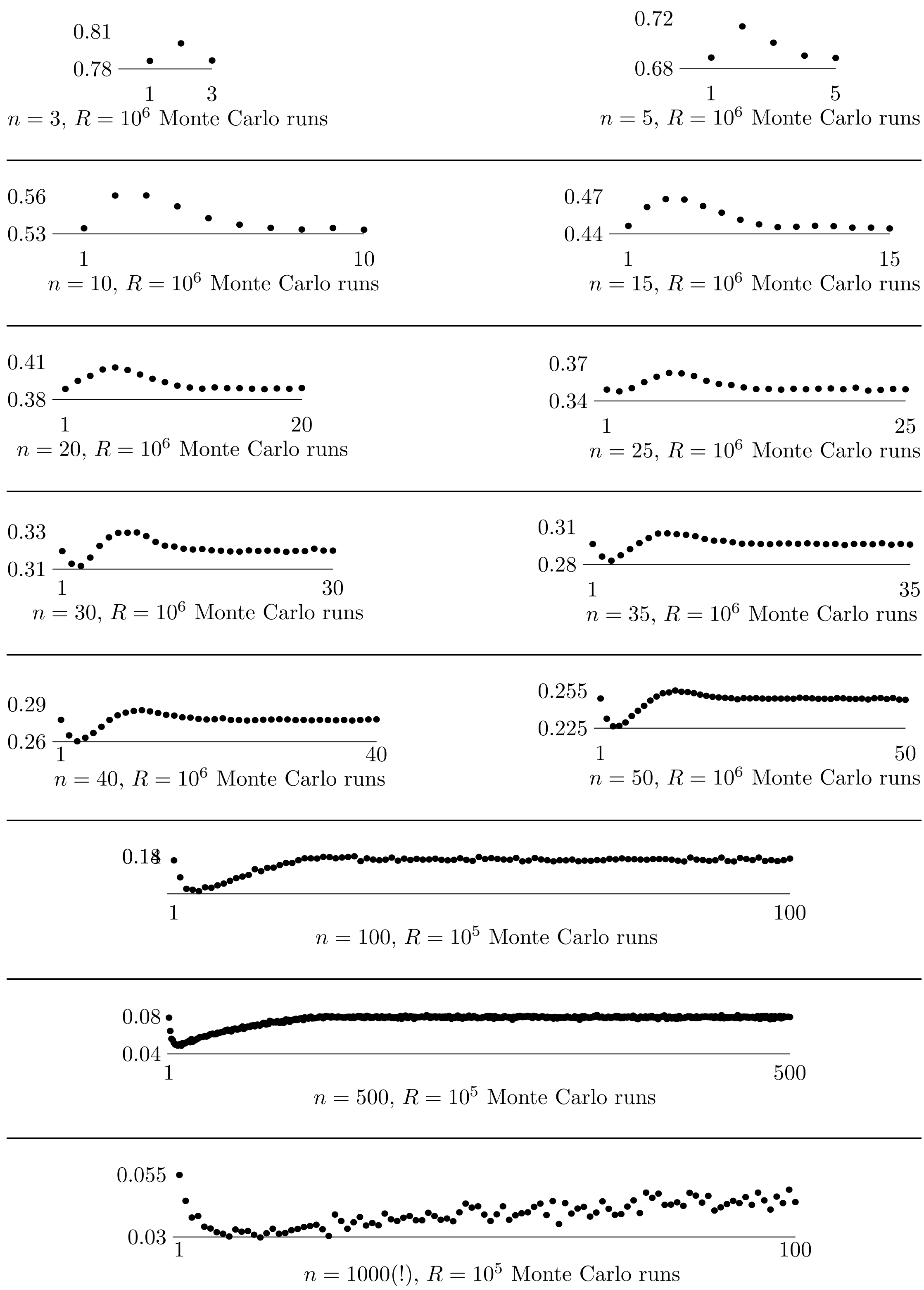

Figure 1: Hitting ratios for certain values of n as a function of k in Model 1.1. R denotes the number of Monte Carlo simulations. Expert precision: $\Gamma = \Delta = 0.5$.

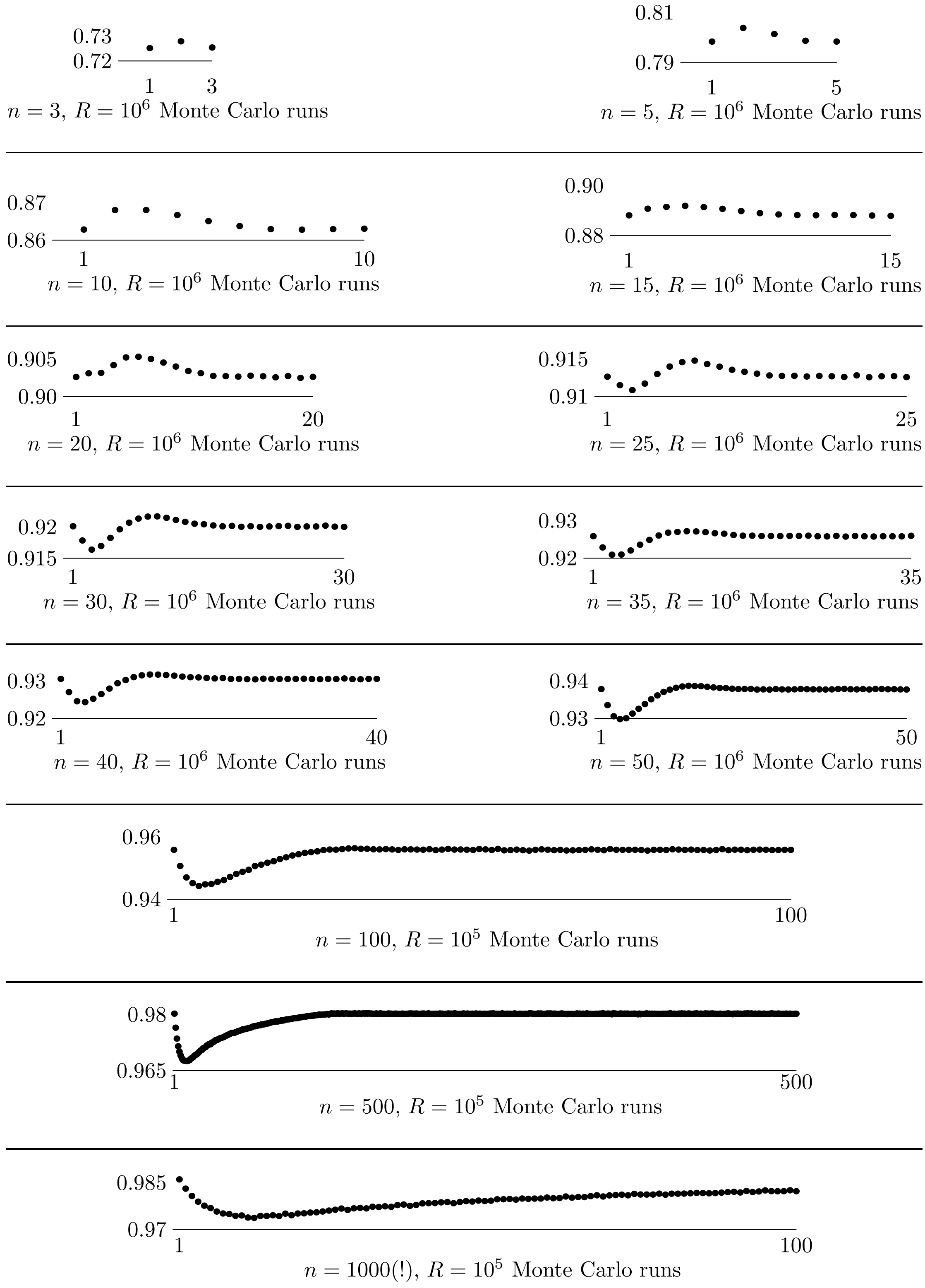

Figure 2: Empirical expected results for certain values of n as a function of k in Model 1.1. R denotes the number of Monte Carlo simulations. Expert precision: $\Gamma = \Delta = 0.5$.

n	$\Gamma=\Delta$	k	$h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{\Gamma} k \xrightarrow{\Delta} 1)$
4	18	1	0.272242
		2	0.272196
	20	1	0.269985
		2	0.269711
5	4	1	0.297805
		2	0.296171
	6	1	0.266224
		2	0.263944
	8	1	0.250017
		2	0.248300
	10	1	0.240564
		2	0.238069

n	$\Gamma=\Delta$	k	$h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{\Gamma} k \xrightarrow{\Delta} 1)$
6	4	1	0.265661
		2	0.258706
	6	1	0.234392
		2	0.227524
	8	1	0.217386
		2	0.212565
	10	1	0.207273
		2	0.202148

Table 1: Empirical hitting ratios in simulations with $R = 10^6$ Monte Carlo runs

2. For $n \geqslant 25$, the empirical hitting ratio first decreases with increasing k (e. g. $n = 30$, $k \leqslant 3$). We call this phenomenon a *Shortlisting Valley*:

Definition 1.2. Model Mod has a *shortlisting valley* for the parameter value $\pi \in \Pi$ if there is some $k \in \{2, \dots, n-1\}$ such that

$$h_{\text{Mod}}(n \xrightarrow{\pi} k \xrightarrow{\pi} 1) < h_{\text{Mod}}(n \xrightarrow{\pi} 1 \xrightarrow{\pi} 1) = h_{\text{Mod}}(n \xrightarrow{\pi} n \xrightarrow{\pi} 1).$$

Model Mod has an *elementary shortlisting valley* for the parameter value $\pi \in \Pi$ if

$$h_{\text{Mod}}(n \xrightarrow{\pi} 2 \xrightarrow{\pi} 1) < h_{\text{Mod}}(n \xrightarrow{\pi} 1 \xrightarrow{\pi} 1) = h_{\text{Mod}}(n \xrightarrow{\pi} n \xrightarrow{\pi} 1). \qquad \Diamond$$

We continue our comments on the empirical results:

3. The shortlisting valley appears for smaller n if Γ (and/or Δ) grows; see for instance Table 1.

4. Consider Table 2, which contains the empirical hitting ratio for $k = 1$ and $k = n$, as well as the value of k for which the empirical hitting ratio was minimal ($k_{\min}$) and maximal ($k_{\max}$).

   Why did we include $h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{0.5} 1 \xrightarrow{0.5} 1)$ and $h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{0.5} n \xrightarrow{0.5} 1)$? Since both experts are equally precise, we will theoretically have

$$h_{\text{Cont}}(n \xrightarrow{\Gamma} 1 \xrightarrow{\Gamma} 1) = h_{\text{Cont}}(n \xrightarrow{\Gamma} 1) = h_{\text{Cont}}(n \xrightarrow{\Gamma} n \xrightarrow{\Gamma} 1),$$

   since both hitting ratios correspond to only one expert making the decision without any input from the other. In our empirical data, we cannot expect equality, since we generate new full data sets for every value of k. However, comparing $h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{\Gamma} 1 \xrightarrow{\Gamma} 1)$ and $h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{\Gamma} n \xrightarrow{\Gamma} 1)$ allows us to gauge the number of significant digits. We make the following observations:

n	$h^{\text{emp}}(1)$	$h^{\text{emp}}(n)$	k_{min}	$h^{\text{emp}}(k_{\text{min}})$	k_{max}	$h^{\text{emp}}(k_{\text{max}})$
3	0.78644	0.786775	1	0.78644	2	0.800503
5	0.688571	0.688247	5	0.688247	2	0.713598
10	0.534477	0.533338	10	0.533338	3	0.560842
15	0.446454	0.444373	15	0.444373	3	0.467969
20	0.388478	0.389213	17	0.388258	5	0.40571
25	0.34913	0.349396	2	0.347683	6	0.362517
30	0.319602	0.319883	3	0.311679	9	0.329605
35	0.296374	0.295996	3	0.283003	9	0.30493
40	0.277593	0.277909	3	0.260313	11	0.285257
50	0.248823	0.247779	3	0.226486	13	0.255211
100	0.17577	0.17769	5	0.14275	30	0.18011
500	0.07878	0.07939	11	0.04874	365	0.08165
1000	0.0549	—	14	0.0298	—	—

Table 2: Empirical hitting ratios based on the data used in Figure 1. We abbreviate $h^{\text{emp}}(k) := h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{0.5} k \xrightarrow{0.5} 1)$. For $n = 1000$, we only simulated $k \leqslant 100$ to reduce computing time.

- Up to $n \approx 20$, the minimal empirical hitting ratio appears for $k = 1$ or $k = n$.
- Starting at $n \approx 25$, the minimal empirical hitting ratio appears at small intermediate values of k, and k_{min} grows slowly with n.
- The maximum empirical hitting ratio $h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{0.5} k_{\text{max}} \xrightarrow{0.5} 1)$ is not much higher than $h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{0.5} 1 \xrightarrow{0.5} 1)$ and $h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{0.5} n \xrightarrow{0.5} 1)$ for larger n. Actually, for $n = 500$, the peak could practically not be distinguished from the "white noise" of our random experiments, which accounts for $k_{\text{max}} = 365$ in this case.
- For $n = 1000$, the minimum empirical hitting ratio $h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{0.5} k_{\text{min}} \xrightarrow{0.5} 1)$ is nearly only half as high as $h^{\text{emp}}_{\text{Cont}}(n \xrightarrow{0.5} k_{\text{max}} \xrightarrow{0.5} 1)$.

1.2 Modifications of the Basic Model

1.2.1 Flexible Shortlist Length

We can improve the results by allowing Alice to reduce the number of indices she passes on: since the index i^* of the maximum x_i cannot be among those indices j for which

$$x_j + \gamma_j < \max_{1 \leqslant i \leqslant n} (x_i + \gamma_i) - \Gamma,$$

Alice only passes on the indices of those of her k largest observations which lie above the "bar" defined by $\max(x_i + \gamma_i) - \Gamma$.

Of course, for this strategy we have to assume that Alice knows her own precision Γ.

Figure 3 shows the results of this changed strategy. We note in particular that for small k the results of the strategy with constant and flexible shortlist size are practically the same. For larger k, the “flexible” strategy performs slightly better.

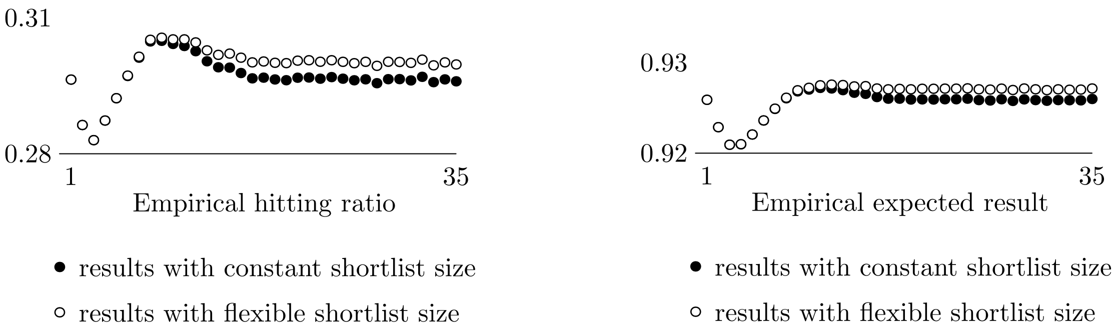

Figure 3: Results with constant vs. flexible shortlist size in the basic model. $n = 35$, $\Delta = \Gamma = 0.5$, $R = 10^6$ Monte Carlo runs

1.2.2 Normally Distributed Imprecisions

We conducted an experiment using normally distributed imprecisions for the two experts. Apart from the distributions of the imprecisions, we followed Model 1.1. After generating random numbers which were uniformly distributed in $[0, 1]$, we obtained normally distributed imprecisions $\gamma_i \sim \mathcal{N}(0, \Gamma)$ and $\delta_i \sim \mathcal{N}(0, \Gamma)$ by applying the ratio method as discovered by Kinderman and Monahan (1976, 1977) – see also Knuth (1998, section 3.4.1).

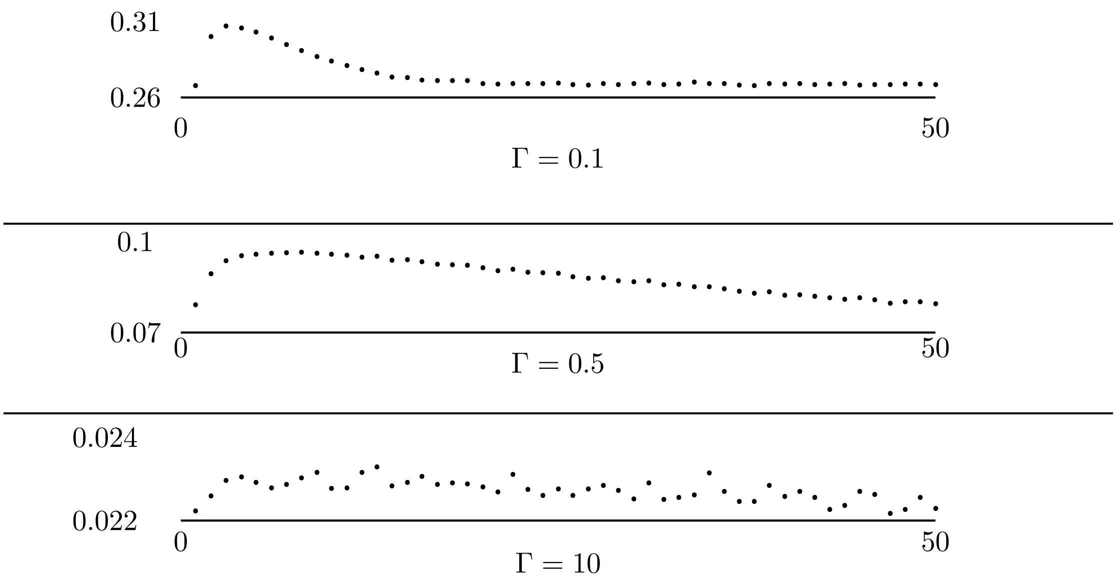

Figure 4: Empirical hitting ratios as a function of k for $n = 50$, $\gamma_i, \delta_i \sim \mathcal{N}(0, \Gamma)$, $\Gamma \in \{0.1, 0.5, 10\}$ and $R = 10^6$ Monte Carlo runs

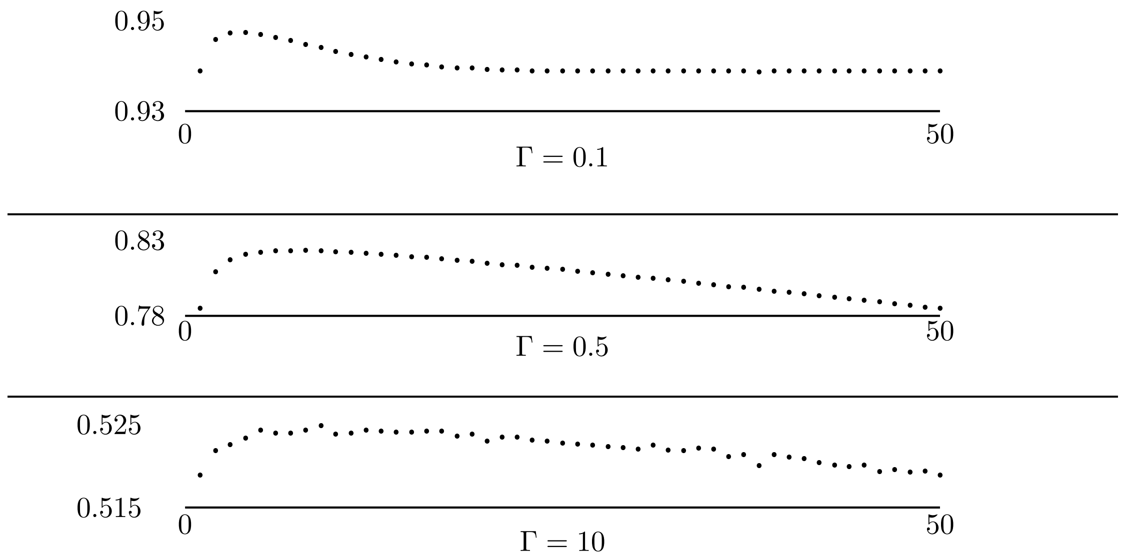

Figure 5: Empirical expected results as a function of k for $n = 50$, $\gamma_i, \delta_i \sim \mathcal{N}(0, \Gamma)$, $\Gamma \in \{0.1, 0.5, 10\}$ and $R = 10^6$ Monte Carlo runs

We simulated the setting with $n = 50$, $\Gamma \in \{0.1, 0.5, 10\}$ and $R = 10^6$ Monte Carlos runs. Results are shown in Figures 4 and 5. As far as we can see, no shortlisting valleys appear. We conjecture that valleys will appear for larger values of Γ. However, simulations become difficult to interpret when the graph of the hitting ratio becomes more ragged.

1.2.3 Maximizing a Convex Combination

In a different model, Alice and Bob try to maximize a convex combination with imprecise knowledge as to the weights of the components:

Model 1.3. Consider n pairs (x_i, y_i) of random variables, $x_i, y_i \sim U[0, 1]$, as well as a weight $\gamma \sim U[\frac{1}{2} - \Gamma, \frac{1}{2} + \Gamma]$ for Alice and a weight $\delta \sim U[\frac{1}{2} - \Delta, \frac{1}{2} + \Delta]$ for Bob, where $0 \leqslant \Gamma, \Delta \leqslant \frac{1}{2}$.

Alice constructs a shortlist $\mathcal{S}$ consisting of those k indices i for which $\gamma x_i + (1 - \gamma) y_i$ is maximal.

Bob chooses the index from $\mathcal{S}$ for which $\delta x_i + (1 - \delta) y_i$ is maximal.

The true objective is to maximize $\frac{1}{2} x_i + \frac{1}{2} y_i$. ◊

In this model, Alice and Bob are unbiased in their subjective weights γ and δ. Our Monte Carlo simulations for $\Gamma = \Delta = \frac{1}{2}$ and $n \in \{30, 50, 100\}$ with $R = 10^6$ show no shortlisting valley, cf. Figure 6.

However, we also investigated the following biased model:

Model 1.4 (Model BiasConv). We modify Model 1.3 by choosing $\gamma \sim U[0, \Gamma]$ and $\delta \sim U[0, \Delta]$, where $0 \leqslant \Gamma, \Delta \leqslant \frac{1}{2}$. ◊

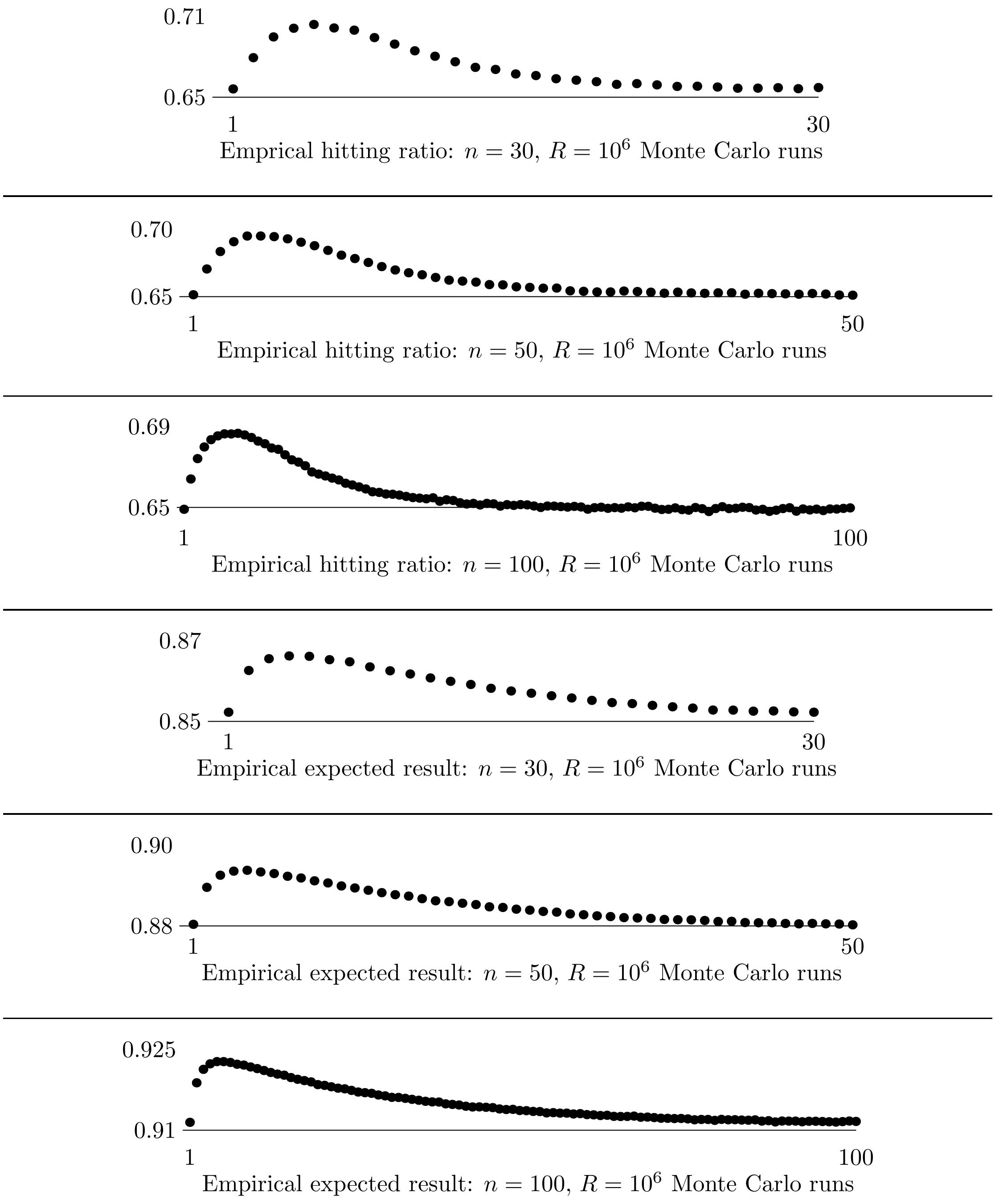

Figure 6: Empirical results for Model 1.3: unbiased experts maximizing a convex combination

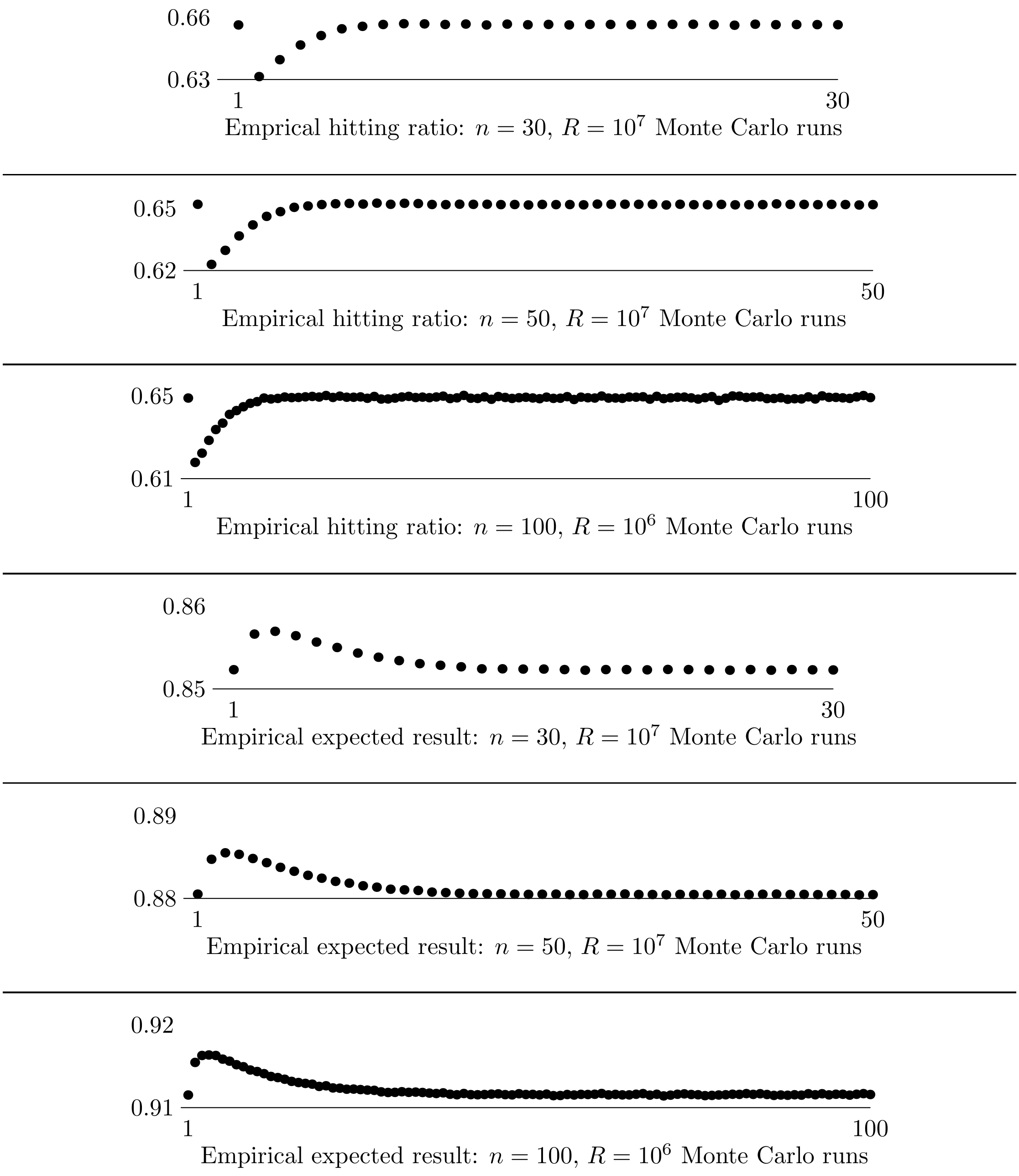

Figure 7: Hitting ratios of Monte Carlo simulations for Model 1.4, $n \in \{30, 50, 100\}$ pairs of random variables, $\Gamma = \Delta = 0.5$

In this model, both Alice and Bob subjectively assign a lower than the real weight to the x-components.

The results of Monte Carlo simulations for $n \in \{30, 50, 100\}$ and $\Gamma = \Delta = 0.5$ are shown in Figure 7. We note a shortlisting valley for the hitting ratio, but not for the average result. More specifically, the empirical hitting ratio $h^{\text{emp}}_{\text{BiasConv}}(n \xrightarrow{0.5} 1 \xrightarrow{0.5} 1)$ is clearly larger than $h^{\text{emp}}_{\text{BiasConv}}(n \xrightarrow{0.5} 2 \xrightarrow{0.5} 1)$, but $h^{\text{emp}}_{\text{BiasConv}}(n \xrightarrow{0.5} k \xrightarrow{0.5} 1)$ increases monotonically for $k \geqslant 2$.

2 Numerical Computation

We compute some of the hitting ratios involved in Model 1.1. Since our random variables are uniformly distributed, this leads to the problem of determining the volume of a polytope. More precisely, given a set of linear inequalities, we need to calculate the volume of the common solution set of the inequalities. For the numerical calculations, we use an algorithm originally proposed by Lasserre (1983) and implemented by Büeler and Enge (2003) – see also Büeler, Enge and Fukuda (2000).

Let P_k denote a probability with an intermediate shortlist of size k. We note that by symmetry, it is sufficient to consider the first variable x_1:

$$
\begin{aligned}
&h_{\text{Cont}}(n \xrightarrow{\Gamma} k \xrightarrow{\Delta} 1)\\
=&P_k(x_1 \text{ correct and chosen } \vee \cdots \vee x_n \text{ correct and chosen})\\
=&\sum_{i=1}^{n} P_k(x_i \text{ correct and chosen})\\
=&n \cdot P_k(x_1 \text{ correct and chosen})
\end{aligned}
$$

Let $\operatorname{vol}_n A$ denote the n-dimensional volume of some set $A \subset \mathbb{R}^n$. We have

$$
\begin{aligned}
&P_1(\text{the correct index is chosen})\\
=&n \cdot P_1(x_1 \text{ is correct and chosen})\\
=&n \cdot P(x_1 > x_2 \,\wedge\, \ldots \,\wedge\, x_1 > x_n \,\wedge\, x_1 + \gamma_1 > x_2 + \gamma_2 \,\wedge\, \ldots x_1 + \gamma_1 > x_n + \gamma_n)\\
=&\frac{n}{\Gamma^n} \operatorname{vol}_{2n} A_1
\end{aligned}
$$

for

$$
\begin{aligned}
A_1 :=\big\{(x,\gamma) \in [0,1]^n \times [0,\Gamma]^n \,\big|\, & x_1 > x_2 \,\wedge\, \ldots \,\wedge\, x_1 > x_n \,\wedge\\
& \wedge\, x_1 + \gamma_1 > x_2 + \gamma_2 \,\wedge\, \ldots \,\wedge\, x_1 + \gamma_1 > x_n + \gamma_n\big\}.
\end{aligned} \tag{1}
$$

For $k > 1$, we can once more use the inherent symmetry to simplify matters. We assume once again that x_1 is the maximum value and only consider the case $\mathcal{S} = \{1, \ldots, k\}$. We have to multiply with the number of possibilities to choose $k - 1$ elements (the indices $2, \ldots, k$) from a set of cardinality $n - 1$ (the indices $2, \ldots, n$):

$$P_k(\text{the correct index is chosen})$$

$$
\begin{aligned}
&= n \cdot P_k(x_1 \text{ is correct and chosen}) \\
&= n \cdot \binom{n-1}{k-1} \cdot P\big(\underbrace{x_1 > x_2 \ \wedge \ \ldots \ \wedge \ x_1 > x_n}_{n-1 \text{ linear inequalities}} \wedge \\
&\qquad \wedge \ \underbrace{\forall i \in \{1, \ldots, k\} \colon \forall j \in \{k+1, \ldots, n\} \colon x_i + \gamma_i > x_j + \gamma_j}_{k \cdot (n-k) \text{ linear inequalities}} \wedge \\
&\qquad \wedge \ \underbrace{\forall i \in \{2, \ldots, k\} \colon x_1 + \delta_1 > x_i + \delta_i}_{k-1 \text{ linear inequalities}} \big) \\
&= \frac{n}{\Gamma^n \Delta^k} \binom{n-1}{k-1} \operatorname{vol}_{2n+k} A_k
\end{aligned}
$$

for

$$
\begin{aligned}
A_k := \big\{ (x, \gamma, \delta) \in [0,1]^n \times [0, \Gamma]^n \times [0, \Delta]^k \,\big|\, & x_1 > x_2 \ \wedge \ \ldots \ \wedge \ x_1 > x_n \wedge \\
& \wedge \ \forall i \in \{1, \ldots, k\} \colon \forall j \in \{k+1, \ldots, n\} \colon x_i + \gamma_i > x_j + \gamma_j \wedge \\
& \wedge \ x_1 + \delta_1 > x_2 + \delta_2 \ \wedge \ \ldots \ \wedge \ x_1 + \delta_1 > x_k + \delta_k \big\} \qquad \text{for } k > 1.
\end{aligned} \tag{2}
$$

Our probabilities can be determined by calculating the volume of an appropriate polytope A_k which is described by the following number of linear inequalities:

- for $k = 1$, there are $2n$ inequalities given in (1). In addition, we need the $3n$ inequalities

$$
\begin{array}{ll}
x_1 < 1 & \text{(which implies } x_2 < 1, \ldots, x_n < 1) \\
x_2 > 0 & \\
\vdots & \\
x_n > 0 & \text{(which implies } x_1 > 0) \\
\gamma_1 > 0 & \quad \gamma_1 < \Gamma \\
\vdots & \quad \vdots \\
\gamma_n > 0 & \quad \gamma_n < \Gamma
\end{array}
$$

  for a total of $5n$ inequalities.

- For $k > 1$, there are $n - 1 + k(n-k) + k - 1$ inequalities given in (2). In addition, we need the same $3n$ inequalities as in the case $k = 1$ as well as the $2k$ inequalities

$$
\begin{array}{ll}
\delta_1 > 0 & \delta_1 < \Delta \\
\vdots & \vdots \\
\delta_n > 0 & \delta_n < \Delta
\end{array}
$$

  for a total of

$$
n - 1 + k(n-k) + k - 1 + 3n + 2k = 4n + kn + 3k - k^2 - 2
$$

  inequalities.

$\Gamma = \Delta$	$k = 1$		$k = 2$	
	Vinci	MC	Vinci	MC
0.33	0.849689	0.849566	0.856692	0.857063
0.75	0.705859	0.706221	0.728763	0.728965
1.5	0.556173	0.556280	0.585160	0.584844
3	0.452160	0.452534	0.471316	0.471379

Table 3: Numerical Results for the hitting ratio in Model 1.1, $n = 3$. Given are exact probabilities as calculated by the `Vinci` package and Monte Carlo (MC) results ($R = 10^6$).

$\Gamma = \Delta$	$k = 1$		$k = 2$		$k = 3$	
	Vinci	MC	Vinci	MC	Vinci	MC
0.25	0.851007	0.851375	0.858450	0.858505	0.852095	0.851554
0.416667	0.769444	0.769789	0.785830	0.785863	0.773667	0.773079
0.583333	0.700186	0.700289	0.724908	0.724887	0.709504	0.709002
0.833333	0.616321	0.616713	0.648242	0.648780	0.633947	0.633771
1.25	0.520324	0.520465	0.549984	0.550130	0.543656	0.543708
1.75	0.453074	0.453586	0.474840	0.475134	0.474606	0.474981
2.5	0.397352	0.397799	0.410973	0.411029	0.414104	0.414480
4	0.344980	0.345050	0.351453	0.351361	0.355605	0.355755

Table 4: Numerical results for the hitting ratio in Model 1.1, $n = 4$. Given are exact probabilities as calculated by the `Vinci` package and Monte Carlo (MC) results ($R = 10^6$).

Unfortunately, giving an explicit formula for the probabilities involved by determining the volume of our polytopes seems to be an unfeasibly complex task. We content ourselves with numerically computing some examples for small n, using the `Vinci` package as implemented by Büeler and Enge (2003).

Results are shown in Tables 3, 4 and 5. The values of $\Gamma = \Delta$ considered were chosen as the midpoints of the intervals used in the discretized model considered in Section 3 (see Table 6).

Not being able to calculate the hitting ratios directly is rather unsatisfying, especially in view of the small n that can be considered numerically. Lasserre and Zeron (2001) may offer a possibility for abstract calculation and an alternative for the numerics.

3 Discretization

In this section, we discretize Model 1.1 based on the following observation:

Remark 3.1. Given n independent random variables which are uniformly distributed in $[0, 1]$, the mean of the k^{th}-smallest variable is $\frac{k}{n+1}$. □

This is intuitively clear: the n variables divide the interval $[0, 1]$ into $n+1$ subintervals, which – given the uniform distribution – should be of equal length.

$\Gamma = \Delta$	$k = 1$		$k = 2$		$k = 3$	
	Vinci	MC	Vinci	MC	Vinci	MC
0.2	0.851960	0.852599	0.859492	0.859310	0.853275	0.853218
0.291667	0.795300	0.795657	0.808723	0.808669	0.798831	0.799263
0.416667	0.727694	0.727780	0.748914	0.748949	0.736020	0.736195
0.583333	0.652487	0.653096	0.681298	0.681392	0.669043	0.669058
0.708333	0.605565	0.605856	0.637122	0.637472	0.628003	0.628143
0.875	0.553505	0.553913	0.584925	0.585291	0.581141	0.581709
1.16667	0.485402	0.485767	0.510459	0.510947	0.513608	0.514065
1.41667	0.444077	0.444280	0.462922	0.463614	0.468636	0.468877
1.75	0.404355	0.404530	0.416748	0.417377	0.423414	0.423317
2.5	0.349464	0.349405	0.353752	0.354122	0.359691	0.359985
3.5	0.310020	0.309904	0.310030	0.310244	0.314526	0.314346
5	0.278813	0.278706	0.276805	0.276969	0.279947	0.279726

Table 5: Numerical results for the hitting ratio in Model 1.1, $n = 5$. Given are exact probabilities as calculated by the `Vinci` package and Monte Carlo (MC) results ($R = 10^6$).

In fact, Remark 3.1 follows directly from Equation (2.2.19) in Arnold, Balakrishnan and Nagaraja (1992), which gives the m^{th} moment of the k^{th}-smallest variable.

In view of the obstacles we encounter in the exact analysis of Model 1.1, we simplify our task by replacing our set of n uniformly distributed random variables in the interval $[0, 1]$ with its expected value: by Remark 3.1, this is a random permutation of the discrete set $\{\frac{1}{n+1}, \dots, \frac{n}{n+1}\}$. To be more precise, the "real values" are given by a random permutation $\sigma \in S_n$, and our experts' imprecisions are given by random permutations $\pi, \varrho \in S_n$. Thus, the Alice observes

$$\frac{\sigma(i)}{n+1} + \frac{\pi(i)}{n+1} \cdot \Gamma \text{ for } 1 \leqslant i \leqslant n, \tag{3}$$

and Bob observes

$$\frac{\sigma(i)}{n+1} + \frac{\varrho(i)}{n+1} \cdot \Delta \text{ for } i \in \mathcal{S}. \tag{4}$$

However, we are interested in the case $\Delta = \Gamma$, we can assume without loss of generality that $\sigma = \text{id}$ (Alice and Bob do not know this, of course!), and after multiplying the expressions in Equations (3) and (4) by $(n + 1)$, we obtain the following discretization of Model 1.1:

Model 3.2 (Model Disc). Let $\Gamma > 0$. Consider two random permutations π, ϱ, independently uniformly distributed in S_n, associated to two experts Alice and Bob. Alice observes $i + \Gamma \cdot \pi(i)$ for $1 \leqslant i \leqslant n$ and constructs a shortlist $\mathcal{S} \subseteq \{1, \dots n\}$ consisting of the indices of her k largest observations.

Next, Bob observes $i + \Gamma \cdot \varrho(i)$ for $i \in \mathcal{S}$ and selects the index of his largest observation. $\Diamond$

The correct index Alice and Bob are looking for is n.

In this model, we can perform an exhaustive analysis for small n by simply evaluating every possible permutation.

We first observe:

Remark 3.3. Ties between observations can occur if and only if $\Gamma = \frac{a}{b}$ with $1 \leqslant a, b \leqslant n-1$.

Proof. A tie occurs if and only if

$$i + \Gamma \cdot \pi(i) = j + \Gamma \cdot \pi(j) \iff \Gamma = \frac{i-j}{\pi(j) - \pi(i)} \quad \text{for } i \neq j$$

or the analogue for ϱ. ■

To avoid complications, we will therefore avoid these values for Γ: define

$$T_n := \left\{ \frac{a}{b} \,\middle|\, 1 \leqslant a, b \leqslant n-1 \right\},$$

and restrict the parameter values for $\Gamma = \Delta$ to $\mathbb{R}_{\geqslant 0} \setminus T_n$.

Observation 3.4. *If $T_n = \{\Gamma_1, \ldots, \Gamma_r\}$ with $\Gamma_1 < \Gamma_2 < \cdots < \Gamma_r$, then $h_{Disc}(n \xrightarrow{\Gamma} k \xrightarrow{\Gamma} 1)$ is constant for $\Gamma_i < \Gamma < \Gamma_{i+1}$.* □

We summarize our subsequent results on shortlisting valleys (SVs):

condition on n	condition on $\Gamma \in \mathbb{R}_{\geqslant 0} \setminus T_n$	shortlisting valley?	reference
all n	$\Gamma < \frac{1}{n-1}$	no SV	Observation 3.6
all n	$\Gamma > n-1$	no SV	Observation 3.6
$n \leqslant 4$	none	no SV	Observation 3.5
$n > 4$	$n-2 < \Gamma < n-1$	elementary SV	Proposition 3.7

For $n < 8$, it is possible to evaluate the hitting ratios for all $\Gamma \in \mathbb{R}_{\geqslant 0} \setminus T_n$ by evaluating all $(2n)!$ possible pairs of permutations:

n	Γ	$k=1$	$k=2$	ESV
3	$\Gamma < 0.5$	1.0	1.0	
	$0.5 < \Gamma < 1$	0.833333	0.861111	
	$1 < \Gamma < 2$	0.5	0.527778	
	$2 < \Gamma$	0.333333	0.333333	
4	$\Gamma < 0.333333$	1.0	1.0	
	$0.333333 < \Gamma < 0.5$	0.916667	0.923611	
	$0.5 < \Gamma < 0.666667$	0.75	0.8125	
	$0.666667 < \Gamma < 1$	0.708333	0.770833	
	$1 < \Gamma < 1.5$	0.458333	0.496528	

Table 6: Exact hitting ratios for $n \leqslant 8$, $\Gamma \in \mathbb{R}_{\geqslant 0} \setminus T_n$ and $k = 1, 2$. Elementary shortlisting valleys (ESV) are marked with a $\times$

n	Γ	$k=1$	$k=2$	ESV
	$1.5 < \Gamma < 2$	0.416667	0.427083	
	$2 < \Gamma < 3$	0.333333	0.333333	
	$3 < \Gamma$	0.25	0.25	
5	$\Gamma < 0.25$	1.0	1.0	
	$0.25 < \Gamma < 0.333333$	0.95	0.9525	
	$0.333333 < \Gamma < 0.5$	0.85	0.8725	
	$0.5 < \Gamma < 0.666667$	0.683333	0.753333	
	$0.666667 < \Gamma < 0.75$	0.633333	0.706667	
	$0.75 < \Gamma < 1$	0.625	0.680417	
	$1 < \Gamma < 1.333333$	0.425	0.457917	
	$1.333333 < \Gamma < 1.5$	0.416667	0.42875	
	$1.5 < \Gamma < 2$	0.383333	0.389167	
	$2 < \Gamma < 3$	0.3	0.293333	$\times$
	$3 < \Gamma < 4$	0.25	0.245	$\times$
	$4 < \Gamma$	0.2	0.2	
6	$\Gamma < 0.2$	1.0	1.0	
	$0.2 < \Gamma < 0.25$	0.966667	0.967778	
	$0.25 < \Gamma < 0.333333$	0.9	0.91	
	$0.333333 < \Gamma < 0.4$	0.8	0.84	
	$0.4 < \Gamma < 0.5$	0.783333	0.823333	
	$0.5 < \Gamma < 0.6$	0.633333	0.706667	
	$0.6 < \Gamma < 0.666667$	0.627778	0.690833	
	$0.666667 < \Gamma < 0.75$	0.580556	0.645741	
	$0.75 < \Gamma < 0.8$	0.566667	0.615926	
	$0.8 < \Gamma < 1$	0.565278	0.604491	
	$1 < \Gamma < 1.25$	0.398611	0.423519	
	$1.25 < \Gamma < 1.333333$	0.397222	0.412593	
	$1.333333 < \Gamma < 1.5$	0.386111	0.392315	
	$1.5 < \Gamma < 1.666667$	0.355556	0.363056	
	$1.666667 < \Gamma < 2$	0.35	0.342407	$\times$
	$2 < \Gamma < 2.5$	0.283333	0.280648	$\times$
	$2.5 < \Gamma < 3$	0.266667	0.254167	$\times$
	$3 < \Gamma < 4$	0.233333	0.223611	$\times$
	$4 < \Gamma < 5$	0.2	0.194444	$\times$
	$5 < \Gamma$	0.166667	0.166667	
7	$\Gamma < 0.166667$	1.0	1.0	
	$0.166667 < \Gamma < 0.2$	0.97619	0.976757	
	$0.2 < \Gamma < 0.25$	0.928571	0.933673	
	$0.25 < \Gamma < 0.333333$	0.857143	0.877551	
	$0.333333 < \Gamma < 0.4$	0.752381	0.803401	
	$0.4 < \Gamma < 0.5$	0.728571	0.780045	
	$0.5 < \Gamma < 0.6$	0.592857	0.659977	
	$0.6 < \Gamma < 0.666667$	0.583333	0.639796	
	$0.666667 < \Gamma < 0.75$	0.539683	0.591393	

Table 6: Exact hitting ratios for $n \leqslant 8$, $\Gamma \in \mathbb{R}_{\geqslant 0} \setminus T_n$ and $k = 1, 2$. Elementary shortlisting valleys (ESV) are marked with a $\times$

n	Γ	$k=1$	$k=2$	ESV
	$0.75 < \Gamma < 0.8$	0.523413	0.564002	
	$0.8 < \Gamma < 0.833333$	0.519841	0.549159	
	$0.833333 < \Gamma < 1$	0.519643	0.545403	
	$1 < \Gamma < 1.2$	0.376786	0.395026	
	$1.2 < \Gamma < 1.25$	0.376587	0.391619	
	$1.25 < \Gamma < 1.333333$	0.373413	0.381321	
	$1.333333 < \Gamma < 1.5$	0.361111	0.364031	
	$1.5 < \Gamma < 1.666667$	0.333333	0.332615	$\times$
	$1.666667 < \Gamma < 2$	0.32619	0.317763	$\times$
	$2 < \Gamma < 2.5$	0.266667	0.258333	$\times$
	$2.5 < \Gamma < 3$	0.252381	0.240476	$\times$
	$3 < \Gamma < 4$	0.214286	0.20102	$\times$
	$4 < \Gamma < 5$	0.190476	0.180952	$\times$
	$5 < \Gamma < 6$	0.166667	0.161565	$\times$
	$6 < \Gamma$	0.142857	0.142857	
8	$\Gamma < 0.142857$	1.0	1.0	
	$0.142857 < \Gamma < 0.166667$	0.982143	0.982462	
	$0.166667 < \Gamma < 0.2$	0.946429	0.949298	
	$0.2 < \Gamma < 0.25$	0.892857	0.904337	
	$0.25 < \Gamma < 0.285714$	0.821429	0.853316	
	$0.285714 < \Gamma < 0.333333$	0.8125	0.844388	
	$0.333333 < \Gamma < 0.4$	0.71131	0.769239	
	$0.4 < \Gamma < 0.428571$	0.684524	0.743729	
	$0.428571 < \Gamma < 0.5$	0.680952	0.735395	
	$0.5 < \Gamma < 0.571429$	0.559524	0.620323	
	$0.571429 < \Gamma < 0.6$	0.558631	0.614929	
	$0.6 < \Gamma < 0.666667$	0.547024	0.594622	
	$0.666667 < \Gamma < 0.714286$	0.506548	0.547906	
	$0.714286 < \Gamma < 0.75$	0.506349	0.545173	
	$0.75 < \Gamma < 0.8$	0.489484	0.521254	
	$0.8 < \Gamma < 0.833333$	0.484177	0.506304	
	$0.833333 < \Gamma < 0.857143$	0.483333	0.500159	
	$0.857143 < \Gamma < 1$	0.483309	0.499153	
	$1 < \Gamma < 1.166667$	0.358309	0.371278	
	$1.166667 < \Gamma < 1.2$	0.358284	0.370382	
	$1.2 < \Gamma < 1.25$	0.35749	0.365797	
	$1.25 < \Gamma < 1.33333$	0.353075	0.355769	
	$1.333333 < \Gamma < 1.4$	0.340675	0.340528	$\times$
	$1.4 < \Gamma < 1.5$	0.340476	0.337686	$\times$
	$1.5 < \Gamma < 1.666667$	0.314881	0.311067	$\times$
	$1.666667 < \Gamma < 1.75$	0.306845	0.298384	$\times$
	$1.75 < \Gamma < 2$	0.305952	0.292071	$\times$
	$2 < \Gamma < 2.333333$	0.253571	0.245886	$\times$
	$2.333333 < \Gamma < 2.5$	0.25	0.235778	$\times$

Table 6: Exact hitting ratios for $n \leqslant 8$, $\Gamma \in \mathbb{R}_{\geqslant 0} \setminus T_n$ and $k = 1, 2$. Elementary shortlisting valleys (ESV) are marked with a $\times$

n	Γ	$k=1$	$k=2$	ESV
	$2.5 < \Gamma < 3$	0.238095	0.2228	×
	$3 < \Gamma < 3.5$	0.205357	0.194005	×
	$3.5 < \Gamma < 4$	0.196429	0.180804	×
	$4 < \Gamma < 5$	0.178571	0.166295	×
	$5 < \Gamma < 6$	0.160714	0.152158	×
	$6 < \Gamma < 7$	0.142857	0.138393	×
	$7 < \Gamma$	0.125	0.125	

Table 6: Exact hitting ratios for $n \leqslant 8$, $\Gamma \in \mathbb{R}_{\geqslant 0} \setminus T_n$ and $k = 1, 2$. Elementary shortlisting valleys (ESV) are marked with a ×

Table 6 yields:

Observation 3.5. *There are no elementary shortlisting valleys for any Γ if $n \leqslant 4$.*

The hitting ratios for $n = 4$ and $1 \leqslant k \leqslant 3$ show that there is no shortlisting valley at all for $n \leqslant 4$:

Γ	$k=1$	$k=2$	$k=3$
$\Gamma < 0.333333$	1.0	1.0	1.0
$0.333333 < \Gamma < 0.5$	0.916667	0.923611	0.916667
$0.5 < \Gamma < 0.666667$	0.75	0.8125	0.75
$0.666667 < \Gamma < 1$	0.708333	0.770833	0.727431
$1 < \Gamma < 1.5$	0.458333	0.496528	0.494792
$1.5 < \Gamma < 2$	0.416667	0.427083	0.440972
$2 < \Gamma < 3$	0.333333	0.333333	0.340278
$3 < \Gamma$	0.25	0.25	0.25

□

Observation 3.6. *If $\Gamma < \frac{1}{n-1}$, the hitting ratio is $h_{Disc}(n \xrightarrow{\Gamma} k \xrightarrow{\Gamma} 1) = 1$ for all k.*

If $\Gamma > n - 1$, the hitting ratio is $h_{Disc}(n \xrightarrow{\Gamma} k \xrightarrow{\Gamma} 1) = \frac{1}{n}$ for all k.

Proof. In the first case, we have

$$n + \Gamma \cdot \pi(n) > i + \Gamma \cdot \pi(i)$$

for all i. Thus, Alice will always pass the index n on in her shortlist, and Bob will always choose n as the index of his largest observation.

In the second case, on the other hand, we have

$$i + \Gamma \cdot n > j + \Gamma \cdot \pi(j)$$

for all $i \neq j$. Thus, the "real value" has no influence at all, and Alice and Bob only choose based on π and ϱ. We have a hit if and only if

$$n \in \mathcal{S} = \left\{\pi^{-1}(n), \ldots, \pi^{-1}(n-k+1)\right\} \quad \text{and} \quad \forall i \in \mathcal{S} \setminus \{n\} \colon \varrho(n) > \varrho(i).$$

Now,

$$\#\{\pi \in S_n \,|\, \pi(n) \in \{n-k+1, \dots, n\}\} = k \cdot (n-1)!$$

and for $n \in \mathcal{S}$, the condition for a hit becomes after some obvious renaming that $\varrho'(k) = k$ for $\varrho' \in S_k$ associated to ϱ. We obtain a hitting ratio of

$$h_{\text{Disc}}(n \xrightarrow{\Gamma} k \xrightarrow{\Gamma} 1) = \frac{k \cdot (n-1)!}{n!} \cdot \frac{1}{k} = \frac{1}{n}. \qquad \blacksquare$$

Proposition 3.7. *If $n > 4$ and $n-2 < \Gamma < n-1$, then an elementary shortlisting valley occurs.*

Proof. We note that the condition on Γ implies that $1 < \frac{n-1}{\Gamma} < 2$ and $\frac{n-i}{\Gamma} < 1$ for $i \geqslant 2$. The condition $n > 4$ will only be used at the very end of our proof.

We need to prove that $h_{\text{Disc}}(n \xrightarrow{\Gamma} 1 \xrightarrow{\Gamma} 1) > h_{\text{Disc}}(n \xrightarrow{\Gamma} 2 \xrightarrow{\Gamma} 1)$.

For $k = 1$, we have

$$\begin{aligned}
&h_{\text{Disc}}(n \xrightarrow{\Gamma} 1 \xrightarrow{\Gamma} 1)\\
&= P\big(n + \Gamma \cdot \pi(n) > 1 + \Gamma \cdot \pi(1) \ \wedge\ \dots \ \wedge\ n + \Gamma \cdot \pi(n) > n-1 + \Gamma \cdot \pi(n-1)\big)\\
&= P\Big(\pi(n) > \pi(1) - \frac{n-1}{\Gamma} \ \wedge\ \dots \ \wedge\ \pi(n) > \pi(n-1) - \frac{1}{\Gamma}\Big)\\
&= P\big(\pi(n) \geqslant \pi(1) - 1 \ \wedge\ \pi(n) > \pi(2) \ \wedge\ \dots \ \wedge\ \pi(n) > \pi(n-1)\big)\\
&= \frac{\#\Big\{\pi \in S_n \,\Big|\, \pi(n) = n \vee \big(\pi(n) = n-1 \ \wedge\ \pi(1) = n\big)\Big\}}{\#S_n}\\
&= \frac{\#S_{n-1} + \#S_{n-2}}{\#S_n}\\
&= \frac{(n-1)! + (n-2)!}{n!}\\
&= \frac{1}{n} + \frac{1}{n(n-1)}\\
&= \frac{1}{n-1}.
\end{aligned}$$

For $k = 2$, we obtain

$$\begin{aligned}
&h_{\text{Disc}}(n \xrightarrow{\Gamma} 2 \xrightarrow{\Gamma} 1)\\
&= \sum_{j=1}^{n-1} P\big(\text{Alice shortlists } \mathcal{S} = \{j, n\}, \text{ Bob chooses } n\big)\\
&= \sum_{j=1}^{n-1} \Big(P\big(n + \Gamma \cdot \varrho(n) > j + \Gamma \cdot \varrho(j) \ \wedge\ n + \Gamma \cdot \pi(n) > j + \Gamma \cdot \pi(j) \ \wedge\\
&\qquad \wedge\ \forall i \neq j, n \colon i + \Gamma \cdot \pi(i) < j + \Gamma \cdot \pi(j)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ P\big(n + \Gamma \cdot \varrho(n) > j + \Gamma \cdot \varrho(j) \ \wedge\ n + \Gamma \cdot \pi(n) < j + \Gamma \cdot \pi(j) \wedge \\
&\quad \wedge\ \forall i \neq j, n \colon i + \Gamma \cdot \pi(i) < n + \Gamma \cdot \pi(n)\big)\Big) \\
= &\sum_{j=1}^{n-1} \Bigg(P\Bigg(\varrho(j) < \varrho(n) + \frac{n-j}{\Gamma} \ \wedge\ \pi(j) < \pi(n) + \frac{n-j}{\Gamma} \wedge \\
&\quad \wedge\ \forall i \neq j, n \colon \pi(i) < \pi(j) + \frac{j-i}{\Gamma}\Bigg) \\
&+ P\Bigg(\varrho(j) < \varrho(n) + \frac{n-j}{\Gamma} \ \wedge\ \pi(j) > \pi(n) + \frac{n-j}{\Gamma} \wedge \\
&\quad \wedge\ \forall i \neq j, n \colon \pi(i) < \pi(n) + \frac{n-i}{\Gamma}\Bigg)\Bigg) \\
= &P\big(\underbrace{\varrho(1) \leqslant \varrho(n) + 1 \ \wedge\ \pi(1) \leqslant \pi(n) + 1 \ \wedge\ \forall i \neq 1, n \colon \pi(i) < \pi(1)}_{=:T_1} \big) \\
&+ \sum_{j=2}^{n-1} P\big(\underbrace{\varrho(j) < \varrho(n) \ \wedge\ \pi(j) < \pi(n) \ \wedge\ \forall i \neq j, n \colon \pi(i) < \pi(j)}_{=:T_2} \big) \\
&+ P\big(\varrho(1) \leqslant \varrho(n) + 1 \ \wedge\ \underbrace{\pi(1) > \pi(n) + 1 \ \wedge\ \forall i \neq 1, n \colon \pi(i) < \pi(n)}_{\text{this is impossible!}} \big) \\
&+ \sum_{j=2}^{n-1} P\big(\underbrace{\varrho(j) < \varrho(n) \ \wedge\ \pi(j) > \pi(n) \ \wedge\ \pi(1) \leqslant \pi(n) + 1 \ \wedge\ \forall i \neq 1, j, n \colon \pi(i) < \pi(n)}_{=:T_3} \big).
\end{aligned}
$$

We consider the three terms one by one:

T_1: For term T_1, we first examine the condition $\varrho(1) \leqslant \varrho(n) + 1$: this is satisfied by

$$
\begin{aligned}
&\underbrace{(n-1)!}_{\varrho(n)=n} + \sum_{j=1}^{n-1} \underbrace{\overbrace{(n-2)!}^{\varrho(1)=j+1} + \overbrace{(j-1) \cdot (n-2)!}^{\varrho(1)<j}}_{\varrho(n)=j} \\
&= (n-1)! + (n-2)! \sum_{j=1}^{n-1} j \\
&= (n-1)! + (n-2)! \frac{n(n-1)}{2} \\
&= (n-1)! + \frac{n!}{2}
\end{aligned}
$$

permutations ϱ. On the other hand, the condition on π is equivalent to $\pi(1) = n-1$ and $\pi(n) = n$ or the other way around. All in all, term T_1 is satisfied by

$$
\left((n-1)! + \frac{n!}{2}\right) \cdot 2(n-2)!
$$

pairs of permutations (π, ϱ).

T_2: Term T_2 is equivalent to

$$\varrho(j) < \varrho(n) \ \wedge\ \pi(j) = n-1 \ \wedge\ \pi(n) = n.$$

This is the case for $\frac{n!}{2} \cdot (n-2)!$ pairs of permutations $(\pi, \varrho) \in S_n^2$.

T_3: The condition on ϱ is again satisfied by $\frac{n!}{2}$ permutations $\varrho \in S_n$. The condition on π is equivalent to

$$\big(\pi(j) = n \ \wedge\ \pi(n) = n-1\big) \vee \big(\pi(1) = n-1 \ \wedge\ \pi(j) = n \ \wedge\ \pi(n) = n-2\big).$$

Thus T_3 is satisfied by $\frac{n!}{2} \cdot \big((n-2)! + (n-3)!\big)$ permutations $(\pi, \varrho) \in S_n^2$.

All in all, we obtain a hitting ratio of

$$\begin{aligned}
&h_{\text{Disc}}(n \xrightarrow{\Gamma} 2 \xrightarrow{\Gamma} 1) \\
&= \frac{2}{n^2(n-1)} + \frac{1}{n(n-1)} + \frac{n-2}{2n(n-1)} + \frac{n-2}{2n(n-1)} + \frac{1}{2n(n-1)} \\
&= \frac{2}{n^2(n-1)} + \frac{1}{n-1} - \frac{1}{2n(n-1)} \\
&< \frac{1}{n-1} \quad \text{since } n > 4 \\
&= h_{\text{Disc}}(n \xrightarrow{\Gamma} 1 \xrightarrow{\Gamma} 1).
\end{aligned}$$

■

Conjecture 3.8. *For all $n > 4$, there is a Γ_n such that an elementary shortlisting valley occurs exactly for $\Gamma_n < \Gamma < n-1$.*

Γ_n decreases with increasing n, and $\lim_{n\to\infty} \Gamma_n = 0$. □

Remark 3.9. Table 6 yields that Conjecture 3.8 is correct for $n \leqslant 8$, with

$$\Gamma_5 = 2, \quad \Gamma_6 = \frac{5}{3}, \quad \Gamma_7 = \frac{3}{2}, \quad \Gamma_8 = \frac{4}{3}.$$

□

4 Length-Weighted Permutations

4.1 Identical Experts

Another possible discretization uses the concept of the *length* of a permutation (Little, 2002):

Definition 4.1. The *length* $\ell(\pi)$ of a permutation π is the minimum number of transpositions of neighboring elements needed to transform the identity permutation into π. ◊

The length of a permutation π gives a notion of the distance between π and id: the higher $\ell(\pi)$, the farther away π is from id.

Lemma 4.2 (Theorem 1.2 in Little, 2002). *The length of a permutation σ is equal to the number of inversions of σ:*

$$\ell(\sigma) = \#\big\{(i,j) \in \{1,\dots,n\}^2 \,\big|\, i < j \text{ and } \sigma(i) > \sigma(j)\big\}. \qquad \square$$

Lemma 4.2 shows why the length of a permutation is often called its *inversion number*.

We consider the following model in this section:

Model 4.3 (Model Len). Choose a parameter $0 \leqslant \lambda \leqslant 1$ and let $D_n(\lambda) := \sum_{\pi \in S_n} \lambda^{\ell(\pi)}$.

Consider two experts Alice and Bob. Alice randomly chooses some $\pi \in S_n$ and Bob some $\varrho \in S_n$, where each $\sigma \in S_n$ is chosen with probability $\frac{\lambda^{\ell(\sigma)}}{D_n(\lambda)}$.

Alice and Bob order alternatives $1, \dots, n$, where 1 is the worst and n the best alternative, as indicated by π and ϱ. Alice passes the shortlist of indices

$$\mathcal{S} = \{\pi^{-1}(n), \dots, \pi^{-1}(n-k+1)\}$$

on to Bob, who then chooses the index in $\mathcal{S}$ that is mapped to the largest value under ϱ. $\diamondsuit$

Thus, for $\lambda = 1$, all permutations have the same probability $\frac{1}{n!}$. For smaller λ, permutations of small length (i.e. permutations that are "close" to the identity permutation) are more and more probable. If we use the convention $0^0 = 1$, then for $\lambda = 0$, the identity permutation is chosen with certainty: our experts deterministically choose the correct index.

We find the correct renorming factor for our probabilities in Stanley (1997, Corollary 1.3.10):

Lemma 4.4. *We have*

$$\begin{aligned} D_n(\lambda) = \sum_{\sigma \in S_n} \lambda^{\ell(\sigma)} &= (1+\lambda)(1+\lambda+\lambda^2)\cdots(1+\lambda+\lambda^2+\cdots+\lambda^{n-1}) \\ &= (1+\lambda+\lambda^2+\cdots+\lambda^{n-1}) \cdot D_{n-1}(\lambda). \end{aligned}$$

The maximum length of a permutation on n elements is $\frac{n(n-1)}{2}$. $\square$

For a given pair (π, ϱ) of permutations, Alice and Bob choose the correct index if and only if the following two conditions are met:

- Alice passes the index n on, i.e. $\pi(n) \geqslant n-k+1$.
- If $n \in \mathcal{S} = \{i_1, \dots, i_k\}$, then Bob prefers n to all other indices, i.e. $\varrho(n) = \max_{1 \leqslant j \leqslant k} \varrho(i_j)$.

This model offers the advantage that we can enumerate all possible pairs of permutations for small n. We checked the above conditions for all pairs for $3 \leqslant n \leqslant 9$. Since the probability that a given pair (π, ϱ) is chosen is $\frac{\lambda^{\ell(\pi)+\ell(\varrho)}}{D_n(\lambda)^2}$, it is sufficient to note the sum of the lengths of the two permutations. We let $L := \ell(\pi) + \ell(\varrho)$ and show our results

L	0	1	2	3	4	5	6
$k=1$	1	3	4	3	1	0	0
$k=2$	1	3	5	3	0	0	0
total	1	4	8	10	8	4	1

Table 7: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=3$

L	0	1	2	3	4	5	6	7	8	9	10	11	12
$k=1$	1	5	13	23	30	30	23	13	5	1	0	0	0
$k=2$	1	5	14	26	35	35	22	6	0	0	0	0	0
$k=3$	1	5	13	25	35	34	22	8	1	0	0	0	0
total	1	6	19	42	71	96	106	96	71	42	19	6	1

Table 8: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=4$

L	0	1	2	3	4	5	6	7	8	9	10
$k=1$	1	7	26	68	139	234	334	411	440	411	334
$k=2$	1	7	27	73	154	267	389	481	504	440	308
$k=3$	1	7	26	70	150	263	387	487	519	452	306
$k=4$	1	7	26	68	142	248	367	461	489	435	320
total	1	8	34	102	241	474	801	1186	1558	1830	1930

L	11	12	13	14	15	16	17	18	19	20
$k=1$	234	139	68	26	7	1	0	0	0	0
$k=2$	162	57	10	0	0	0	0	0	0	0
$k=3$	151	50	10	1	0	0	0	0	0	0
$k=4$	190	88	30	7	1	0	0	0	0	0
total	1830	1558	1186	801	474	241	102	34	8	1

Table 9: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=5$

in Tables 7, 8, 9, 13 and 14 (some of which have been relegated to Section A in the Appendix). In these tables, we do not show the values for $k = n$ since, as in most of our models, they are identical to those for $k = 1$.

Letting

$$H(n, k, L) := \#\big\{(\pi, \varrho) \in S_n^2 \,\big|\, (\pi, \varrho) \text{ successful for } k \text{ and } \ell(\pi) + \ell(\varrho) = L\big\},$$

we have the hitting ratio

$$h_{\text{Len}}(n \xrightarrow{\lambda} k \xrightarrow{\lambda} 1) = \sum_{L=0}^{n(n-1)} H(n, k, L)\lambda^L.$$

Reading the values for $H(n, k, L)$ from Tables 7, 8, 9, 13 and 14, we plot some hitting ratios in Figures 8, 9, 10 and 11.

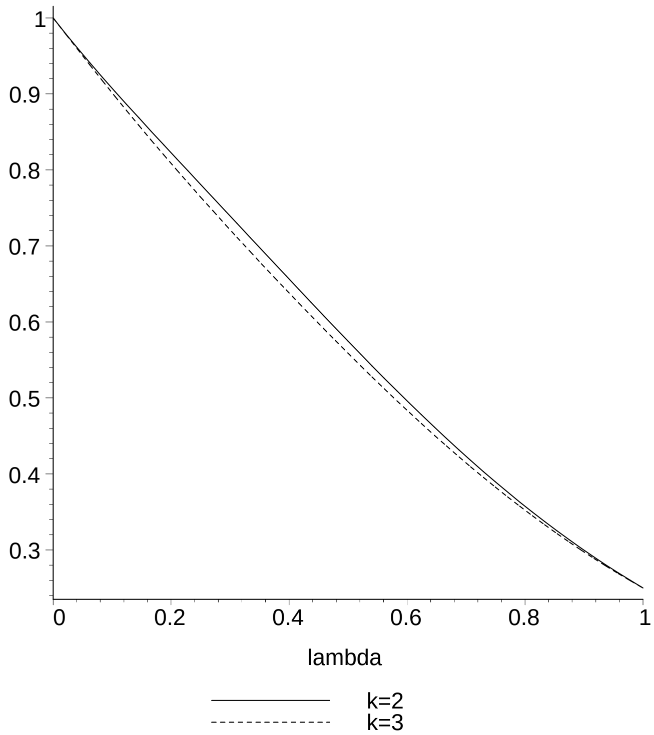

Figure 8: The hitting ratio for Model 4.3 with $n = 4$

We first state Descartes' sign rule and a technical lemma:

Proposition 4.5 (Descartes' sign rule (Henrici, 1974)). *Let $p(x) := a_0 + a_1 x + \cdots + a_n x^n \not\equiv 0$ be a real polynomial. Let v denote the number of sign changes in the sequence of its coefficients, and let r denote the number of its real positive zeros, counted with multiplicities. Then $v - r$ is even and non-negative.* □

We remark that the *exact* number of zeros of a polynomial in a given interval can be calculated using Sturm sequences.

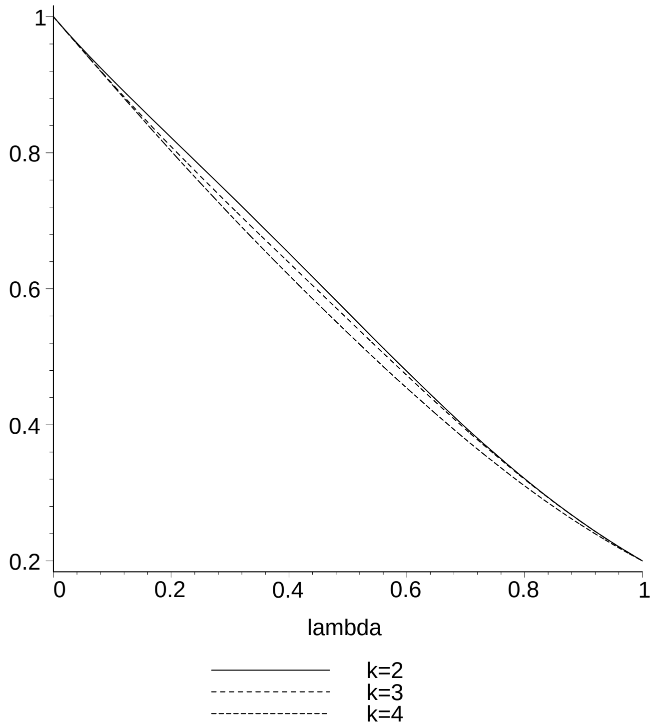

Figure 9: The hitting ratio for Model 4.3 with $n = 5$

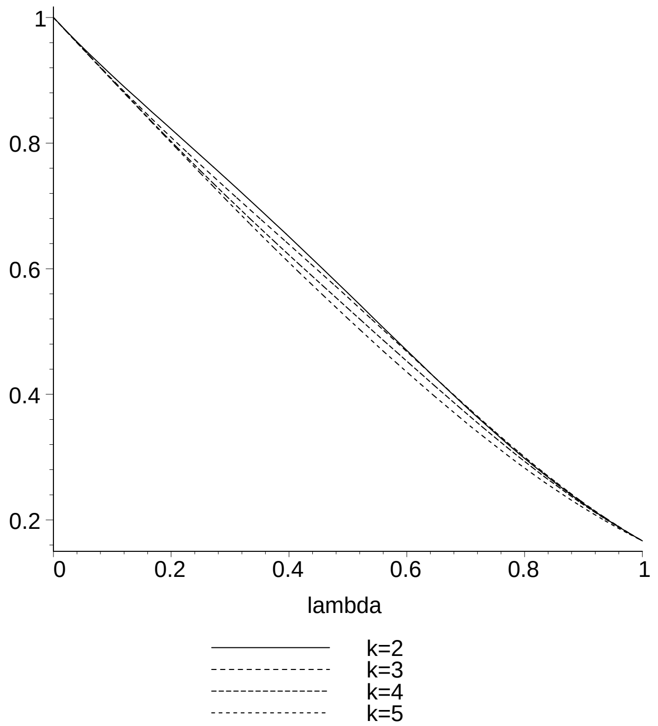

Figure 10: The hitting ratio for Model 4.3 with $n = 6$

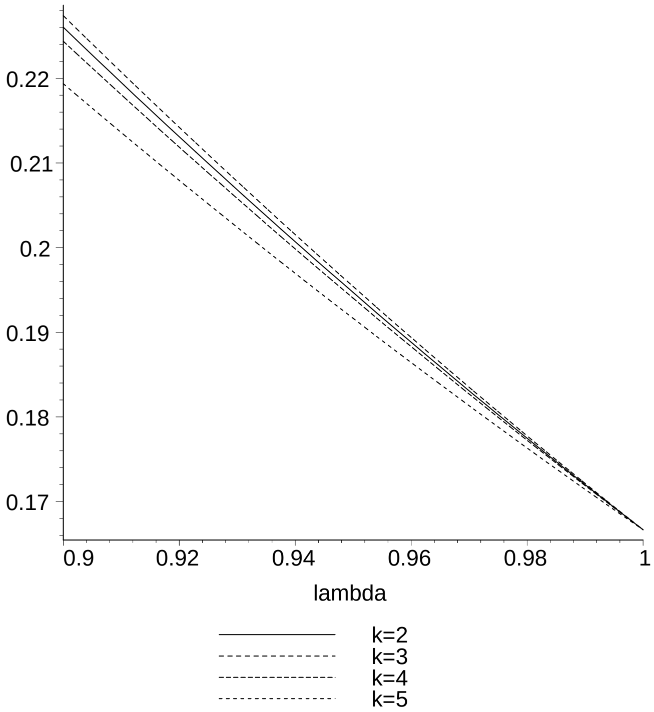

Figure 11: The hitting ratio for Model 4.3 with $n = 6$ and $\lambda > 0.9$

Lemma 4.6. *A sufficient condition that* $h_{Len}(n \xrightarrow{\lambda} k^* \xrightarrow{\lambda} 1) > h_{Len}(n \xrightarrow{\lambda} k \xrightarrow{\lambda} 1)$ *for all* $\lambda \in]0,1[$ *is that*

$$\sum_{L=0}^{L^*} H(n, k^*, L) \geqslant \sum_{L=0}^{L^*} H(n, k, L)$$

for all $L^* = 0, \ldots, n(n-1)$*, with a strict inequality for at least one* L^*.

Proof. To show that k^* is better than k is equivalent to showing that

$$\begin{aligned} &h_{\text{Len}}(n \xrightarrow{\lambda} k^* \xrightarrow{\lambda} 1) - h_{\text{Len}}(n \xrightarrow{\lambda} k \xrightarrow{\lambda} 1) \\ =&\frac{1}{D_n(\lambda)^2} \sum_{L=0}^{n(n-1)} \big(H(n, k^*, L) - H(n, k, L)\big)\lambda^L \overset{!}{>} 0. \end{aligned}$$

The monotonicity of $L \mapsto \lambda^L$ implies the result, since $\lambda \in]0,1[$. ∎

Figure 12 shows which values of k dominate others for small n.

Our results on Model 4.3 are as follows:

Proposition 4.7. *For* $k = 1$ *and* $k = n$*, the hitting ratio is*

$$h_{Len}(n \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 1) = h_{Len}(n \xrightarrow{\lambda} n \xrightarrow{\lambda} 1) = \frac{D_{n-1}(\lambda)}{D_n(\lambda)} = \frac{1}{1 + \lambda + \cdots + \lambda^{n-1}}.$$

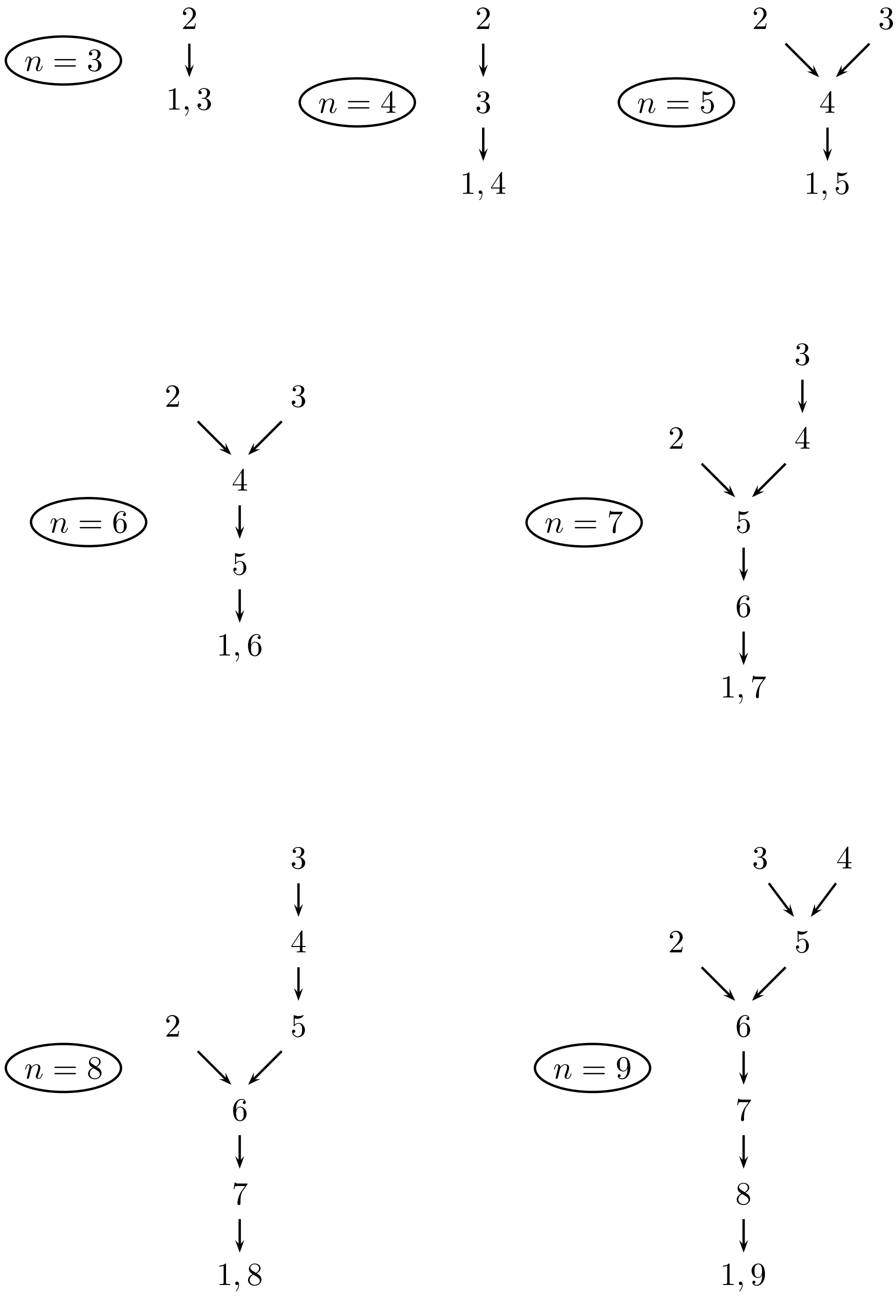

Figure 12: Values of k which dominate other values by Lemma 4.6. Since domination is transitive, we only show the transitive reductions of domination hierarchies as directed graphs.

Proof. For $k = 1$, we have a hit if and only if $\pi(n) = n$. Since the rest of the permutation π does not matter, we have

$$\sum_{\substack{\pi \in S_n \\ \pi(n)=n}} \lambda^{\ell(\pi)} = \sum_{\pi' \in S_{n-1}} \lambda^{\ell(\pi')} = D_{n-1}(\lambda)$$

and thus

$$\begin{aligned} h_{\mathrm{Len}}(n \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 1) =& \frac{1}{D_n(\lambda)^2} \sum_{\substack{(\pi,\varrho) \in S_n^2 \\ \pi(n)=n}} \lambda^{\ell(\pi)+\ell(\varrho)} \\ =& \frac{1}{D_n(\lambda)^2} \cdot D_n(\lambda) \cdot \sum_{\substack{\pi \in S_n \\ \pi(n)=n}} \lambda^{\ell(\pi)} = \frac{D_{n-1}(\lambda)}{D_n(\lambda)}. \end{aligned}$$

Lemma 4.4 concludes the case $k = 1$, and the observation that, as usual, $h_{\mathrm{Len}}(n \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 1) = h_{\mathrm{Len}}(n \xrightarrow{\lambda} n \xrightarrow{\lambda} 1)$ completes the proof. ■

Proposition 4.8. *The total number of hitting pairs of permutations is independent of k:*

$$\sum_{L=0}^{n(n-1)} H(n, k, L) = n! \cdot (n-1)!.$$

Proof. We have

$$\begin{aligned} h_{\mathrm{Len}}(n \xrightarrow{\lambda} k \xrightarrow{\lambda} 1) &= \frac{1}{D_n(\lambda)^2} \sum_{L=0}^{n(n-1)} H(n, k, L) \lambda^L \\ \Longrightarrow \sum_{L=0}^{n(n-1)} H(n, k, L) &= D_n(1)^2 \cdot h_{\mathrm{Len}}(n \xrightarrow{1} k \xrightarrow{1} 1). \end{aligned}$$

However, for $\lambda = 1$, all permutations are equally probable, and therefore, all indices have the same probability of being returned by Alice and Bob. Thus, $h_{\mathrm{Len}}(n \xrightarrow{1} k \xrightarrow{1} 1) = \frac{1}{n}$, and

$$\sum_{L=0}^{n(n-1)} H(n, k, L) = (n!)^2 \cdot \frac{1}{n} = n! \cdot (n-1)!.$$ ■

However, the following more "constructive" proof of Proposition 4.8 along the lines of the proof of Proposition 4.9 may be more helpful in tackling Conjecture 4.18:

Alternative proof of Proposition 4.8. Given $1 \leqslant k < n$, we have a disjoint union

$$S_n^2 = P_{n,k}^{(1)} \,\dot\cup\, P_{n,k}^{(2)} \,\dot\cup\, P_{n,k}^{(3)} \,\dot\cup\, P_{n,k}^{(4)}$$

with

$$
\begin{aligned}
P_{n,k}^{(1)} &= \big\{(\pi,\varrho) \in S_n^2 \,\big|\, (\pi,\varrho) \text{ successful for } k \text{ and } k+1\big\}\\
P_{n,k}^{(2)} &= \big\{(\pi,\varrho) \in S_n^2 \,\big|\, (\pi,\varrho) \text{ successful for } k \text{ and not for } k+1\big\}\\
P_{n,k}^{(3)} &= \big\{(\pi,\varrho) \in S_n^2 \,\big|\, (\pi,\varrho) \text{ successful for } k+1 \text{ and not for } k\big\}\\
P_{n,k}^{(4)} &= \big\{(\pi,\varrho) \in S_n^2 \,\big|\, (\pi,\varrho) \text{ successful neither for } k \text{ nor for } k+1\big\}.
\end{aligned}
$$

We need to show that $\#P_{n,k}^{(2)} = \#P_{n,k}^{(3)}$. For this, we explicitly give mappings $\Phi\colon P_{n,k}^{(2)} \to P_{n,k}^{(3)}$ and $\Psi\colon P_{n,k}^{(3)} \to P_{n,k}^{(2)}$ that are inverse to each other.

We denote the shortlist of size k given by Alice's permutation π by

$$
\mathcal{S}_k^\pi := \big\{\pi^{-1}(n-k+1), \ldots, \pi^{-1}(n)\big\}.
$$

Let $\sigma_{i,j}$ denote the permutation that only exchanges i and j.

Φ: Let $(\pi,\varrho) \in P_{n,k}^{(2)}$, i. e. (π,ϱ) is successful for k, but not for $k+1$. Let $i_0 := \pi^{-1}(n-k)$. Then

$$
\varrho(i_0) > \varrho(n) = \max_{i \in \mathcal{S}_k^\pi} \big\{\varrho(i)\big\}.
$$

Let

$$
\Phi(\pi,\varrho) := (\pi \circ \sigma_{i_0,n}, \varrho \circ \sigma_{i_0,n}) =: (\pi', \varrho').
$$

We show that $\Phi\colon P_{n,k}^{(2)} \to P_{n,k}^{(3)}$ is well-defined: first, we have

$$
\pi'(n) = \pi \circ \sigma_{i_0,n}(n) = \pi(i_0) = n-k.
$$

Therefore, $n \in \mathcal{S}_{k+1}^\pi$, but $n \notin \mathcal{S}_k^{\pi'}$, and (π',ϱ') is not successful for k. On the other hand,

$$
\begin{aligned}
\varrho'(n) =& \varrho(i_0) = \max\big\{\varrho(n), \varrho(i_0)\big\} = \max\left\{\max_{i\in\mathcal{S}_k^\pi}\big\{\varrho(i)\big\}, \varrho(i_0)\right\}\\
=& \max_{i \in \mathcal{S}_{k+1}^{\pi'} = \mathcal{S}_k^\pi \cup \{i_0\}} \big\{\varrho(i)\big\} = \max_{i \in \mathcal{S}_{k+1}^{\pi'}} \big\{\varrho'(i)\big\},
\end{aligned}
$$

and (π',ϱ') is successful for $k+1$.

Ψ: Consider $(\pi,\varrho) \in P_{n,k}^{(3)}$, i. e. (π,ϱ) is successful for $k+1$, but not for k. Then $\pi(n) = n-k$ and $\varrho(n) = \max_{i\in\mathcal{S}_{k+1}^\pi}\big\{\varrho(i)\big\}$. Let i_0 denote the index that satisfies

$$
\varrho(i_0) = \max_{i \in \mathcal{S}_{k+1}^\pi \setminus \{n\} = \mathcal{S}_k^\pi} \big\{\varrho(i)\big\}
$$

and set

$$
\Psi(\pi,\varrho) := (\pi \circ \sigma_{i_0,n}, \varrho \circ \sigma_{i_0,n}) =: (\pi', \varrho').
$$

We show that $(\pi', \varrho') \in P^{(2)}_{n,k}$: we have $\pi'(n) = \pi(i_0) > n - k$, thus $n \in \mathcal{S}^{\pi'}_k \subset \mathcal{S}^{\pi'}_{k+1}$. Next,

$$\varrho'(n) = \varrho(i_0) = \max_{i \in \mathcal{S}^{\pi}_{k+1} \setminus \{n\} = \mathcal{S}^{\pi}_k} \{\varrho(i)\} \quad \Longrightarrow \quad \varrho'(n) = \max_{i \in \mathcal{S}^{\pi'}_k} \{\varrho'(i)\},$$

and (π', ϱ') is successful for k. On the other hand, $\pi'(i_0) = \pi(n) = n - k$ and $i_0 \in \mathcal{S}^{\pi'}_{k+1}$ with $\varrho'(i_0) = \varrho(n)$ implies that (π', ϱ') is not successful for $k + 1$.

We easily see that $\Psi \circ \Phi = \mathrm{id}_{P^{(2)}_{n,k}}$ and $\Phi \circ \Psi = \mathrm{id}_{P^{(3)}_{n,k}}$. ■

Proposition 4.9. *Model Len has no shortlisting valleys.*

Proof. We have to show that

$$h_{\mathrm{Len}}(n \xrightarrow{\lambda} k \xrightarrow{\lambda} 1) \geqslant h_{\mathrm{Len}}(n \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 1) = h_{\mathrm{Len}}(n \xrightarrow{\lambda} 1)$$

for all k. Consider some $k \neq 1, n$. We have a disjoint union $S^2_n = P_1 \dot\cup P_2 \dot\cup P_3 \dot\cup P_4$, where

$$\begin{aligned} P_1 :=& \big\{(\pi, \varrho) \in S^2_n \,\big|\, (\pi, \varrho) \text{ successful for 1 and } k\big\} \\ P_2 :=& \big\{(\pi, \varrho) \in S^2_n \,\big|\, (\pi, \varrho) \text{ successful for 1 and not for } k\big\} \\ P_3 :=& \big\{(\pi, \varrho) \in S^2_n \,\big|\, (\pi, \varrho) \text{ successful for } k \text{ and not for } 1\big\} \\ P_4 :=& \big\{(\pi, \varrho) \in S^2_n \,\big|\, (\pi, \varrho) \text{ successful neither for 1 nor for } k\big\}. \end{aligned}$$

We will exhibit an injective function $\Phi\colon P_2 \to P_3$ which reduces total length:

$$\ell(\pi') + \ell(\varrho') \leqslant \ell(\pi) + \ell(\varrho) \quad \text{for } (\pi', \varrho') = \Phi(\pi, \varrho) \tag{5}$$

with a strict inequality for at least one pair $(\pi, \varrho) \in P_2$.

By Proposition 4.8, Φ will be a bijection, and we can conclude with Lemma 4.6.

Recall that $\mathcal{S}$ denotes Alice's shortlist of size k. Now, let $(\pi, \varrho) \in P_2$, i. e. $\pi(n) = n$ and $\varrho(n) \neq \max_{j \in \mathcal{S}} \varrho(j)$. Let $i_0 := \pi^{-1}(n-1)$ and $j_0 \in \mathcal{S} \setminus \{n\}$ such that $\varrho(j_0) = \max_{j \in \mathcal{S}} \varrho(j)$. Now, set

$$\Phi(\pi, \varrho) := (\pi', \varrho') = (\pi \circ \sigma_{i_0,n}, \varrho \circ \sigma_{j_0,n}).$$

Then π' yields the same shortlist of size k as π, and $(\pi', \varrho') \in P_3$, i. e. Φ is well-defined and obviously injective.

To show Inequality (5), recall from Lemma 4.2 that

$$\ell(\pi) = \#I_\pi \quad \text{for } I_\pi := \big\{(i, j) \in \{1, \ldots, n\}^2 \,\big|\, i < j \text{ and } \pi(i) > \pi(j)\big\}.$$

We note that $I_{\pi'} = I_\pi \dot\cup \big\{(i_0, n)\big\}$, therefore $\ell(\pi') = \ell(\pi) + 1$. On the other hand, let us examine I_ϱ and $I_{\varrho'}$ in detail:

$$\text{for } j < j_0, \qquad\qquad \text{if } \varrho(j) < \varrho(n), \text{ then } (j, j_0) \notin I_\varrho \text{ and } (j, n) \notin I_\varrho$$

$$
\begin{aligned}
& & & (j,j_0)\notin I_{\varrho'} \text{ and } (j,n)\notin I_{\varrho'} \\
& & \text{if } \varrho(n)<\varrho(j)<\varrho(j_0), \text{ then } & (j,j_0)\notin I_{\varrho} \text{ and } (j,n)\in I_{\varrho} \\
& & & (j,j_0)\in I_{\varrho'} \text{ and } (j,n)\notin I_{\varrho'} \\
& & \text{if } \varrho(j_0)<\varrho(j), \text{ then } & (j,j_0)\in I_{\varrho} \text{ and } (j,n)\in I_{\varrho} \\
& & & (j,j_0)\in I_{\varrho'} \text{ and } (j,n)\in I_{\varrho'} \\
& \text{for } j_0<j<n, & \text{if } \varrho(j)<\varrho(n), \text{ then } & (j_0,j)\in I_{\varrho} \text{ and } (j,n)\notin I_{\varrho} \\
& & & (j_0,j)\in I_{\varrho'} \text{ and } (j,n)\notin I_{\varrho'} \\
& & \text{if } \varrho(n)<\varrho(j)<\varrho(j_0), \text{ then } & (j_0,j)\in I_{\varrho} \text{ and } (j,n)\in I_{\varrho} \qquad (6) \\
& & & (j_0,j)\notin I_{\varrho'} \text{ and } (j,n)\notin I_{\varrho'} \qquad (7) \\
& & \text{if } \varrho(j_0)<\varrho(j), \text{ then } & (j_0,j)\notin I_{\varrho} \text{ and } (j,n)\in I_{\varrho} \\
& & & (j_0,j)\notin I_{\varrho'} \text{ and } (j,n)\in I_{\varrho'}.
\end{aligned}
$$

Thus $\ell(\varrho') = \#I_{\varrho'} \leqslant \#I_{\varrho} = \ell(\varrho)$, and if there is some $j > j_0$ such that $\varrho(n) < \varrho(j) < \varrho(j_0)$, then (6) and (7) imply that even $\ell(\varrho') < \ell(\varrho)$.

However, we also have $(j_0, n) \in I_{\varrho}$, but $(j_0, n) \notin I_{\varrho'}$, therefore even $\ell(\varrho') \leqslant \ell(\varrho) - 1$.

Thus Inequality (5) is verified, and $h_{\text{Len}}(n \xrightarrow{\lambda} k \xrightarrow{\lambda} 1) \geqslant h_{\text{Len}}(n \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 1)$ for all $\lambda \in [0,1]$.

To show that actually $h_{\text{Len}}(n \xrightarrow{\lambda} k \xrightarrow{\lambda} 1) > h_{\text{Len}}(n \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 1)$ for all $\lambda \in]0,1[$, it is sufficient to exhibit a pair $(\pi, \varrho) \in P_2$ for which we have a strict inequality in (5). For $k = 2$, the existence of such a pair follows from Proposition 4.13. For $2 < k < n$, consider the pair (π, ϱ), where

$$
\pi := \text{id}, \qquad \varrho(j) := \begin{cases} n & \text{if } j = n-k+1 \\ n-k+1 & \text{if } j = n \\ j & \text{otherwise.} \end{cases}
$$

Then $(\pi, \varrho) \in P_2$, $\ell(\pi) = 0$, $\ell(\varrho) = 2 \cdot (k-2) + 1$, $\varrho' = \text{id}$ and

$$
\ell(\pi') + \ell(\varrho') = 1 + 0 = 1 < 2k - 3 = 0 + 2 \cdot (k-2) + 1 = \ell(\pi) + \ell(\varrho). \qquad \blacksquare
$$

Remark 4.10. For $k = 2$, Proposition 4.9 follows directly from Proposition 6.1, since Model 4.3 is "nice" in the sense of Section 6. □

Proposition 4.11. *For $3 \leqslant n \leqslant 5$ and $0 < \lambda < 1$, the value $k = 2$ is optimal.*

For $n = 6, 7, 8, 9$, there exists $\lambda_n^{(2)} \in]0,1[$ such that for $\lambda < \lambda_n^{(2)}$ the value $k = 2$ and for $\lambda > \lambda_n^{(2)}$ the value $k = 3$ is optimal.

Proof. By Lemma 4.6, only the top leaves of the trees depicted in Figure 12 can be optimal. This already proves the cases $n = 3, 4$.

The criterion of Lemma 4.6 does not work to show that $k = 2$ is better than $k = 3$ for $n = 5$. However, a direct approach works:

$$\begin{aligned}
&h_{\mathrm{Len}}(5 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 1) - h_{\mathrm{Len}}(5 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 1)\\
=&\frac{1}{D_5(\lambda)^2} \sum_{L=0}^{20} \bigl(H(5,2,L) - H(5,3,L)\bigr)\lambda^L\\
=&\frac{1}{D_5(\lambda)^2}\bigl(\lambda^2 + 3\lambda^3 + 4\lambda^4 + 4\lambda^5 + 2\lambda^6 - 6\lambda^7 - 15\lambda^8\\
&\quad - 12\lambda^9 + 2\lambda^{10} + 11\lambda^{11} + 7\lambda^{12} - \lambda^{14}\bigr)\\
=&\frac{\lambda^2}{D_5(\lambda)^2}(-\lambda^5 + 2\lambda^4 + 3\lambda^3 + 2\lambda^2 + \lambda + 1)(\lambda^2 + \lambda + 1)(\lambda - 1)^2(\lambda + 1)^3\\
>&0.
\end{aligned}$$

For $n = 6, 7, 8$, we have to compare $k = 2$ and $k = 3$: for $n = 6$, we have

$$h_{\mathrm{Len}}(6 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 1) - h_{\mathrm{Len}}(6 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 1) = \underbrace{\frac{\lambda^2}{D_6(\lambda)^2}(1-\lambda)(\lambda^2 + \lambda + 1)^2(\lambda + 1)^4}_{>0} \cdot f_6(\lambda)$$

for

$$f_6(\lambda) := \lambda^{11} - 8\lambda^{10} - 14\lambda^9 - 13\lambda^8 - 12\lambda^7 - 4\lambda^6 - 4\lambda^5 + \lambda^3 + 2\lambda^2 + 1.$$

The signs of the coefficients of f_6 change twice. Descartes' sign rule (Proposition 4.5) now implies that f_6 has either two or no positive zeros. Since $f_6(0) = 1$ and $f_6(1) = -50$, one of these zeros is at some $\lambda_6^{(2)} \in]0, 1[$, and the other lies to the right of 1. Since $h_{\mathrm{Len}}(6 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 1) > h(6_{\mathrm{Len}} \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 1)$ for $\lambda < \lambda_6^{(2)}$, the value $k = 2$ is optimal for $\lambda < \lambda_6^{(2)}$ and $k = 3$ for $\lambda > \lambda_6^{(2)}$.

For $n = 7$, we have

$$\begin{aligned}
&h_{\mathrm{Len}}(7 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 1) - h_{\mathrm{Len}}(7 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 1)\\
=&\underbrace{\frac{\lambda^2}{D_7(\lambda)^2}(1-\lambda)(\lambda^2 + 1)(\lambda^2 + \lambda + 1)^2(\lambda + 1)^5}_{>0} \cdot f_7(\lambda)
\end{aligned}$$

for

$$\begin{aligned}
f_7(\lambda) :=&\lambda^{18} - 13\lambda^{17} - 44\lambda^{16} - 95\lambda^{15} - 139\lambda^{14} - 169\lambda^{13} - 166\lambda^{12} - 151\lambda^{11}\\
&- 120\lambda^{10} - 84\lambda^9 - 49\lambda^8 - 26\lambda^7 - 7\lambda^6 - \lambda^5 + 4\lambda^4 + 4\lambda^3 + 3\lambda^2 + \lambda + 1.
\end{aligned}$$

Descartes' sign rule once again yields that f_7 has two or no positive zeros. With $f_7(0) = 1$ and $f_7(1) = -1050$ we conclude as above.

For $n = 8$, we have

$$h_{\mathrm{Len}}(8 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 1) - h_{\mathrm{Len}}(8 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 1)$$

$$= \frac{\lambda^2}{D_8(\lambda)^2}(1-\lambda)(\lambda^4+\lambda^3+\lambda^2+\lambda+1)(\lambda^2+1)^2(\lambda^2+\lambda+1)^3(\lambda+1)^6 \cdot f_8(\lambda)$$

for

$$\begin{aligned} f_8(\lambda) :=& \lambda^{21} - 21\lambda^{20} - 51\lambda^{19} - 95\lambda^{18} - 144\lambda^{17} - 169\lambda^{16} - 181\lambda^{15} - 188\lambda^{14} \\ & - 180\lambda^{13} - 142\lambda^{12} - 115\lambda^{11} - 80\lambda^{10} - 54\lambda^9 - 32\lambda^8 - 19\lambda^7 - 3\lambda^6 \\ & - 3\lambda^5 + \lambda^4 + 2\lambda^3 + 2\lambda^2 + 1. \end{aligned}$$

We conclude as above.

Finally, for $n = 9$, we have to consider the cases $k = 2$, $k = 3$ and $k = 4$. As above, we see that $k = 3$ dominates $k = 4$ and that $k = 2$ is better than $k = 3$ if and only if $\lambda < \lambda_9^{(2)}$. ■

Remark 4.12. Maple numerically computes $\lambda_6^{(2)} \approx 0.64805$, $\lambda_7^{(2)} \approx 0.58616$, $\lambda_8^{(2)} \approx 0.56379$ and $\lambda_9^{(2)} \approx 0.55400$. □

Proposition 4.13. *For $n \geqslant 3$, the numbers of hitting pairs of permutations of total length L are as follows:*

	$L = 0$	$L = 1$	$L = 2$
$k = 2$	1	$2n - 3$	$2n^2 - 5n + 2$
$k \neq 2$	1	$2n - 3$	$2n^2 - 5n + 1$

Proof. Having outsourced all counting to Observation 4.14 and Lemma 4.15, we distinguish the different values of L:

$L = 0$: This is only the case for the single pair $(\pi, \varrho) = (\mathrm{id}, \mathrm{id})$. The correct index is chosen, regardless of the value of k.

$L = 1$: We examine two possible cases:

$\pi = \mathrm{id}$ and $\ell(\varrho) = 1$: $k = 1$: All $n - 1$ permutations ϱ yield a hit.
$k \geqslant 2$: A hit occurs exactly for $\varrho(n) = n$, for $n - 2$ permutations ϱ.

$\ell(\pi) = 1$ and $\varrho = \mathrm{id}$: $k = 1$: We have a hit if and only if $\pi(n) = n$, for $n - 2$ permutations π.
$k \geqslant 2$: All $n - 1$ permutations π yield a hit.

Adding up, we have $n - 1 + n - 2 = 2n - 3$ hits for all k.

$L = 2$: Three cases are possible for the pair (π, ϱ):

$\pi = \mathrm{id}$ and $\ell(\varrho) = 2$: $k = 1$: In this case, Alice and Bob always choose correctly, for $\frac{n(n-1)}{2} - 1$ permutations.
$k = 2$: In this case, we have a hit if and only if $\varrho(n-1) = n-2$ and $\varrho(n) = n-1$ or $\varrho(n) = n$, for $1 + \frac{(n-1)(n-2)}{2} - 1 = \frac{(n-1)(n-2)}{2}$ permutations.

$k > 2$: In this case, a hit occurs exactly for $\varrho(n) = n$, for $\frac{(n-1)(n-2)}{2} - 1$ permutations.

$\ell(\pi) = \ell(\varrho) = 1$: $k = 1$: We have a hit exactly for $\pi(n) = n$, for $(n-2)(n-1)$ pairs.

$k = 2$: A hit occurs exactly for $\pi(n-2) = n-1$ and $\pi(n-1) = n-2$ (yielding $\mathcal{S} = \{n-2, n\}$) or for $\pi(n-2) \neq n-1$ or $\pi(n-1) \neq n-2$ (with $\mathcal{S} = \{n-1, n\}$) and $\varrho(n) = n$, which adds up to $n - 1 + (n-2)^2$ pairs.

$k > 2$: In this case, we always have $n-1, n \in \mathcal{S}$, and a hit occurs if and only if $\varrho(n) = n$, for $(n-1)(n-2)$ pairs.

$\ell(\pi) = 2$ and $\varrho = \mathrm{id}$: $k = 1$: A hit occurs exactly for $\pi(n) = n$, for $\frac{(n-1)(n-2)}{2} - 1$ pairs of permutations.

$k = 2$: In this case, we do *not* have a hit if and only if $\pi(n) \neq n$ and $\pi(n) \neq n-1$, which, given that $\ell(\pi) = 2$, is equivalent to $\pi(n-1) = n-2$ and $\pi(n) = n-1$. This condition is satisfied by one permutation. Thus, $\frac{n(n-1)}{2} - 2$ permutations yield a hit.

$k > 2$: In this case, all $\frac{n(n-1)}{2} - 1$ pairs hit.

Summing up, the number of hits is

$$\begin{aligned}
\frac{n(n-1)}{2} - 1 + (n-2)(n-1) + \frac{(n-1)(n-2)}{2} - 1 &= 2n^2 - 5n + 1 \text{ for } k = 1, \\
\frac{(n-1)(n-2)}{2} + (n-1) + (n-2)^2 + \frac{n(n-1)}{2} - 2 &= 2n^2 - 5n + 2 \text{ for } k = 2, \\
\frac{(n-1)(n-2)}{2} - 1 + (n-1)(n-2) + \frac{n(n-1)}{2} - 1 &= 2n^2 - 5n + 1 \text{ for } k > 2. \quad \blacksquare
\end{aligned}$$

Observation 4.14 (Counting certain permutations of length 1). *Obviously,*

$$\begin{aligned}
\#\big\{\sigma \in S_n \,\big|\, \ell(\sigma) = 1\big\} &= n - 1, \\
\#\big\{\sigma \in S_n \,\big|\, \ell(\sigma) = 1, \sigma(n-2) = n-1, \sigma(n-1) = n-2\big\} &= 1, \\
\#\big\{\sigma \in S_n \,\big|\, \ell(\sigma) = 1, \sigma(n-2) \neq n-1 \text{ or } \sigma(n-1) \neq n-2\big\} &= n-2 \quad \text{and} \\
\#\big\{\sigma \in S_n \,\big|\, \ell(\sigma) = 1, \sigma(n) = n\big\} &= n-2. \qquad \square
\end{aligned}$$

Lemma 4.15 (Counting certain permutations of length 2). *We have*

$$\#\big\{\sigma \in S_n \,\big|\, \ell(\sigma) = 2\big\} = \frac{n(n-1)}{2} - 1, \tag{8}$$

$$\#\big\{\sigma \in S_n \,\big|\, \ell(\sigma) = 2, \sigma(n) = n\big\} = \frac{(n-1)(n-2)}{2} - 1 \quad \text{and} \tag{9}$$

$$\#\big\{\sigma \in S_n \,\big|\, \ell(\sigma) = 2, \sigma(n-1) = n-2, \sigma(n) = n-1\big\} = 1. \tag{10}$$

Proof. For Equation (8), let

$$a_n := \#\big\{\sigma \in S_n \,\big|\, \ell(\sigma) = 2\big\}.$$

We will show that $a_n = n - 1 + a_{n-1}$ and $a_3 = 2$.

Given $\sigma' \in S_{n-1}$ with $\ell(\sigma') \leqslant 2$, we proceed as follows to obtain $\sigma \in S_n$ with $\ell(\sigma) = 2$:

- If $\ell(\sigma') = 2$, we just write n at the end of σ'.
- If $\ell(\sigma') = 1$, we write n at the end of σ' and switch n with $\sigma'(n-1)$.
- Finally, if $\ell(\sigma') = 0$ (i. e. $\sigma' = \mathrm{id}$), we write n at the end of σ' and then first switch the last two entries and then the second-to-last two entries.

One easily sees (by considering the three possible cases $\sigma(n) = n$, $\sigma(n) = n-1$ and $\sigma(n) = n-2$) that every $\sigma \in S_n$ with $\ell(\sigma) = 2$ is the image of exactly one $\sigma' \in S_{n-1}$ under the above operation.

Using Observation 4.14, we obtain that

$$a_n = a_{n-1} + n - 2 + 1 = a_{n-1} + n - 1.$$

Obviously $a_3 = 2$, and the result follows.

Equation (9) immediately follows from Equation (8) once we note that

$$\#\big\{\sigma \in S_n \,\big|\, \ell(\sigma) = 2, \sigma(n) = n\big\} = \#\big\{\sigma \in S_{n-1} \,\big|\, \ell(\sigma) = 2\big\}.$$

Finally, for Equation (10), the only way to obtain $\sigma \in S_n$ with $\ell(\sigma) = 2$, $\sigma(n-1) = n-2$ and $\sigma(n) = n-1$ is to take the identity permutation and first switch the last two entries and then the second-to-last two entries. ■

Corollary 4.16. *For all $n \geqslant 3$ there exists a $\lambda_n^{(2)} > 0$ such that for all $\lambda < \lambda_n^{(2)}$, the value $k = 2$ yields the highest hitting ratio.*

Proof. By Proposition 4.13, the hitting ratio is (for $\lambda \to 0$)

$$\begin{aligned} h_{\mathrm{Len}}(n \xrightarrow{\lambda} k \xrightarrow{\lambda} 1) &= \frac{1}{D_n(\lambda)^2} \sum_{(\pi,\varrho)\ \mathrm{hit}} \lambda^{\ell(\pi)+\ell(\varrho)} \\ &= \begin{cases} \dfrac{1}{D_n(\lambda)^2}\big(1 + (2n-3)\lambda + (2n^2 - 5n + 2)\lambda^2 + O(\lambda^3)\big) & \text{if } k = 2 \\[2ex] \dfrac{1}{D_n(\lambda)^2}\big(1 + (2n-3)\lambda + (2n^2 - 5n + 1)\lambda^2 + O(\lambda^3)\big) & \text{if } k \neq 2. \end{cases} \end{aligned}$$

The difference in the coefficient of the term λ^2 yields the result. ■

Remark 4.17. In fact, Proposition 4.11 and Remark 4.12 yield the maximum possible values of $\lambda_n^{(2)}$ for $n \leqslant 9$:

$$\begin{aligned} &\lambda_3^{(2)} = \lambda_4^{(2)} = \lambda_5^{(2)} = 1, \\ &\lambda_6^{(2)} \approx 0.64805, \quad \lambda_7^{(2)} \approx 0.58616, \quad \lambda_8^{(2)} \approx 0.56379, \quad \lambda_9^{(2)} \approx 0.55400. \end{aligned}$$

□

Conjecture 4.18. *We make the following conjectures:*

- *The hitting ratio is unimodal in k for all n and λ.*
- *For all n, there exists a sequence*

$$0 = \lambda_n^{(1)} < \lambda_n^{(2)} < \lambda_n^{(3)} < \cdots < \lambda_n^{(\widetilde{k})} < \lambda_n^{(\widetilde{k}+1)} = 1$$

such that, if $\lambda \in]\lambda_n^{(k^*-1)}, \lambda_n^{(k^*)}[$, the value $k = k^*$ yields the highest hitting ratio. For all k, the values $\lambda_n^{(k)}$ decrease monotonically with increasing n.

In particular, the optimal k increases monotonically with n and λ. □

4.2 A Paradox for Different Experts

We consider the obvious generalisation of Model 4.3 that uses different precision parameters λ and μ for Alice and Bob.

It seems reasonable to expect that for given n, k and μ the hitting ratio decreases monotonically if Alice's parameter λ increases. However, this is not the case: we present values $n, k, \lambda_1, \lambda_2, \mu$ such that $\lambda_1 < \lambda_2$ with $h_{\text{Len}}(n \xrightarrow{\lambda_1} k \xrightarrow{\mu} 1) < h_{\text{Len}}(n \xrightarrow{\lambda_2} k \xrightarrow{\mu} 1)$.

Specifically, set $n = 3$, $k = 2$, $\lambda_1 = 0$, $\lambda_2 = \frac{1}{20}$ and $\mu = \frac{1}{10}$.

For $\lambda_1 = 0$, Alice is perfect and will always pass on $\mathcal{S} = \{2, 3\}$. Bob chooses with probability $\frac{10}{11}$ a permutation that prefers 3 to 2 (see Table 10). The hitting ratio is

$$h_{\text{Len}}\left(3 \xrightarrow{0} 2 \xrightarrow{\frac{1}{10}} 1\right) = \frac{10}{11}.$$

Permutation	Length	Weight	Probability
$(1,2,3)$	0	1	$\frac{1000}{1221}$
$(1,3,2)$	1	$\frac{1}{10}$	$\frac{100}{1221}$
$(2,1,3)$	1	$\frac{1}{10}$	$\frac{100}{1221}$
$(2,3,1)$	2	$\frac{1}{100}$	$\frac{10}{1221}$
$(3,1,2)$	2	$\frac{1}{100}$	$\frac{10}{1221}$
$(3,2,1)$	3	$\frac{1}{1000}$	$\frac{1}{1221}$

Table 10: All permutations for $n = 3$ and their probabilities for $\mu = \frac{1}{10}$

For $\lambda_2 = \frac{1}{20}$, Alice's permutation is distributed according to Table 11. Table 12 shows the probabilities of each shortlist and of Bob's choosing the correct index, given the shortlist. The probability of finally choosing the index n is thus

$$h_{\text{Len}}\left(3 \xrightarrow{\frac{1}{20}} 2 \xrightarrow{\frac{1}{10}} 1\right) = \frac{1}{421} \cdot \frac{0}{1221} + \frac{20}{421} \cdot \frac{1200}{1221} + \frac{400}{421} \cdot \frac{1110}{1221} = \frac{156000}{171347} > \frac{10}{11}.$$

Where does this paradox come from? In most cases in which Alice does not pass on

Permutation	Length	Weight	Probability
$(1,2,3)$	0	1	$\frac{8000}{8841}$
$(1,3,2)$	1	$\frac{1}{20}$	$\frac{400}{8841}$
$(2,1,3)$	1	$\frac{1}{20}$	$\frac{400}{8841}$
$(2,3,1)$	2	$\frac{1}{400}$	$\frac{20}{8841}$
$(3,1,2)$	2	$\frac{1}{400}$	$\frac{20}{8841}$
$(3,2,1)$	3	$\frac{1}{8000}$	$\frac{1}{8841}$

Table 11: All permutations for $n = 3$ and their probabilities for $\lambda = \frac{1}{20}$

Shortlist	Weight	Probability	conditional success probability
$\{1,2\}$	$\frac{21}{8000}$	$\frac{1}{421}$	$\frac{0}{1221}$
$\{1,3\}$	$\frac{21}{400}$	$\frac{20}{421}$	$\frac{1200}{1221}$
$\{2,3\}$	$\frac{21}{20}$	$\frac{400}{421}$	$\frac{1110}{1221}$

Table 12: Which shortlist is passed on, and what is the conditional probability of Bob's choosing the index n, given $\mathcal{S}$? Parameter values are $\lambda_2 = \frac{1}{20}$ and $\mu = \frac{1}{10}$.

the two best indices 2 and 3, she still passes on a shortlist that contains 3. In this case, it is easier for Bob to "separate" the two indices. On the other hand, the small number of cases in which Alice does not pass on the index 3 does not matter.

We remark that so far we have only seen this phenomenon if Alice is more precise than Bob, i.e. $\lambda \leqslant \mu$.

5 Two-Stage Drawing From Urns

In this model, we consider an urn with n balls.

Model 5.1. The i^{th} ball has a payoff of p_i. Whenever we draw an ball from the urn, the probability of drawing ball i is proportional to p_i.

Alice draws k balls from the urn and puts them in a second urn $\mathcal{S}$ which she then passes to Bob. Bob draws a single ball from $\mathcal{S}$. ◊

We investigated the following settings:

- $p_1 = \frac{p}{n-1+p}$ and $p_i = \frac{1}{n-1+p}$ for $p > 1$
- $p_i = p - (i-1) \cdot \delta$
- $p_1 = p$ and $p_i = \frac{p_{i-1}}{2}$
- $p_1 = p$ and $p_i = \sqrt{p_{i-1}}$ for $p > 1$
- $p_1 = q$ and $p_i = p_{i-1}^2$ for $q < 1$

All these payoff vectors were tested with $n \in \{20, 50\}$ balls, $p = 2$, $q = 0.9$ and $\delta = 0.02$ for $R = 10^5$ random experiments. Neither for the hitting ratio nor for the average result did a shortlisting valley appear. In fact, all curves were unimodal (data not shown). We conjecture that the hitting ratio and the average payoff are unimodal in k.

5.1 The Simplest Urn

In the remainder of this Section, we will consider a special case of our urn model. The "simplest urn model" is as follows:

Model 5.2 (Model SUrn). Let $p, q > 0$. The first ball has probability $\frac{p}{n-1+p}$, and the others have probability $\frac{1}{n-1+p}$, such that the probabilities add up to 1. Thus, the "good" ball is p times as likely to be chosen as any one of the "bad" ones. However, we add a touch of non-symmetry: for Bob, the first ball is q times as likely to be drawn as each one of the others. ◊

5.2 Unimodality

Proposition 5.3. *The hitting ratio is unimodal in k if $p \neq 1$.*

In particular, Model SUrn has no shortlisting valleys.

Proof. The probability for Alice to pass on the correct ball in $\mathcal{S}$ is the complement of the probability of her passing on only "bad" balls:

$$1 - \frac{n-1}{n-1+p} \cdot \frac{n-2}{n-2+p} \cdot \ldots \cdot \frac{n-k}{n-k+p}$$

The hitting ratio therefore is

$$h_{\mathrm{SUrn}}(n \xrightarrow{p} k \xrightarrow{q} 1) = \left(1 - \frac{n-1}{n-1+p} \cdot \ldots \cdot \frac{n-k}{n-k+p}\right) \cdot \frac{q}{k-1+q}. \tag{11}$$

First, for fixed n and $p > 1$, we determine the value q_k^* for which

$$h_{\mathrm{SUrn}}(n \xrightarrow{p} k \xrightarrow{q_k^*} 1) = h_{\mathrm{SUrn}}(n \xrightarrow{p} k+1 \xrightarrow{q_k^*} 1).$$

Define

$$F_k := \frac{n-1}{n-1+p} \cdot \ldots \cdot \frac{n-k}{n-k+p} \quad \text{and} \quad f_{k+1} := \frac{n-(k+1)}{n-(k+1)+p}.$$

Now,

$$\begin{aligned}
& & h_{\mathrm{SUrn}}(n \xrightarrow{p} k \xrightarrow{q_k^*} 1) &= h_{\mathrm{SUrn}}(n \xrightarrow{p} k+1 \xrightarrow{q_k^*} 1) \\
&\iff & (1-F_k)\frac{q_k^*}{k-1+q_k^*} &= (1-F_k f_{k+1})\frac{q_k^*}{k+q_k^*} \qquad (12) \\
&\iff & (1-F_k)(k+q_k^*) &= (1-F_k f_{k+1})(k-1+q_k^*) \\
&\iff & 1-F_k f_{k+1} &= (F_k - F_k f_{k+1})(k+q_k^*) \\
&\iff & q_k^* &= \frac{1-F_k f_{k+1}}{F_k - F_k f_{k+1}} - k.
\end{aligned}$$

For large q, the hitting ratio $h_{\mathrm{SUrn}}(n \xrightarrow{p} k \xrightarrow{q} 1)$ behaves like $1 - F_k$, and $h_{\mathrm{SUrn}}(n \xrightarrow{p} k+1 \xrightarrow{q} 1)$ like $1 - F_k f_{k+1}$. Since $f_{k+1} < 1$, we have $h_{\mathrm{SUrn}}(n \xrightarrow{p} k \xrightarrow{q} 1) < h_{\mathrm{SUrn}}(n \xrightarrow{p} k+1 \xrightarrow{q} 1)$ for $q > q_k^*$ and $h_{\mathrm{SUrn}}(n \xrightarrow{p} k \xrightarrow{q} 1) > h_{\mathrm{SUrn}}(n \xrightarrow{p} k+1 \xrightarrow{q} 1)$ for $q < q_k^*$.

Analogously, we obtain

$$q_{k+1}^* = \frac{1-F_k f_{k+1} f_{k+2}}{F_k f_{k+1} - F_k f_{k+1} f_{k+2}} - (k+1) = \frac{1-F_k f_{k+1}}{F_k f_{k+1} - F_k f_{k+1} f_{k+2}} - k$$

for $1 \leqslant k \leqslant n-2$.

Comparing q_k^* and q_{k+1}^*, we have

$$\begin{aligned}
& & q_k^* &< q_{k+1}^* \\
&\iff & \frac{1-F_k f_{k+1}}{F_k - F_k f_{k+1}} - k &< \frac{1-F_k f_{k+1}}{F_k f_{k+1} - F_k f_{k+1} f_{k+2}} - k \\
&\iff & \frac{1}{F_k - F_k f_{k+1}} &< \frac{1}{F_k f_{k+1} - F_k f_{k+1} f_{k+2}} \\
&\iff & 1 - f_{k+2} &< \frac{1}{f_{k+1}} - 1 \\
&\iff & 1 - \frac{n-(k+2)}{n-(k+2)+p} &< \frac{n-(k+1)+p}{n-(k+1)} - 1 \\
&\iff & \frac{p}{n-(k+2)+p} &< \frac{p}{n-(k+1)},
\end{aligned}$$

which is true for $p > 1$. Therefore we have

$$q_1^* < q_2^* < \cdots < q_{n-3}^* < q_{n-2}^*.$$

Now suppose that $q_{k^*-1}^* < q < q_{k^*}^*$. Then, for $k < k^*$, we have $q > q_k^*$ and therefore $h_{\mathrm{SUrn}}(n \xrightarrow{p} k+1 \xrightarrow{q} 1) > h_{\mathrm{SUrn}}(n \xrightarrow{p} k \xrightarrow{q} 1)$. On the other hand, $k \geqslant k^*$ implies that $q < q_k^*$ and $h_{\mathrm{SUrn}}(n \xrightarrow{p} k \xrightarrow{q} 1) > h_{\mathrm{SUrn}}(n \xrightarrow{p} k+1 \xrightarrow{q} 1)$. Thus, we have

$$\begin{aligned} h_{\mathrm{SUrn}}(n \xrightarrow{p} 1 \xrightarrow{q} 1) < h_{\mathrm{SUrn}}(n \xrightarrow{p} 2 \xrightarrow{q} 1) < \cdots < h_{\mathrm{SUrn}}(n \xrightarrow{p} k^* \xrightarrow{q} 1) \\ \text{and } h_{\mathrm{SUrn}}(n \xrightarrow{p} k^* \xrightarrow{q} 1) > h_{\mathrm{SUrn}}(n \xrightarrow{p} k^*+1 \xrightarrow{q} 1) > \cdots > h_{\mathrm{SUrn}}(n \xrightarrow{p} n \xrightarrow{q} 1) \end{aligned}$$

and $h_{\mathrm{SUrn}}(n \xrightarrow{p} k \xrightarrow{q} 1)$ is unimodal in k with a maximum at k^*.

If $p < 1$, an analogous argument shows that the hitting ratio has a single local minimum at k^* (it is "downward unimodal"). To maximize the hitting ratio, the only possible values of k are 1 and n. If $q > p$, we choose $k = n$, otherwise $k = 1$.

If $p = 1$, we have $q_1^* = \cdots = q_n^* = 1$. If now $q > 1$, then $h_{\mathrm{SUrn}}(n \xrightarrow{1} k \xrightarrow{q} 1)$ increases monotonically in k, for $q < 1$ it decreases monotonically, and $h_{\mathrm{SUrn}}(n \xrightarrow{1} k \xrightarrow{1} 1)$ is constant. ■

5.3 The Case $n = 3$

If $n = 3$, we have the following hitting ratios:

$$h_{\mathrm{SUrn}}(3 \xrightarrow{p} 1 \xrightarrow{q} 1) = h_{\mathrm{SUrn}}(3 \xrightarrow{q} 3 \xrightarrow{p} 1) = \frac{p}{2+p}$$

$$h_{\mathrm{SUrn}}(3 \xrightarrow{p} 3 \xrightarrow{q} 1) = h_{\mathrm{SUrn}}(3 \xrightarrow{q} 1 \xrightarrow{p} 1) = \frac{q}{2+q}$$

$$h_{\mathrm{SUrn}}(3 \xrightarrow{p} 2 \xrightarrow{q} 1) = \frac{3+p}{2+p}\frac{pq}{(1+p)(1+q)} \quad (13)$$

$$h_{\mathrm{SUrn}}(3 \xrightarrow{q} 2 \xrightarrow{p} 1) = \frac{3+q}{2+q}\frac{pq}{(1+p)(1+q)} \quad (14)$$

The Order of Alice and Bob is Fixed

If the order of Alice and Bob is fixed (first p, then q), then we can choose between the first three equations above.

We have

$$p > q \iff h_{\mathrm{SUrn}}(3 \xrightarrow{p} 1 \xrightarrow{q} 1) > h_{\mathrm{SUrn}}(3 \xrightarrow{p} 3 \xrightarrow{q} 1).$$

We now have to consider the case $k = 2$. If $p < q$, then we have to compare $h_{\mathrm{SUrn}}(3 \xrightarrow{p} 2 \xrightarrow{q} 1)$ with $h_{\mathrm{SUrn}}(3 \xrightarrow{p} 3 \xrightarrow{q} 1)$. Some calculations yield that

$$h_{\mathrm{SUrn}}(3 \xrightarrow{p} 3 \xrightarrow{q} 1) < h_{\mathrm{SUrn}}(3 \xrightarrow{p} 2 \xrightarrow{q} 1) \iff q < \frac{p^2+3p-2}{2}.$$

If $p \geqslant q$, then we have to compare $h_{\text{SUrn}}(3 \xrightarrow{p} 2 \xrightarrow{q} 1)$ with $h_{\text{SUrn}}(3 \xrightarrow{p} 1 \xrightarrow{q} 1)$. In this case, some calculations yield that

$$h_{\text{SUrn}}(3 \xrightarrow{p} 1 \xrightarrow{q} 1) < h_{\text{SUrn}}(3 \xrightarrow{p} 2 \xrightarrow{q} 1) \iff q > \frac{p+1}{2}.$$

For $p \geqslant 1$, the result is as follows:

- If $q \leqslant \frac{p+1}{2}$, then Alice should decide alone.
- If $\frac{p+1}{2} < q < \frac{p^2+3p-2}{2}$, then Alice should shortlist to $k = 2$ candidates.
- If $q \geqslant \frac{p^2+3p-2}{2}$, then Bob should decide alone.

We show an overview of the possibilities in Figure 13.

Alice and Bob Can Switch Places

If we allow the possibility of switching the order of Alice and Bob, Equation (14) enters the picture. Examining Equations (13) and (14) shows that the better expert (with the higher parameter) should be in the second position if $k = 2$:

$$h_{\text{SUrn}}(3 \xrightarrow{q} 2 \xrightarrow{p} 1) > h_{\text{SUrn}}(3 \xrightarrow{p} 2 \xrightarrow{q} 1) \iff p > q.$$

If $k = 1$ or $k = 3$, the order does not matter, since the better expert should do all the deciding anyway. Thus, the best option is to put the better expert in the second position.

Now we can argue by symmetry (or calculate explicitly) that if $p > q$,

$$h_{\text{SUrn}}(3 \xrightarrow{q} 2 \xrightarrow{p} 1) > h_{\text{SUrn}}(3 \xrightarrow{p} 1 \xrightarrow{q} 1) = h_{\text{SUrn}}(3 \xrightarrow{q} 3 \xrightarrow{p} 1) \iff p < \frac{q^2+3q-2}{2}.$$

The situation in this case is shown in Figure 14.

5.4 Asymptotic Behavior of k^* for Large n

For general p, the hitting ratio as in Equation (11) can be described by means of the Gamma function. An exhaustive analysis of the asymptotic behavior of the optimal value k^* by analytical methods becomes rather involved, but for certain values of p, many terms in (11) cancel out.

Since $k = 1$ or $k = n$ is optimal for $q < 1$, we restrict our attention to the case $q > 1$.

Proposition 5.4. *For $q > 1$ and $p = 2$, we have $k^* \sim \sqrt{2(q-1)n}$.*

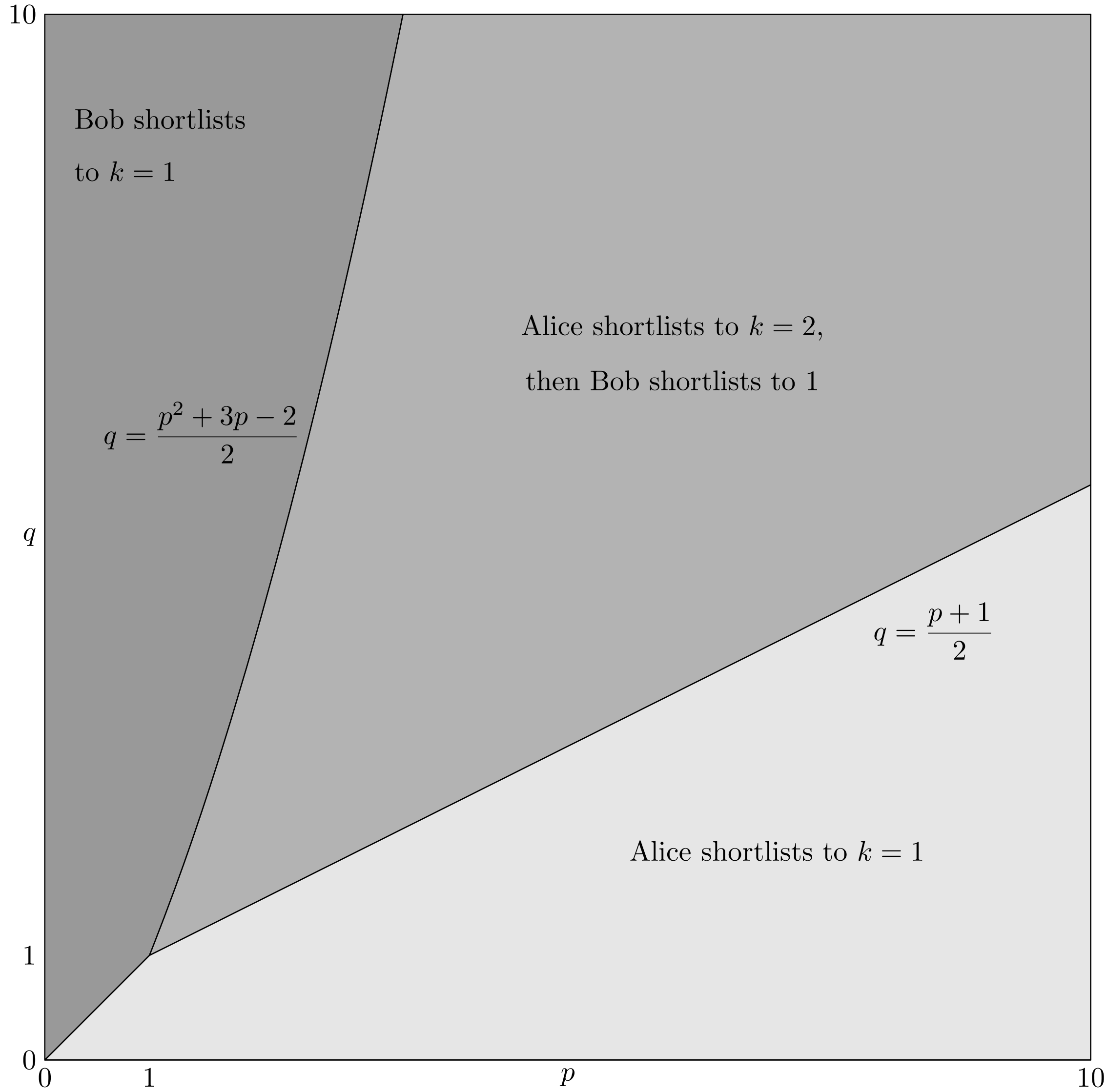

Figure 13: Optimal choices in the simplest urn model if the order of Alice and Bob is fixed

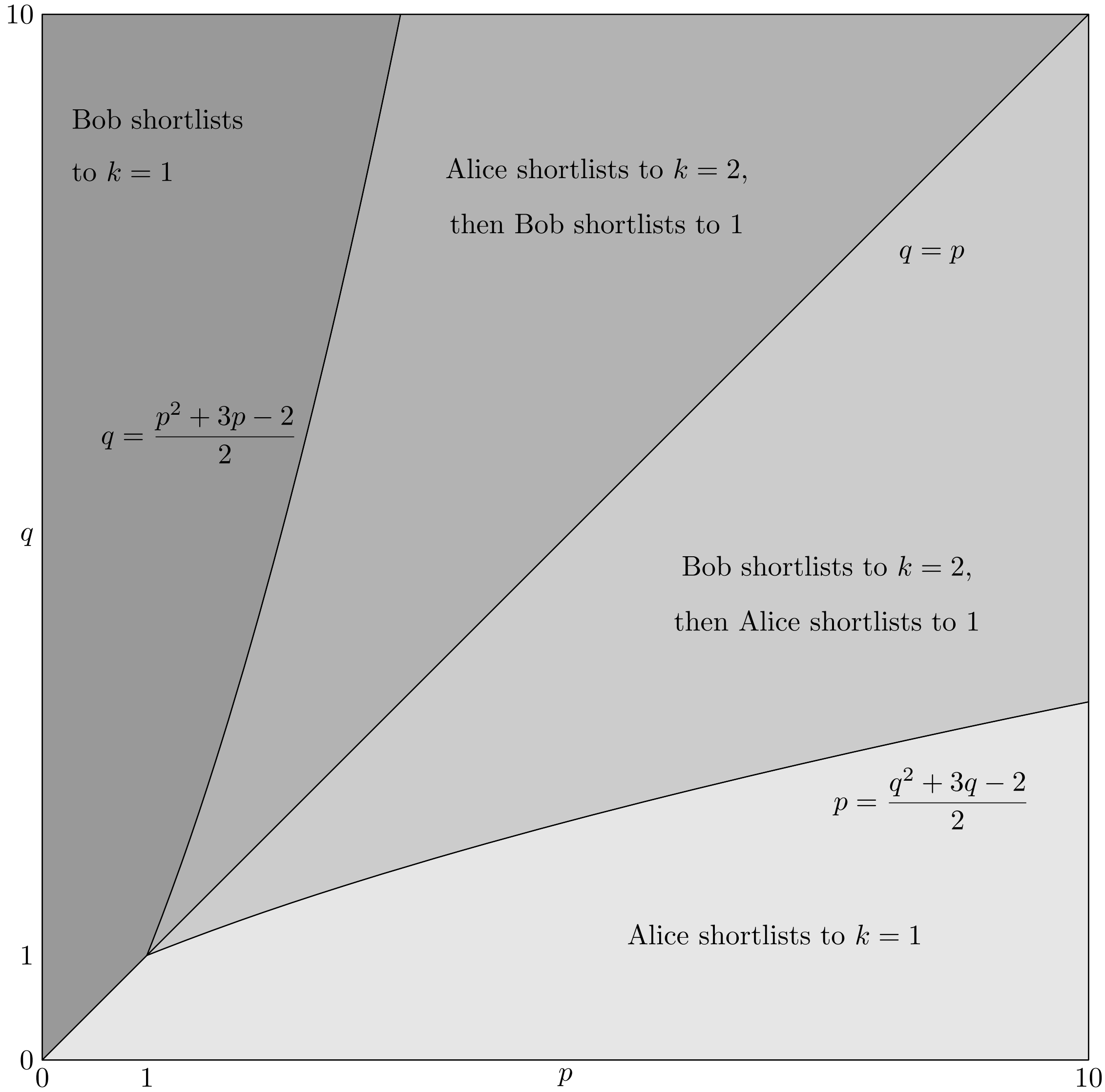

Figure 14: Optimal choices in the simplest urn model if the order of Alice and Bob is not fixed

Proof. We have

$$h_{\text{SUrn}}(n \xrightarrow{2} k \xrightarrow{q} 1) = \left(1 - \frac{(n+1-k)(n-k)}{(n+1)n}\right)\frac{q}{k-1+q}.$$

Some calculation yields that

$$\frac{\partial^2}{\partial k^2} h_{\text{SUrn}}(n \xrightarrow{2} k \xrightarrow{q} 1) = \underbrace{\frac{2q}{n(n+1)(k-1+q)^3}}_{>0} \cdot \underbrace{(q - q^2 + 2n - 2nq)}_{<0} < 0.$$

Therefore $h_{\text{SUrn}}(n \xrightarrow{2} \cdot \xrightarrow{q} 1)$ is concave, and unimodality, which we proved only for $k \in \mathbb{N}$ in Proposition 5.3, even holds for $k \in \mathbb{R}$.

We now look for the value $\overline{k} \in \mathbb{R}$ for which

$$h_{\text{SUrn}}(n \xrightarrow{2} \overline{k} \xrightarrow{q} 1) = h_{\text{SUrn}}(n \xrightarrow{2} \overline{k}+1 \xrightarrow{q} 1).$$

Then $k^* \in [\overline{k}, \overline{k}+1]$ by unimodality. Note that if $\overline{k} \in \mathbb{N}$, then k^* is not uniquely determined.

Some more calculation yields that

$$\overline{k} = -\frac{2q-1}{2} + \sqrt{\frac{(2q-1)^2}{4} + 2n(q-1)}.$$

Since by assumption $q > 1$, we have $k^* \sim \sqrt{2(q-1)n}$. ∎

Proposition 5.5. *For $q > 1$ and $p = 3$, we have $k^* \sim \sqrt{(q-1)n}$.*

Proof. In this case, the hitting ratio is

$$h_{\text{SUrn}}(n \xrightarrow{3} k \xrightarrow{q} 1) = \left(1 - \frac{(n+2-k)(n+1-k)(n-k)}{(n+2)(n+1)n}\right)\frac{q}{k-1+q}.$$

Some calculation yields that

$$\begin{aligned} & h_{\text{SUrn}}(n \xrightarrow{3} \overline{k} \xrightarrow{q} 1) = h_{\text{SUrn}}(n \xrightarrow{3} \overline{k}+1 \xrightarrow{q} 1) \\ \iff\quad & \underbrace{2\overline{k}^3 - 3(n-q+1)\overline{k}^2 - \big((6n+3)(q-1)+3n+2\big)\overline{k} + 3(n^2+n)(q-1)}_{=:f_3(\overline{k})} \stackrel{!}{=} 0. \end{aligned}$$

Descartes' sign rule (Proposition 4.5) once again yields that f_3 has either 2 or no positive zeros. Since $f_3(0) > 0$ and $f_3(n) = -n^3 - 3n^2 - 2n < 0$, the polynomial has a single root $\overline{k} \in]0, n[$.

We show that $\overline{k} \in o(n)$: if $\overline{k} = cn + o(n)$, we would have

$$2c^3n^3 \sim 3(n-q+1)c^2n^2 \quad\Longrightarrow\quad \overline{k} \sim cn \sim \frac{3}{2}(n-q+1),$$

thus $\overline{k} > n$, which is impossible.

Since $\overline{k} \in o(n)$, the second term in f_k dominates the first one and the fourth the third, and asymptotically we have

$$\overline{k} \sim \sqrt{\frac{n(n+1)(q-1)}{n-q+1}},$$

thus $k^* \sim \sqrt{(q-1)n}$. ∎

5.5 More than Two Experts with an Urn

Proposition 5.6. *Consider n balls in an urn, one of them with weight 2, the others with weight 1. A queue of r experts reduce the number of balls successively from $k_0 = n$ to $k_1 \leqslant k_0$, $k_2 \leqslant k_1$, ..., $k_r = 1 \leqslant k_{r-1}$ balls.*

Asymptotically for $n \to \infty$, the hitting ratio is optimal for $k_i \sim n^{\frac{r-i}{r}}$.

Proof. Lemma 5.7 implies that asymptotically, $k_i \sim \sqrt{k_{i-1}k_{i+1}}$ for $1 \leqslant i \leqslant r-1$.

It is sufficient to prove the following statement: Let $x_0 = 1$ and $x_r = n$. If $x_1, \ldots, x_{r-1} \in [1, n]$ with $x_i = \sqrt{x_{i-1}x_{i+1}}$ for $1 \leqslant i \leqslant r-1$, then $x_i = n^{\frac{i}{r}}$ (note that we re-ordered the indices for ease of exposition – before, we had $k_r = 1$ and $k_0 = n$). We proceed by induction:

$r = 2$: From $x_0 = 1$ and $x_2 = n$ with $x_1 = \sqrt{x_0 x_2}$, we immediately deduce that $x_1 = n^{\frac{1}{2}}$.

$r \rightsquigarrow r+1$: Let $x_0 = 1$ and $x_{r+1} = n$. We treat x_r as a free variable. Once x_r is fixed, the induction hypothesis implies that $x_i = x_r^{\frac{i}{r}}$ for $1 \leqslant i \leqslant r-1$, so it remains to show that $x_r = n^{\frac{r}{r+1}}$. However, this immediately follows from

$$x_r = \sqrt{x_{r-1}x_{r+1}} = \sqrt{x_r^{\frac{r-1}{r}}\, n}.$$ ∎

Lemma 5.7. *Consider the setting in Proposition 5.6.*

Then a necessary condition for the hitting ratio to be optimal is that for $1 \leqslant i \leqslant r-1$,

$$k_i \begin{cases} = \left\lceil \sqrt{(k_{i-1}+1)(k_{i+1}+1)+\frac{1}{4}} - \frac{1}{2} \right\rceil, & \textit{if } \sqrt{(k_{i-1}+1)(k_{i+1}+1)+\frac{1}{4}} - \frac{1}{2} \notin \mathbb{N} \\ \in \left\{ \sqrt{(k_{i-1}+1)(k_{i+1}+1)+\frac{1}{4}} - \frac{1}{2}, \sqrt{k_{i-1}+1)((k_{i+1}+1)+\frac{1}{4}} + \frac{1}{2} \right\} & \textit{otherwise.} \end{cases}$$

Once again, we note that the k_i are not necessarily uniquely determined.

Proof. We generalize the notion of "hitting ratio" as follows: if we shortlist from n with an intermediate shortlist of size k to a final list of size ℓ, then we define $h_{\text{Mod}}(n \dot{\to} k \dot{\to} \ell)$ to be the probability that the "correct" ball (here the one with weight 2) is in the final list.

We have

$$\begin{aligned}
&h_{\text{SUrn}}(k_{i-1} \xrightarrow{2} k_i \xrightarrow{2} k_{i+1}) \\
=&\left(1 - \frac{(k_{i-1} - k_i + 1)(k_{i-1} - k_i)}{(k_{i-1} + 1)k_{i-1}}\right) \cdot \left(1 - \frac{(k_i - k_{i+1} + 1)(k_i - k_{i+1})}{(k_i + 1)k_i}\right) \\
=&\frac{k_{i+1}}{k_{i-1}(k_{i-1} + 1)} \cdot \frac{(2k_{i-1} + 1 - k_i)(2k_i + 1 - k_{i+1})}{k_i + 1}.
\end{aligned}$$

Now suppose that k_{i-1} and k_{i+1} are fixed. As in Proposition 5.3, $h_{\text{SUrn}}(k_{i-1} \xrightarrow{2} \cdot \xrightarrow{2} k_{i+1})$ is unimodal. We compute the value $k_i^* \in \mathbb{R}$ for which

$$h_{\text{SUrn}}(k_{i-1} \xrightarrow{2} k_i^* \xrightarrow{2} k_{i+1}) = h_{\text{SUrn}}(k_{i-1} \xrightarrow{2} k_i^* + 1 \xrightarrow{2} k_{i+1})$$

and obtain

$$k_i^* = \sqrt{k_{i-1} + 1)(k_{i+1} + 1) + \frac{1}{4}} - \frac{1}{2}.$$

The result follows immediately from the unimodality. ■

6 How Do Valleys Appear?

All the shortlisting valleys we observed until now already started between $k = 1$ and $k = 2$. We will therefore investigate these two cases in more detail.

To be able to speak of a "valley" instead of a "drop", we assume identical experts. This ensures that $k = 1$ and $k = n$ yield the same hitting ratio.

The hitting ratios are as follows:

$$\begin{aligned}
h_1 &= \sum_{j=1}^{n-1} P_{\text{Alice}}\big(\mathcal{S} = \{j, n\}\big) \cdot P\big(n \succ_{\text{Alice}} j \,|\, \mathcal{S} = \{j, n\}\big) \\
h_2 &= \sum_{j=1}^{n-1} P_{\text{Alice}}\big(\mathcal{S} = \{j, n\}\big) \cdot P\big(n \succ_{\text{Bob}} j \,|\, \mathcal{S} = \{j, n\}\big).
\end{aligned}$$

Here, we denote by $P_{\text{Alice}}\big(\mathcal{S} = \{j, n\}\big)$ the probability that Alice constructs the shortlist $\mathcal{S} = \{j, n\}$ and by $P\big(n \succ_{\text{Alice}} j \,|\, \mathcal{S} = \{j, n\}\big)$ the probability that given the shortlist $\mathcal{S} = \{j, n\}$, Alice prefers n to j.

If we assume independent experts, we have

$$P\big(n \succ_{\text{Bob}} j \,|\, \mathcal{S} = \{j, n\}\big) = P(n \succ_{\text{Bob}} j),$$

and since Alice and Bob are identical,

$$P\big(n \succ_{\text{Bob}} j \,|\, \mathcal{S} = \{j, n\}\big) = P(n \succ_{\text{Bob}} j) = P(n \succ_{\text{Alice}} j).$$

The hitting ratios become

$$h_1 = \sum_{j=1}^{n-1} P_{\text{Alice}}\big(\mathcal{S} = \{j, n\}\big) \cdot P\big(n \succ_{\text{Alice}} j \,|\, \mathcal{S} = \{j, n\}\big)$$
$$h_2 = \sum_{j=1}^{n-1} P_{\text{Alice}}\big(\mathcal{S} = \{j, n\}\big) \cdot P(n \succ_{\text{Alice}} j).$$

In a "nice" model we should always have

$$P(n \succ_{\text{Alice}} j) \geqslant P\big(n \succ_{\text{Alice}} j \,|\, \mathcal{S} = \{j, n\}\big),$$

since the additional information that Alice likes n and j best of all alternatives should not help her decide which one of them she prefers.

We have shown the following Proposition:

Proposition 6.1. *If these three conditions are met – independence, identical distributions and niceness – then $h_{Mod}(n \dot{\rightarrow} 2 \dot{\rightarrow} 1) \geqslant h_{Mod}(n \dot{\rightarrow} 1 \dot{\rightarrow} 1)$.* ■

6.1 A Non-Nice Model

Consider the following model:

Model 6.2. Let $n \geqslant 4$ be even. Alice and Bob each believe that half of the alternatives are much better than the other half, independently of each other and of the true ranking.

Within each of the groups, both Alice and Bob sort by the true rankings. ◊

Now we conduct the usual shortlisting process with $k \leqslant m := \frac{n}{2}$.

Alice passes the best alternative on if and only if it is in her "better" group.

In which cases does Bob choose the best alternative from the shortlist of size k? He does this if and only if he considers the best alternative "good" or the shortlist only consists of "bad" alternatives.

Thus the probability that the best alternative is selected is

$$\frac{1}{2} \cdot \left(\frac{1}{2} + \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \ldots \cdot \frac{m-k+1}{n-k+1} \right).$$

For $k > m$ Alice will always pass on the best alternative, and Bob will select it if and only if he considers it "good", i. e. with probability $\frac{1}{2}$.

The hitting ratio therefore decreases from $\frac{1}{2}$ for $k = 1$ to slightly more than $\frac{1}{4}$ for $k = m$ and then jumps to $\frac{1}{2}$.

Where does this valley come from? Alice and Bob are identical and independently distributed. However, we have

$$P(n \succ_{\text{Alice}} j) = \frac{3}{4},$$

since Alice only prefers j to n if n is "bad" and j "good". On the other hand,

$$P\big(n \succ_{\text{Alice}} j \,|\, \mathcal{S} = \{j, n\}\big) = 1,$$

since Alice will only create a shortlist of size $k = 2$ including n and j if both are "good" – and in this case, she will always prefer n to j.

Model 6.2 is therefore not nice.

6.2 Model 1.1 Lacks Independence

In our basic Model 1.1 with $\Delta = \Gamma$, our experts are identical. However, they are not independent, since they are linked by the true values x_i. If the imprecision parameter Γ increases, this link becomes weaker.

7 Three Experts

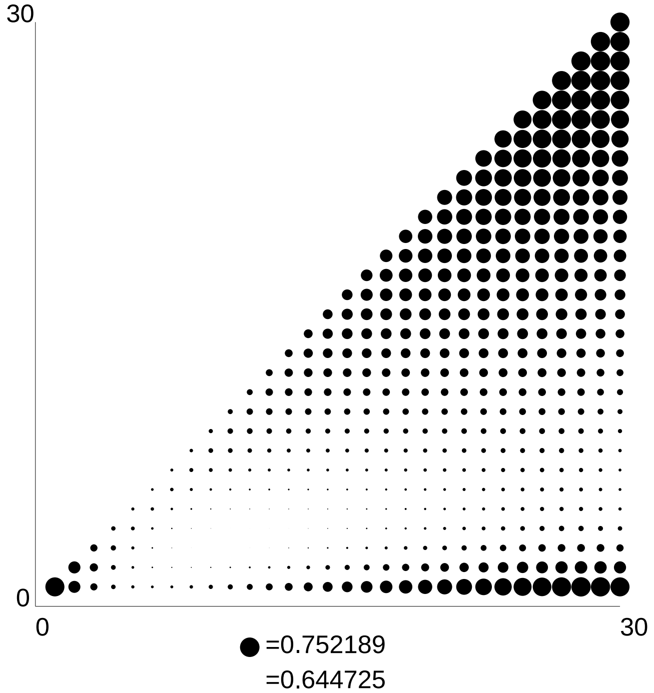

Figure 15: Empirical average result: $n = 30$ random variables, $R = 10^6$ Monte Carlo runs, $\Gamma = \Delta = \mathrm{E} = 5$

The basic model is easily extended to a *three* stage shortlisting process. The results are shown in Figures 15, 16, 17, 18, 19 and 20. In these Figures, we show the number k_1 to which the first expert shortlists on the x axis and the number k_2 to which the second expert shortlists on the y axis. The third expert, of course, shortlists to one index. Since $k_2 \leqslant k_1$ (the second expert cannot pass on more indices than he receives), the

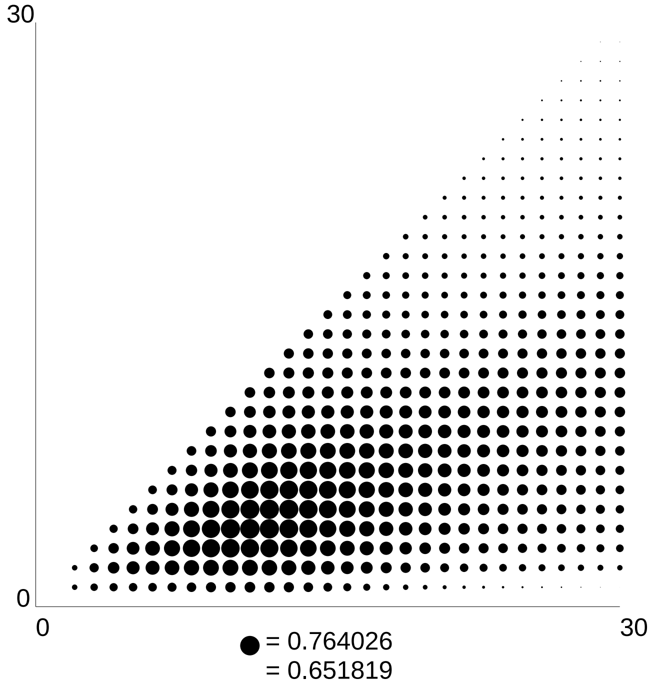

Figure 16: Empirical average result: $n = 30$, $R = 10^6$ Monte Carlo runs, $\gamma_i, \delta_i, \varepsilon_i \sim \mathcal{N}(0, 1)$

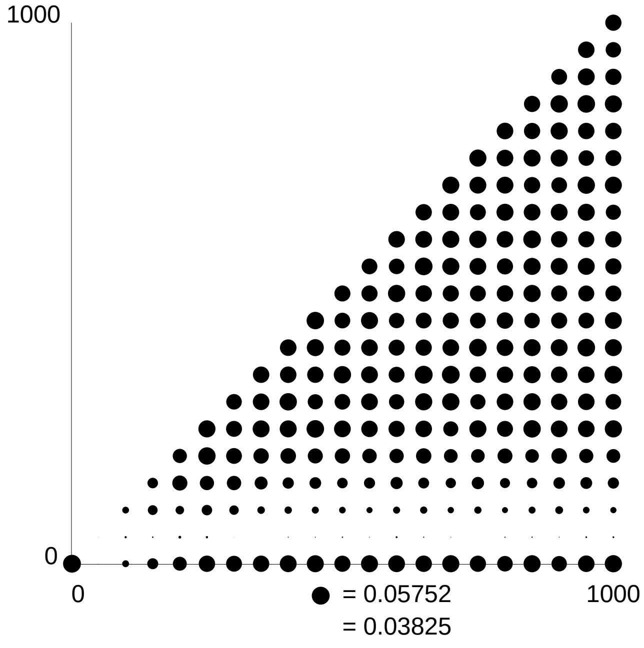

Figure 17: Empirical hitting ratio: n=1000, $R = 10^5$ Monte Carlo runs, $\Gamma = \Delta = \mathrm{E} = 0.5$. Shown are the results for $k_1, k_2 \in \{1, 50, 100, 150, \ldots, 1000\}$.

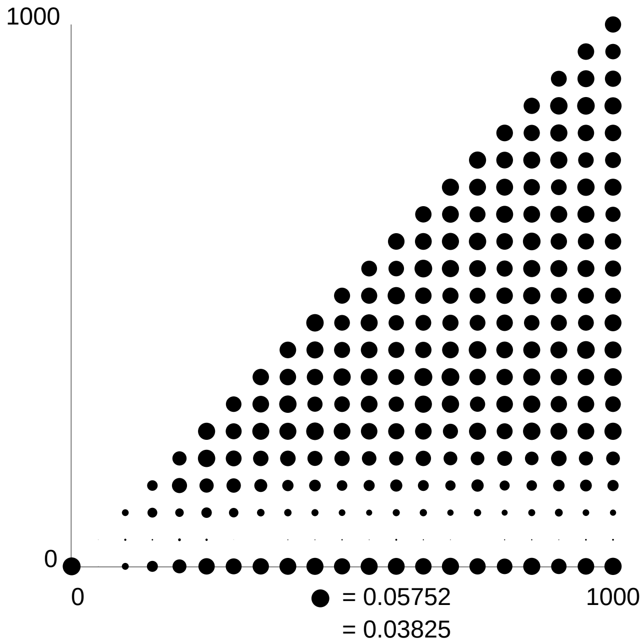

Figure 18: Empirical average result: n=1000, $R = 10^5$ Monte Carlo runs, $\Gamma = \Delta = \mathrm{E} = 0.5$. Shown are the results for $k_1, k_2 \in \{1, 50, 100, 150, \ldots, 1000\}$.

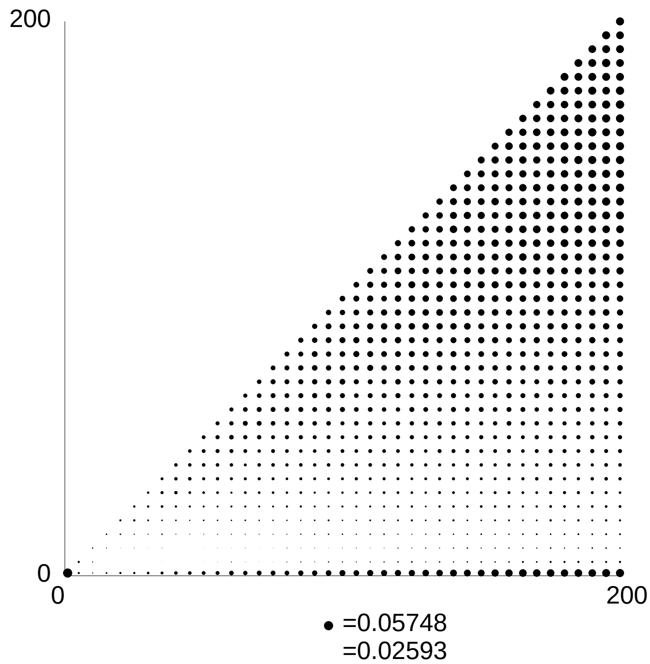

Figure 19: Hitting ratio: n=1000, $R = 10^5$ Monte Carlo runs, $\Gamma = \Delta = \mathrm{E} = 0.5$. Shown are the results for $k_1, k_2 \in \{1, 5, 10, 15, \ldots, 200\}$.

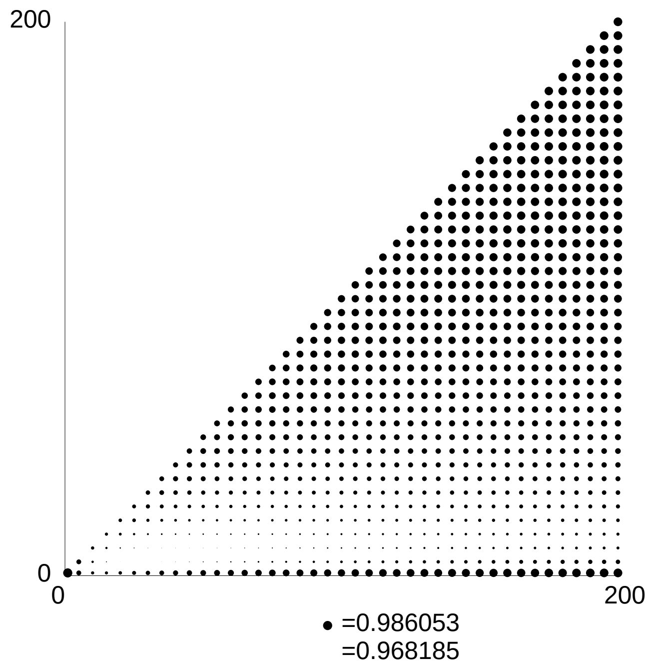

Figure 20: Average result: n=1000, $R = 10^5$ Monte Carlo runs, $\Gamma = \Delta = \mathrm{E} = 0.5$. Shown are the results for $k_1, k_2 \in \{1, 5, 10, 15, \ldots, 200\}$.

data only fill the triangle on the bottom right. The size of the disks in the diagrams shows the hitting ratio or the average result. More specifically, the maximum entry is visualized by the largest disk and the minimum entry by a "zero disk." Intermediate values are linearly mapped to circle radii (*not* to area!).

The three sides of the triangle are of interest. On the right, we have $k_1 = n$, along the diagonal $k_2 = k_1$ and on the bottom $k_2 = 1$. Thus, the three sides correspond, respectively, to the first, the second and the third expert doing nothing at all – the original Model 1.1 with two experts! Since the three experts have the same precision, we expect that the sequences of disks from the bottom right to the top right, from the bottom left to the top right and from the bottom left to the bottom right are identical and correspond to our data in Figures 1 and 2. In fact, this is the case.

8 Conclusion

Our models show that an intermediate shortlisting step can significantly improve performance when we choose one out of many alternatives. However, even for our idealized settings, it is usually infeasible to compute the optimal shortlist size k^*. In some models, we even find our counterintuitive shortlisting valleys.

How could research continue? On the one hand, there are many intuitively appealing models we could still check, e. g. mixed models in which some variables are continuous and others discrete. On the other hand, the theoretical analysis of our models is still incomplete and may offer a fertile ground for further results. And finally, it may be worthwhile to consider less idealized models which are more in touch with real-world applications.

Much remains to be done, yet the results of this paper already show that the field of shortlisting by imprecise agents offers rich opportunities for the researcher.

Acknowledgements

We wish to thank Ingo Althöfer for many helpful discussions.

A More Data for Length-Weighted Permutations

In this section, we give the complete data for Model 4.3 in Section 4 for small n.

L	0	1	2	3	4	5	6	7	8
$k=1$	1	9	43	145	386	860	1660	2838	4362
$k=2$	1	9	44	152	414	943	1858	3235	5048
$k=3$	1	9	43	147	401	919	1825	3208	5060
$k=4$	1	9	43	145	389	883	1750	3078	4857
$k=5$	1	9	43	145	386	864	1687	2934	4600
total	1	10	53	198	584	1444	3103	5932	10251

L	9	10	11	12	13	14	15	16	17
$k=1$	6090	7779	9135	9892	9892	9135	7779	6090	4362
$k=2$	7122	9127	10640	11270	10802	9300	7111	4751	2709
$k=3$	7215	9333	10963	11668	11167	9476	6990	4377	2266
$k=4$	6937	9016	10668	11442	11038	9485	7181	4734	2686
$k=5$	6544	8475	10004	10761	10534	9360	7519	5430	3499
total	16196	23589	31864	40096	47150	51923	53612	51923	47150

L	18	19	20	21	22	23	24	25	26
$k=1$	2838	1660	860	386	145	43	9	1	0
$k=2$	1271	462	116	15	0	0	0	0	0
$k=3$	942	304	73	12	1	0	0	0	0
$k=4$	1297	527	177	47	9	1	0	0	0
$k=5$	1992	989	421	150	43	9	1	0	0
total	40096	31864	23589	16196	10251	5932	3103	1444	584

L	27	28	29	30
$k=1$	0	0	0	0
$k=2$	0	0	0	0
$k=3$	0	0	0	0
$k=4$	0	0	0	0
$k=5$	0	0	0	0
total	198	53	10	1

Table 13: Number $H(n, k, L)$ of successful pairs of permutations with given total length L for $n = 6$

L	0	1	2	3	4	5	6	7
$k=1$	1	11	64	262	846	2290	5393	11324
$k=2$	1	11	65	271	891	2454	5875	12528
$k=3$	1	11	64	264	865	2385	5731	12288
$k=4$	1	11	64	262	849	2319	5541	11851
$k=5$	1	11	64	262	846	2294	5432	11521
$k=6$	1	11	64	262	846	2290	5398	11368
total	1	12	76	338	1184	3474	8867	20190
L	8	9	10	11	12	13	14	15
$k=1$	21565	37708	61099	92379	131031	175078	221069	264430
$k=2$	24204	42887	70317	107383	153472	205970	260143	309579
$k=3$	23900	42664	70494	108489	156226	211166	268408	321031
$k=4$	23036	41146	68102	105102	151895	206129	263087	316047
$k=5$	22254	39581	65316	100567	145101	196800	251397	302690
$k=6$	21770	38380	62831	96095	137888	186174	236990	284774
total	41744	79388	140225	231758	360499	530184	739929	982794
L	16	17	18	19	20	21	22	23
$k=1$	300157	323718	331950	323718	300157	264430	221069	175078
$k=2$	347231	366891	364723	340383	297317	242060	182693	126924
$k=3$	361148	381553	377572	348569	298573	235639	169897	110803
$k=4$	357272	379474	377557	350317	301465	239189	173970	115398
$k=5$	343389	366588	367564	345310	303132	247935	188371	132532
$k=6$	323242	346664	351206	335920	303044	257492	205686	154100
total	1245243	1507862	1747433	1940120	2065199	2108560	2065199	1940120
L	24	25	26	27	28	29	30	31
$k=1$	131031	92379	61099	37708	21565	11324	5393	2290
$k=2$	80416	45887	23179	10111	3668	1040	205	21
$k=3$	64630	33292	14934	5742	1856	490	100	14
$k=4$	69504	37864	18577	8155	3167	1067	302	68
$k=5$	86076	51431	28159	14051	6337	2549	896	267
$k=6$	107963	70475	42670	23828	12184	5650	2344	852
total	1747433	1507862	1245243	982794	739929	530184	360499	231758
L	32	33	34	35	36	37	38	39
$k=1$	846	262	64	11	1	0	0	0
$k=2$	0	0	0	0	0	0	0	0
$k=3$	1	0	0	0	0	0	0	0
$k=4$	11	1	0	0	0	0	0	0
$k=5$	64	11	1	0	0	0	0	0
$k=6$	262	64	11	1	0	0	0	0
total	140225	79388	41744	20190	8867	3474	1184	338

Table 14: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=7$

L	40	41	42
$k=1$	0	0	0
$k=2$	0	0	0
$k=3$	0	0	0
$k=4$	0	0	0
$k=5$	0	0	0
$k=6$	0	0	0
total	76	12	1

Table 14: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=7$

L	0	1	2	3	4	5	6
$k=1$	1	13	89	427	1611	5085	13952
$k=2$	1	13	90	438	1677	5370	14938
$k=3$	1	13	89	429	1634	5224	14545
$k=4$	1	13	89	427	1614	5120	14167
$k=5$	1	13	89	427	1611	5089	13999
$k=6$	1	13	89	427	1611	5085	13957
$k=7$	1	13	89	427	1611	5085	13952
total	1	14	103	530	2141	7226	21178
L	7	8	9	10	11	12	13
$k=1$	34142	75885	155261	295410	526830	886145	1412855
$k=2$	37034	83331	172484	331780	597762	1014997	1632247
$k=3$	36155	81670	169853	328470	595185	1016609	1644652
$k=4$	35067	79017	164157	317461	575777	985128	1597366
$k=5$	34433	77144	159523	307419	556130	949759	1538159
$k=6$	34201	76247	156803	300536	540988	919923	1484331
$k=7$	34148	75950	155633	296906	531568	898701	1441739
total	55320	131204	286452	581773	1108176	1992710	3400480
L	14	15	16	17	18	19	20
$k=1$	2143917	3106521	4310020	5738494	7345702	9054064	10758764
$k=2$	2495664	3639276	5073881	6776332	8681818	10681854	12630089
$k=3$	2529645	3710350	5201891	6983604	8989548	11104717	13169665
$k=4$	2463584	3624360	5097783	6867091	8870510	10996400	13086750
$k=5$	2370895	3488099	4908909	6619307	8562191	10632842	12682317
$k=6$	2280619	3346067	4698348	6324696	8172991	10147413	12111191
$k=7$	2202995	3215732	4494588	6025962	7760597	9610782	11454126
total	5530445	8602824	12836959	18420192	25470484	33997718	43870337
L	21	22	23	24	25	26	27

Table 15: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=8$

$k=1$	12337140	13662410	14619736	15121926	15121926	14619736	13662410
$k=2$	14356679	15690032	16482648	16636160	16120119	14979894	13331230
$k=3$	14995111	16386045	17172207	17239300	16553858	15175052	13249098
$k=4$	14950958	16389578	17224930	17333199	16671479	15293094	13345508
$k=5$	14529984	15984856	16873708	17071956	16531044	15294632	13496832
$k=6$	13898103	15332636	16256324	16555400	16183412	15172698	13630934
$k=7$	13144694	14530626	15475592	15880090	15698143	14945642	13698310
total	54794622	66313115	77826330	88638236	98021668	105295702	109904048
L	28	29	30	31	32	33	34
$k=1$	12337140	10758764	9054064	7345702	5738494	4310020	3106521
$k=2$	11342045	9205143	7107764	5204576	3599586	2339859	1420744
$k=3$	10985929	8622485	6381032	4432825	2876101	1732936	963470
$k=4$	11047985	8651603	6391034	4442158	2898440	1771584	1012155
$k=5$	11340051	9056677	6865014	4932279	3354575	2156889	1308874
$k=6$	11722818	9641068	7573857	5676525	4053635	2753911	1776753
$k=7$	12079590	10241048	8339490	6515486	4877285	3492470	2387683
total	111482424	109904048	105295702	98021668	88638236	77826330	66313115
L	35	36	37	38	39	40	41
$k=1$	2143917	1412855	886145	526830	295410	155261	75885
$k=2$	799520	412792	192929	80146	28834	8641	2025
$k=3$	490763	227202	94728	35173	11455	3197	736
$k=4$	539115	266726	121904	51053	19365	6544	1924
$k=5$	747925	401149	201038	93559	40093	15639	5464
$k=6$	1086237	627514	341288	173894	82466	36085	14401
$k=7$	1554943	961939	563397	311102	161107	77707	34589
total	54794622	43870337	33997718	25470484	18420192	12836959	8602824
L	42	43	44	45	46	47	48
$k=1$	34142	13952	5085	1611	427	89	13
$k=2$	330	28	0	0	0	0	0
$k=3$	131	16	1	0	0	0	0
$k=4$	475	93	13	1	0	0	0
$k=5$	1671	432	89	13	1	0	0
$k=6$	5157	1617	427	89	13	1	0
$k=7$	14029	5092	1611	427	89	13	1
total	5530445	3400480	1992710	1108176	581773	286452	131204
L	49	50	51	52	53	54	55

Table 15: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=8$

$k = 1$	1	0	0	0	0	0	0
$k = 2$	0	0	0	0	0	0	0
$k = 3$	0	0	0	0	0	0	0
$k = 4$	0	0	0	0	0	0	0
$k = 5$	0	0	0	0	0	0	0
$k = 6$	0	0	0	0	0	0	0
$k = 7$	0	0	0	0	0	0	0
total	55320	21178	7226	2141	530	103	14

L	56
$k = 1$	0
$k = 2$	0
$k = 3$	0
$k = 4$	0
$k = 5$	0
$k = 6$	0
$k = 7$	0
total	1

Table 15: Number $H(n, k, L)$ of successful pairs of permutations with given total length L for $n = 8$

L	0	1	2	3	4	5	6
$k = 1$	1	15	118	648	2789	10015	31193
$k = 2$	1	15	119	661	2880	10469	32996
$k = 3$	1	15	118	650	2816	10206	32141
$k = 4$	1	15	118	648	2792	10056	31487
$k = 5$	1	15	118	648	2789	10019	31248
$k = 6$	1	15	118	648	2789	10015	31198
$k = 7$	1	15	118	648	2789	10015	31193
$k = 8$	1	15	118	648	2789	10015	31193
total	1	16	134	782	3571	13586	44779

L	7	8	9	10	11	12	13
$k = 1$	86513	217717	504168	1085927	2194000	4186180	7584519
$k = 2$	92557	235449	550822	1197938	2442571	4701051	8587504
$k = 3$	90231	229992	539613	1177721	2410981	4660376	8551991
$k = 4$	87985	223502	523194	1140425	2333633	4512201	8287539
$k = 5$	86910	219719	512080	1112002	2268510	4375795	8022636
$k = 6$	86582	218217	506695	1095915	2226841	4279544	7820163
$k = 7$	86519	217800	504765	1088906	2205595	4223685	7689374
$k = 8$	86513	217724	504258	1086536	2196886	4196927	7617973
total	131292	349009	853176	1939088	4132970	8318502	15900232

L	14	15	16	17	18	19	20

Table 16: Number $H(n, k, L)$ of successful pairs of permutations with given total length L for $n = 9$

$k=1$	13107738	21689384	34471023	52760011	77944043	111359988	154122149
$k=2$	14956274	24927976	39885065	61422313	91240184	130975441	181973392
$k=3$	14964317	25060127	40287847	62336231	93028182	134145571	187187120
$k=4$	14521801	24362641	39249077	60871994	91074242	131680695	184258529
$k=5$	14040129	23536860	37907138	58796005	88005949	127335637	178350600
$k=6$	13646031	22818657	36671292	56776275	84858187	122643074	171642486
$k=7$	13367926	22273413	35673825	55057711	82049926	118268759	165124891
$k=8$	13198176	21907200	34946899	53715647	79725550	114466452	159218599
total	28997955	50656146	85040656	137582950	215022825	325296886	477225035
L	21	22	23	24	25	26	27
$k=1$	206924061	269836696	342132581	422167993	507352702	594228212	678661776
$k=2$	245019039	320058071	405950327	500304855	599444121	698533485	791890804
$k=3$	253075124	331842157	422343118	522047642	626969636	731781769	830141860
$k=4$	249821812	328510370	419287747	519713501	625848067	732341634	832740258
$k=5$	242098623	318797046	407533517	506033560	610553856	715954736	815985919
$k=6$	232883769	306613272	392013644	486986328	588052482	690418163	788232639
$k=7$	223573740	293845461	375189121	465676680	562115378	660111236	754310049
$k=8$	214826421	281457587	358381925	443811194	534840462	627523344	717096570
total	679962916	942215093	1271239936	1671718545	2144600224	2686068425	3286786158
L	28	29	30	31	32	33	34
$k=1$	756146482	822180193	872681273	904389826	915201732	904389826	872681273
$k=2$	873462300	937419271	978802059	994119511	981809309	942479534	878883392
$k=3$	915226232	980426107	1020123955	1030436507	1009799044	959277943	882536463
$k=4$	920012355	987259054	1028528877	1039620726	1018740041	966879283	887828186
$k=5$	903787258	972567310	1016385124	1030931280	1014186608	966834704	892323651
$k=6$	875043246	944418116	990672467	1009600640	999093521	959519715	893776535
$k=7$	838816019	907755394	955919258	979393830	976075996	945981498	891283749
$k=8$	798346257	866082691	915667438	943521384	947538904	927343566	884343459
total	3931572652	4599630696	5265387908	5899941038	6473010189	6955232022	7320560593
L	35	36	37	38	39	40	41
$k=1$	822180193	756146482	678661776	594228212	507352702	422167993	342132581
$k=2$	795621961	698615872	594426506	489532057	389668298	299327287	221473637
$k=3$	785437129	675331591	560150603	447447773	343559819	253019498	178300829
$k=4$	787778174	674560268	556632943	441997393	337242776	246899251	173202595
$k=5$	796520516	686979700	571938537	459231041	355340428	264774930	189859007
$k=6$	806973576	705786686	597593748	489553402	387797170	296870848	219493952
$k=7$	816067507	725833746	626837910	525372372	427108702	336595889	256972903
$k=8$	821575425	743361178	654827840	561365022	468096650	379437709	298786411
total	7548512574	7625997280	7548512574	7320560593	6955232022	6473010189	5899941038
L	42	43	44	45	46	47	48

Table 16: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=9$

$k=1$	269836696	206924061	154122149	111359988	77944043	52760011	34471023
$k=2$	157495273	107363477	69944385	43386913	25512115	14143729	7343505
$k=3$	119903891	76717081	46545926	26680502	14389406	7268332	3420690
$k=4$	116270640	74592434	45669490	26642477	14780340	7777443	3868483
$k=5$	130917994	86741040	55164398	33629170	19616689	10924076	5790385
$k=6$	156628592	107785158	71457865	45582350	27932502	16410681	9220910
$k=7$	189908265	135737042	93736062	62465811	40111979	24776389	14689414
$k=8$	228375749	169280775	121554652	84451430	56688199	36703099	22876152
total	5265387908	4599630696	3931572652	3286786158	2686068425	2144600224	1671718545
L	49	50	51	52	53	54	55
$k=1$	21689384	13107738	7584519	4186180	2194000	1085927	504168
$k=2$	3540898	1568633	629517	224552	69322	17804	3570
$k=3$	1490915	597375	217944	71459	20667	5131	1050
$k=4$	1810570	792612	322029	120205	40679	12260	3210
$k=5$	2910480	1380712	614515	254659	97305	33843	10529
$k=6$	4939927	2513698	1209162	546553	230382	89672	31810
$k=7$	8337532	4515907	2324952	1132194	518317	221314	87226
$k=8$	13693640	7850429	4295655	2234150	1098712	507570	218428
total	1271239936	942215093	679962916	477225035	325296886	215022825	137582950
L	56	57	58	59	60	61	62
$k=1$	217717	86513	31193	10015	2789	648	118
$k=2$	497	36	0	0	0	0	0
$k=3$	166	18	1	0	0	0	0
$k=4$	704	122	15	1	0	0	0
$k=5$	2859	653	118	15	1	0	0
$k=6$	10099	2795	648	118	15	1	0
$k=7$	31291	10022	2789	648	118	15	1
$k=8$	86617	31201	10015	2789	648	118	15
total	85040656	50656146	28997955	15900232	8318502	4132970	1939088
L	63	64	65	66	67	68	69
$k=1$	15	1	0	0	0	0	0
$k=2$	0	0	0	0	0	0	0
$k=3$	0	0	0	0	0	0	0
$k=4$	0	0	0	0	0	0	0
$k=5$	0	0	0	0	0	0	0
$k=6$	0	0	0	0	0	0	0
$k=7$	0	0	0	0	0	0	0
$k=8$	1	0	0	0	0	0	0
total	853176	349009	131292	44779	13586	3571	782
L	70	71	72				

Table 16: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=9$

$k=1$	0	0	0
$k=2$	0	0	0
$k=3$	0	0	0
$k=4$	0	0	0
$k=5$	0	0	0
$k=6$	0	0	0
$k=7$	0	0	0
$k=8$	0	0	0
total	134	16	1

Table 16: Number $H(n,k,L)$ of successful pairs of permutations with given total length L for $n=9$

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