On approximation of solutions of nonlinear regular hypo-elliptic equations on unbounded domains

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Abstract. In the paper it is proved the existence of solutions of boundary-value problem for nonlinear regular hypo-elliptic equation on unbounded domain. Besides, it is proved an approximation theorem for solutions of the problem on unbounded domain Ω by the solutions of problems on bounded domains $\Omega_R \supset \{\Omega \cap B_R\}$, where B_R denotes the ball of radius R centered at the origin.

In [1] it has been proved the existence of solutions of boundary-value problems for quasi-linear regular hypo-elliptic operator of divergence form

$$\sum_{\alpha\in\Sigma}(-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma^1}u,...,D^{\gamma^N}u) + g_1(x,u) + g_2(x, D^{\beta^1}u,...,D^{\beta^M}u) = f$$

where $\Sigma = \{\gamma^1,...,\gamma^N\}$ is a finite set of multi-indices, the multi-indices $\beta^1,...,\beta^M$ are associated with multi-indices $\{\gamma^1,...,\gamma^N\}$ in some sense, $x \in \Omega$, $\Omega \subset R^n$ is (in general) an unbounded domain and the functions A_α, g_1, g_2 satisfy certain conditions.

In the present paper we prove the existence of solutions of boundary-value problems for more general regular equations of the form

$$\sum_{\alpha\in\aleph}(-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma^1}u,...,D^{\gamma^N}u) + \sum_{\alpha\in\aleph^{(l)}}(-1)^{|\alpha|} D^{\alpha} g_{\alpha}(x, D^{\beta^1}u,...,D^{\beta^M}u) = f ,$$

where the functions A_α satisfy the conditions of [1], the conditions with respect to $\aleph, \aleph^{(l)}$ and g_α will be clarified later. Besides, it is proved an approximation theorem for solutions of the problem on unbounded Ω by solutions of problems on bounded $\Omega_R \supset \{\Omega \cap B_R\}$, where B_R denotes the ball of radius R centered at the origin.

Results of such type for nonlinear elliptic equations are obtained in [2].

§ 1. Existence of solutions

Let us recall some definitions, which can be found in the paper [3].

Definition 1. The characteristic polyhedron (c.p.) of a set of multi-indices Σ is defined to be the smallest convex polyhedron $\aleph = \aleph(\Sigma)$ in R^n containing all points of Σ **.**

Definition 2. A polyhedron $\aleph$ is said to be complete, if $\aleph$ has vertices at the origin and at all coordinate axes N_0^n, where N_0^n is the set of n-tuples with non-negative components.

Definition 3. A complete polyhedron $\aleph$ is said to be regular, if the projection operators on the coordinate hyperplanes do not remove points of $\aleph$ from $\aleph$.

Definition 4. A complete polyhedron is said to be completely regular (CR) if all the coordinates of outward normals of noncoordinate $(n-1)$ - dimentional faces of $\aleph$ are positive.

Obviously any (CR) polyhedron is regular. Let us denote by $\aleph^{(0)}$ the interior part of the polyhedron $\aleph$, i.e. $\aleph^{(0)} = \aleph \setminus \partial'\aleph$, where $\partial'\aleph$ is the set of vertices of noncoordinate faces.

Next we specify the form of the operator considered in the paper. Let $\aleph$ be the (CR) polyhedron of the set $\{\gamma^1,...,\gamma^N\}$ and

$$Pu = Au + \sum_{\alpha\in\aleph^{(l)}} (-1)^{|\alpha|} D^\alpha g_\alpha(x, D^{\beta^1}u,...,D^{\beta^M}u) , \qquad (1)$$

where $Au = \sum_{\alpha\in\aleph} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^{\gamma^1}u,...,D^{\gamma^N}u)$, $\beta^1,...,\beta^M \in \aleph^{(0)}$, $\aleph^{(l)} \subseteq \aleph^{(0)}$ ($l \geq 0$ - fixed integer) is a polyhedron of multi-indices geometrically similar to $\aleph$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $D^\alpha u = \dfrac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ are distributional S.L. Sobolev derivatives. For the sake of shortness we shall write $A_\alpha(x, D(u))$ instead of $A_\alpha(x, D^{\gamma^1}u,...,D^{\gamma^N}u)$ and $g_\alpha(x, D(u))$ - instead of $g_\alpha(x, D^{\beta^1}u,...,D^{\beta^M}u)$.

Suppose that the (c.p.) $\aleph$ contains the multi-indices $\alpha^1,...,\alpha^N$. We set $\xi = (\xi_1,...,\xi_N) = (\xi_{\alpha^1},...,\xi_{\alpha^N}) \in R^N$, where ξ_i corresponds to derivative D^{α^i} $(i = 1,...,N)$ and $\xi = (\eta, \zeta)$, where η corresponds to derivative D^α with $\alpha \in \aleph^{(0)}$, ζ - to derivative D^α with $\alpha \in \partial'\aleph$, i.e. $\eta \in R^M$, $\zeta \in R^{N-M}$. Furthermore, let $\Lambda \subseteq \aleph^{(0)}$ be the skeleton of (c.p.) $\aleph$ (see [4], §13.1 for a precise definition). Vectors $\eta \in R^M$ will be written in the form $\eta = (\eta', \eta'')$, where

η' corresponds to derivatives D^{α} with $\alpha \in \Lambda$.

We describe the function space where we study the operator P. We denote by $W_p^{\aleph}(\Omega)$ ($p > 1$) the set of functions u such that $D^{\alpha}u \in L_p(\Omega)$ for all $\alpha \in \aleph$. The space $W_p^{\aleph}(\Omega)$ is equipped with the norm

$$\| u \|_{\aleph, \Omega, p} = \sum_{\alpha \in \aleph} \| D^{\alpha} u \|_{L_p(\Omega)} . \tag{2}$$

Furthermore, $\overset{0}{W}{}_p^{\aleph}(\Omega)$ stands for the closure of the space $C_0^{\infty}(\Omega)$ with respect to the norm (2).

Definition 5. Let X and Y be normed spaces. The mapping $T: X \to Y$ is called demi-continuous if for any strongly convergent sequence $(x_n) \in X$, $x_n \to x_0$, the sequence of images $(T(x_n))$ weakly converges to $T(x_0)$.

The mapping $T: X \to Y$ is called hemi-continuous if for any $x, \omega \in X$ $\quad T(x + t\omega) \to T(x)$ weakly when $t \to 0$. Obviously demi-continuous mappings are hemi-continuous.

Definition 6. The mapping $T: X \to X^*$ is called coercive, if

$$\lim_{\|x\| \to \infty} \frac{\langle T(x), x \rangle}{\| x \|} = +\infty .$$

Definition 7. The mapping $T: X \to X^*$ is called monotone, if

$\langle x - y, T(x) - T(y) \rangle \geq 0 \quad \forall x, y \in X$. The mapping is called pseudo-monotone, if for any sequence $(u_j) \in X$ such that $u_j \to u$, $T(u_j) \to y$ in X^* and $\overline{\lim_{j \to \infty}} \langle T(u_j), u_j - u \rangle \leq 0$ follows that $y = Tu$ and $\langle T(u_j), u_j - u \rangle \to 0$ for $j \to \infty$.

Restrictions on the operator A**.** Assume that:

A_0) The functions $A_{\alpha}(x, \xi): \Omega \times R^N \to R^1$, $\alpha \in \aleph$ satisfy the Caratheodory conditions, i.e. they are measurable in x for each fixed $\xi \in R^N$ and continuous in ξ for almost all $x \in \Omega$.

A_1) There exists a constant $c > 0$ such that

$$|A_{\alpha}(x, \xi)| \leq c \, |\xi|^{p-1} + |K(x, \xi)| \ , \quad x \in \Omega \, , \ \xi \in R^N \ , \ \alpha \in \aleph \, ,$$

where $K(x,\xi)$ is a measurable function satisfying the condition: if $\|u\|_{\aleph,\Omega,p} \le c_1$ and $\frac{1}{p}+\frac{1}{q}=1$, then there exists a constant $c_2 = c_2(c_1)$ such that

$$\int_{\Omega} \left|K(x, D^{\gamma^1}u(x),...,D^{\gamma^N}u(x))\right|^q dx \le c_2 \ .$$

A_2) The operator A is coercive in $\overset{0}{W}{}^{\aleph}_p(\Omega)$.

A_3) There exist a function $\psi \in C_0^\infty(\Omega)$ and a function $F(r,\sigma) \ge 0$ ($r > 0$) continuous in σ with the following properties: for any fixed $r > 0$

$$\lim_{y\to 0} \frac{F\ (r, y\sigma)}{y} = 0 \ ,$$

and for any $u,\, v \in \overset{0}{W}{}^{\aleph}_p(\Omega)$, $\|u\|_{\aleph,\Omega,p} \le r$, $\|v\|_{\aleph,\Omega,p} \le r$ the inequality

$$\left\langle A(u)-A(v)\ ,\ u-v\right\rangle \ge -F\ (r, \|(u-v)\psi\|_{\aleph^{(0)},\Omega,p}) .$$

holds.

Restrictions on the operators g_α.

G_1) Let for all $x \in \Omega$, $\eta \in R^M$, $\alpha \in \aleph^{(l)}$

$$g_\alpha(x,\eta) = p_\alpha(x,\eta) + r_\alpha(x,\eta),$$

where p_α, r_α satisfy the Caratheodory conditions and for almost all $x \in \Omega$

$$p_\alpha(x,\eta)\,\xi_\alpha \ge 0,$$

$$|\, r_\alpha(x,\eta)\,| \le h_\alpha(x) \text{ where } h_\alpha \in L_1(\Omega) \cap L_q(\Omega) .$$

G_2) If $g^*_{\alpha,\tau}(x) = \sup_{|\eta|\le\tau,\, |\eta'|\le\tau} |g_\alpha(x,\eta)|$ then $g^*_{\alpha,\tau} \in L_1(\Omega)$ for any τ; $0 < \tau < \infty$.

G_3) If $\|u\|_{\aleph,\Omega,p} \le c_3$, $u \in \overset{0}{W}{}^{\aleph}_p(\Omega)$, then it holds the following inequality

$$\int_{\Omega} |g_\alpha(x, D(u))| \left|D^\alpha u(x)\right| dx \le c_4 \quad \forall \alpha \in \aleph^{(l)}$$

with a certain constant $c_4 = c_4(c_3)$.

Restrictions on the domain Ω.

Ω_1) The domain Ω satisfies the cube condition (see [4], §13.1 for a precise definition).

Ω_2) $\overset{0}{W}{}^{\aleph}_p(\Omega) = \Re = \left\{ f\ ;\ D^\alpha f\big|_{\partial\Omega} = 0\ ,\ \alpha \in \Lambda \right\}$.

$(A.C.)$ (Additional condition) $W_p^{\aleph}(R^n) \subset C^{\Lambda}(R^n)$ (see [5]), where $C^{\Lambda}(R^n)$ is the space with finite norm

$$\| u \|_{C^{\Lambda}(R^n)} = \sum_{\alpha \in \Lambda} \sup_{R^n} \left| D^{\alpha} u \right| .$$

By Lebesgue's theorem and Proposition 1.1 in [1] the following holds.

Proposition 1.1. *Assume that the conditions $\Omega_1), \Omega_2)$ and $(A.C.)$ are satisfied. Then for any function $u \in \overset{0}{W}{}_p^{\aleph}(\Omega)$ there exist a constant $c_5 > 0$ and a sequence $\omega_k \in \overset{0}{W}{}_p^{\aleph}(\Omega) \cap L_{\infty}(\Omega)$ such that*

$$(\omega_k) \text{ weakly converges to } u \text{ in } W_p^{\aleph}(\Omega),$$

$$D^{\alpha} \omega_k \in L_{\infty}(\Omega) \text{ for } \alpha \in \aleph^{(l)},$$

$$\left| D^{\alpha} \omega_k(x) \right| \le c_5 \left| D^{\alpha} u(x) \right| \text{ a.e. (almost everywhere) in } \Omega .$$

Remark 1.1. *It is shown in [1], that for $|\alpha| = 0$ Proposition 1.1. holds. Furthermore, in virtue of $(A.C.)$ for every $\alpha \in \Lambda$ the Proposition 1.1 holds for any closed subspace of $W_p^{\aleph}(\Omega)$.*

Theorem 1.1. *Assume that conditions $A_0) - A_3)$, $G_1) - G_3)$, $\Omega_1), \Omega_2)$ and $(A.C.)$ are satisfied. Then for any $f \in (\overset{\mathbf{0}}{W}{}_p^{\aleph}(\Omega))^*$ there exists a solution $u \in \overset{0}{W}{}_p^{\aleph}(\Omega)$ of the equation $Pu = f$ such that*

$$\sum_{\alpha \in \aleph} \int_{\Omega} A_{\alpha}(x, D(u)) D^{\alpha} v \, dx + \sum_{\alpha \in \aleph^{(l)}} \int_{\Omega} g_{\alpha}(x, D(u)) D^{\alpha} v \, dx = \langle f, v \rangle, \tag{3}$$

$$g_{\alpha}(\cdot, D(u)) \in L_1(\Omega), \tag{4}$$

$$g_{\alpha}(\cdot, D(u)) D^{\alpha} u \in L_1(\Omega) \quad (\alpha \in \aleph^{(l)}), \tag{5}$$

for $v = u$ and all $v \in \overset{0}{W}{}_p^{\aleph}(\Omega)$ satisfying $D^{\alpha} v \in L_{\infty}(\Omega) \quad \forall \alpha \in \aleph^{(l)}$.

In the proof of Theorem 1.1 we use the following auxiliary propositions. Furthermore, we assume that the conditions of Theorem 1.1 are satisfied.

For any $u, v \in \overset{\mathbf{0}}{W}{}_p^{\aleph}(\Omega)$ let

$$\langle T(u),v\rangle = \sum_{\alpha\in\aleph}\int_{\Omega} A_\alpha(x,D(u))D^\alpha v\,dx$$

be the Dirichlet form corresponding to the operator A. The conditions $A_0)$, $A_1)$ imply $T(u)\in(\overset{\mathbf{0}}{W}{}_p^{\aleph}(\Omega))^*$.

Lemma 1.1. *The operator* $T:\ \overset{\mathbf{0}}{W}{}_p^{\aleph}(\Omega)\to(\overset{\mathbf{0}}{W}{}_p^{\aleph}(\Omega))^*$ *is pseudo-monotone.*

Proof. Indeed, let $C>0$ and $H=\left\{u\in\overset{0}{W}{}_p^{\aleph}(\Omega);\ \|u\|_{\aleph,\Omega,p}\le C\right\}$.

Consider on H the operator $T(u)$. We have

$$\langle T(u),v\rangle\le C_1\left(\sum_{\alpha\in\aleph}\int_{\Omega}\sum_{\beta\in\aleph}\left|D^\beta u\right|^{p/q}\left|D^\alpha v\right|\right)dx+\sum_{\alpha\in\aleph}\int_{\Omega}\left|K(x,D^{\gamma^1}u,\ldots,D^{\gamma^N}u)\right|\left|D^\alpha v\right|dx\le$$

$$\le C_1\sum_{\alpha,\beta\in\aleph}\left\|D^\beta u\right\|_{L_p(\Omega)}\left\|D^\alpha v\right\|_{L_p(\Omega)}+$$

$$+\sum_{\alpha\in\aleph}\int_{\Omega}\left|K(x,D^{\gamma^1}u,\ldots,D^{\gamma^N}u)\right|^q dx\int_{\Omega}\left|D^\alpha v\right|^p dx\le C_2\,\|v\|_{\aleph,\Omega,p}\,.$$

We show that $T(u)$ is a continuous mapping on the finite-dimensional space $V_r=\left\{\overline{u_{\mathbf{1}},\ldots,u_r}\right\}$, i.e. if $u^k=c_{\mathbf{1}}^k u_{\mathbf{1}}^k+\ldots+c_r^k u_r^k$ and $c_{\mathbf{1}}^k\to c_{\mathbf{1}}^{\mathbf{0}}$, …, $c_r^k\to c_r^{\mathbf{0}}$ for $k\to\infty$ and $u^{\mathbf{0}}=c_{\mathbf{1}}^{\mathbf{0}}u_{\mathbf{1}}^{\mathbf{0}}+\ldots+c_r^{\mathbf{0}}u_r^{\mathbf{0}}$, then $T(u_k)\to T(u_{\mathbf{0}})$ in V_r^*. It holds

$$\langle T(u^k),v\rangle-\langle T(u^{\mathbf{0}}),v\rangle=\sum_{\alpha\in\aleph}\int_{\Omega}\left(A_\alpha(x,D(u^k))-A_\alpha(x,D(u^{\mathbf{0}}))\right)D^\alpha v\,dx\ .$$

The Caratheodory condition implies $A_\alpha(x,D(u^k))^q\to A_\alpha(x,D(u^{\mathbf{0}}))^q$. Moreover, the condition $A_{\mathbf{1}})$ yields the estimate

$$\int_{\Omega}\left|A_\alpha(x,D(u^0))\right|^q dx\le\left\|u^0\right\|_{\aleph,\Omega,p}+\int_{\Omega}\left|K(x,D^{\gamma^1}u^0,\ldots,D^{\gamma^N}u^0)\right|^q dx\le C_3\,.$$

Applying Lebesgue's theorem we find that the operator T is continuous on finite-dimensional spaces. Hence, it is hemi-continuous. Then, by Lemma 1.2 in [1] it is a pseudo-monotone operator.

The lemma is proved.

Let for any natural number τ

$$g_{\alpha,\tau}(x,\eta)=\chi_\tau(x)\,p_{\alpha,\tau}(x,\eta)+r_\alpha(x,\eta),$$

where $\chi_\tau(x)$ is the characteristic function of the domain $\{x\in\Omega\,;\,|x|<\tau\}$ and

$$p_{\alpha,\tau}(x,\eta)=\begin{cases} p_\alpha(x,\eta)\,, & |p_\alpha(x,\eta)|\le\tau \\ \tau\,\dfrac{p_\alpha(x,\eta)}{|p_\alpha(x,\eta)|}\,, & |p_\alpha(x,\eta)|>\tau \end{cases}.$$

Therefore, by means of $G_1)$

$$\left|g_{\alpha,\tau}(x,\eta)\right|\le\tau\;\chi_\tau(x)+h_\alpha(x)\;(\in L_1(\Omega)\cap L_q(\Omega))\;. \qquad (6)$$

Let for any $u,\,v\in\overset{0}{W}{}^{\aleph}_p(\Omega)$

$$\langle S_\tau(u),v\rangle=\sum_{\alpha\in\aleph^{(l)}}\int_\Omega g_{\alpha,\tau}(x,D(u))D^\alpha v\,dx$$

Inequality (6) implies $S_\tau(u)\in(\overset{0}{W}{}^{\aleph}_p(\Omega))^*$. Using Lebesgue's theorem, by means of conditions $G_1)$, $G_3)$ and (6) we find the following lemma.

Lemma 1.2. *If* $\lim\limits_{j\to\infty}D^\alpha u_j=D^\alpha u$ *a.e. in* Ω *for all* $\alpha\in\aleph^{(l)}$*, then*

$$g_{\alpha,\tau}(\cdot\,,D(u_j))\to g_{\alpha,\tau}(\cdot\,,D(u)) \text{ in the norm of } L_q(\Omega).$$

From $A_1)$, Lemma 1.2 and (6) immediately follows

Lemma 1.3. *The (nonlinear) operator* $T+S_\tau:\ \overset{0}{W}{}^{\aleph}_p(\Omega)\to(\overset{0}{W}{}^{\aleph}_p(\Omega))^*$ *is bounded.*

If the sequence $c^j=(c_1^j,...,c_r^j)\in R^r$ *converges to* $c=(c_1,...,c_r)\in R^r$ *, then for any*

$u_1,...,u_r,\,v\in\overset{0}{W}{}^{\aleph}_p(\Omega)$ *holds*

$$\lim_{j\to\infty}\left\langle(T+S_\tau)(c_1^ju_1+...+c_r^ju_r),v\right\rangle=\left\langle(T+S_\tau)(c_1u_1+...+c_ru_r),v\right\rangle\;.$$

The following lemma can be proved analogous to Lemma 1.4 in [1].

Lemma 1.4. *The operator* $T+S_\tau:\ \overset{0}{W}{}^{\aleph}_p(\Omega)\to(\overset{0}{W}{}^{\aleph}_p(\Omega))^*$ *is pseudo-monotone.*

Lemma 1.5. *For any function* $u\in\overset{0}{W}{}^{\aleph}_p(\Omega)$ *the estimate*

$$\left\langle (T+S_\tau)(u),u \right\rangle \geq c_6 \| u \|^p_{\aleph,\Omega,p} - c_7 - \left(\sum_{\alpha\in\aleph^{(l)}} \| h_\alpha \|_{L_q(\Omega)} \right) \| u \|_{\aleph,\Omega,p}$$

is true. Hence the operator $T+S_\tau$ *is coercive.*

Proof. We have

$$\left\langle S_\tau(u),\, u\right\rangle = \int_\Omega \sum_{\alpha\in\aleph^{(l)}} g_{\alpha,\tau}(x,D(u))D^\alpha u\, dx = \int_\Omega \chi_\tau(x) \sum_{\alpha\in\aleph^{(l)}} p_\alpha(x,D(u))D^\alpha u\, dx +$$

$$+ \int_\Omega \sum_{\alpha\in\aleph^{(l)}} r_\alpha(x,D(u))D^\alpha u\, dx\,.$$

The definition of p_α and condition $G_2)$ show that

$$\chi_\tau(x) p_\alpha(x,D(u))D^\alpha u \geq 0\,,\ \ \alpha\in\aleph^{(l)}\,.$$

Therefore, by means of $G_1)$ with a certain constant $c_8 > 0$

$$\left\langle S_\tau(u),u\right\rangle \geq -\left| \sum_{\alpha\in\aleph^{(l)}} \int_\Omega r_\alpha(x,D(u))D^\alpha u\, dx \right| \geq -c_8 \left(\sum_{\alpha\in\aleph^{(l)}} \| h_\alpha \|_{L_q(\Omega)} \right) \| u \|_{L_p(\Omega)}$$

Condition $A_2)$ implies coerciveness of the operator. So, when $\| u \|_{\aleph,\Omega,p} \to \infty$ then

$$\frac{\left\langle (T+S_\tau)(u),u\right\rangle}{\| u \|_{\aleph,\Omega,p}} = \frac{\left\langle T(u),u\right\rangle}{\| u \|_{\aleph,\Omega,p}} + \frac{\left\langle S_\tau(u),u\right\rangle}{\| u \|_{\aleph,\Omega,p}} \geq \frac{\left\langle T(u),u\right\rangle}{\| u \|_{\aleph,\Omega,p}} - \frac{c_8 \sum_{\alpha\in\aleph^{(l)}} \| h_\alpha \|_{L_q(\Omega)}}{\| u \|_{\aleph,\Omega,p}} \cdot \| u \|_{L_p(\Omega)} \to +\infty$$

This leads to the statements of the lemma.

Lemma 1.6. *Suppose that* $\{u_j\} \subset \overset{0}{W}{}^{\aleph}_p(\Omega)$, (u_j) *converges to* u *in* $\overset{0}{W}{}^{\aleph}_p(\Omega)$ *and*

$$\sum_{\alpha\in\aleph^{(l)}} \int_\Omega \left| g_{\alpha,j}(\,\cdot\,,D(u_j)) \right| \left| D^\alpha u_j \right| dx \leq c_9\ ,$$

where the constant c_9 *does not depend on* j. *Then (4) and (5) are true and there exists a subsequence* (u_{j_k}) *of* (u_j) *such that*

$$\lim_{k\to\infty} D^\beta u_{j_k} = D^\beta u \text{ a.e. in } \Omega \text{ for } \beta\in\aleph^{(0)},$$

$$\lim_{k\to\infty} \| g_{\alpha,j_k}(\,\cdot\,,D(u_{j_k})) - g_\alpha(\,\cdot\,,D(u)) \|_{L_1(\Omega)} = 0\,.$$

The proof is analogous to the proof of Lemma 1.5 of [1] (using Lemma 1.3 of [7]).

Theorem (H. Brezis [6]). *Let X be a reflexive Banach space Let* $A: X \to X^*$ *be a coercive and bounded pseudo-monotone operator which is continuous on finite-dimensional spaces. Then for any* $f \in X^*$ *the equation* $Au = f$ *has at least one solution* $u \in X$ *.*

Therefore, for any $f \in (\overset{0}{W}{}_p^{\aleph}(\Omega))^*$ there exist functions $u_j \in \overset{0}{W}{}_p^{\aleph}(\Omega)$ such that

$$\langle T(u_j), v\rangle + \langle S_j(u_j), v\rangle = \langle f, v\rangle \quad \forall v \in \overset{0}{W}{}_p^{\aleph}(\Omega)$$

holds. First we prove that the sequence (u_j) is bounded. Assume the contrary. If $\|u_j\|_{\aleph,\Omega,p} \to \infty$ then it follows from Lemma 1.5 that

$$\frac{\langle f, u_j\rangle}{\|u_j\|_{\aleph,\Omega,p}} = \frac{\langle (T+S_j)(u_j), u_j\rangle}{\|u_j\|_{\aleph,\Omega,p}} \to \infty$$

On the other hand we have

$$\frac{\langle f, u_j\rangle}{\|u_j\|_{\aleph,\Omega,p}} \le \frac{\|f\|_{L_q(\Omega)} \cdot \|u_j\|_{\aleph,\Omega,p}}{\|u_j\|_{\aleph,\Omega,p}} < \infty$$

This contradiction proves the boundedness of the sequence (u_j).

From the reflexivity of the space $W_p^{\aleph}(\Omega)$ and the boundedness of the operator T it follows that there exists a subsequence of the sequence (u_j) (which we also denote by (u_j)) such that $u_j \to u$, $T(u_j) \to y$ in $W_p^{\aleph}(\Omega)$. The definition and the boundedness of the sequence (u_j) $(T(u_j))$ as well as the estimate $\sum_{\alpha\in\aleph^{(l)}} \int_{\Omega} |r_\alpha(\cdot\,, D(u_j))|\,|u_j|\,dx \le c_2$ lead to

$$\sum_{\alpha\in\aleph^{(l)}} \int_{\Omega} |g_{\alpha,j}(\cdot\,, D(u_j))|\,|u_j|\,dx \le c_1\|f\| + c_1c_2 + c_3 = c < \infty$$

Furthermore, according to $G_3)$ we have

$$\sum_{\alpha\in\aleph^{(l)}} \int_{\Omega} |g_{\alpha,j}(\cdot\,, D(u_j))|\,|D^\alpha u_j|\,dx \le c\,.$$

From this and from Lemma 1.6 it follows that $u \cdot \sum_{\alpha\in\aleph^{(l)}} g_\alpha(\cdot\,, D(u)) \in L_1(\Omega)$ and

$$\left(\sum_{\alpha\in\aleph^{(l)}} g_\alpha(\cdot\,, D(u_j))\right) v \xrightarrow{L_1} \left(\sum_{\alpha\in\aleph^{(l)}} g_\alpha(\cdot\,, D(u))\right) v\,.$$

for any $v \in \overset{0}{W}{}_p^{\aleph}(\Omega) \cap L_\infty(\Omega)$. A limiting argument shows that

$$\langle y,v\rangle+\sum_{\alpha\in\aleph^{(l)}}\int_{\Omega} g_\alpha(\cdot\,,D(u))\,v(x)\,dx=\langle f,v\rangle\quad \forall v\in \overset{0}{W}{}_p^{\aleph}(\Omega)\cap L_\infty(\Omega)$$

Now we prove that $y=T(u)$. Using the pseudo-monotonicity of the operator T, we obtain

$$\limsup_{j\to\infty}\langle T(u_j),\,u_j-u\rangle=\limsup_{j\to\infty}\Big(\langle (f-S_j)(u_j),\,u_j\rangle-\langle T(u_j),\,u\rangle\Big)=$$

$$=\limsup_{j\to\infty}\Big(\langle (f-S_j)(u_j),\,u_j\rangle-\langle y,\,u\rangle\Big)\le\langle f-y,\,u\rangle-\limsup_{j\to\infty}\langle S_j(u_j),\,u_j\rangle\ .$$

Hence, by Fatou's lemma we obtain

$$\limsup_{j\to\infty}\langle T(u_j),\,u_j-u\rangle=\langle f-y,\,u\rangle-\sum_{\alpha\in\aleph^{(l)}}\int_{\Omega} g_\alpha(\cdot\,,D(u))u(x)\,dx\ .$$

Thus for any $\omega\in\overset{0}{W}{}_p^{\aleph}(\Omega)\cap L_\infty(\Omega)$

$$\limsup_{j\to\infty}\langle T(u_j),\,u_j-u\rangle=\langle f-y,\,u-\omega\rangle+\sum_{\alpha\in\aleph^{(l)}}\int_{\Omega} g_\alpha(\cdot\,,D(u))(\omega-u)\,dx\ .$$

Next we construct functions $\omega_j\in\overset{0}{W}{}_p^{\aleph}(\Omega)\cap L_\infty(\Omega)$ such that $\omega_j\to u$ in $W_p^{\aleph}(\Omega)$ and $|\omega_j(x)|\le c|u(x)|$. Then $\langle f-y,u-\omega_j\rangle\to 0$ and

$$\sum_{\alpha\in\aleph^{(l)}}\int_{\Omega} g_\alpha(\cdot\,,D(u))\omega_j(x)\,dx\xrightarrow{L_1}\sum_{\alpha\in\aleph^{(l)}}\int_{\Omega} g_\alpha(\cdot\,,D(u))u(x)\,dx\,.$$

The last relation follows from Lebesgue's theorem and the fact that $\sum_{\alpha\in\aleph^{(l)}}\int_{\Omega} g_\alpha(\cdot\,,D(u))u(x)\,dx\in L_1(\Omega)$ a.e.

As a consequence we have $\limsup_{j\to\infty}\langle T(u_j),\,u_j-u\rangle\le 0$. This property and the pseudo-monotonicity of the operator T imply that $y=T(u)$ and $\langle T(u_j),\,u_j-u\rangle\to 0$. Finally when $v=\omega_j$ and $j\to\infty$ we see that the function u is a weak solution of the equation $Pu=f$ such that (4) and (5) are satisfied.

Theorem 1.2. *If the regular operator A (see deinition 3 of [3]) satisfies the condition A_1), i.e. for some constant $C>0$ and for all $u,v\in\overset{0}{W}{}_p^{\aleph}(\Omega)$*

$$\sum_{\alpha\in\aleph}\int_{\Omega}(A_\alpha(x,D(u))-A_\alpha(x,D(v)))D^\alpha(u-v)\,dx\ge C\|u-v\|_{\aleph,\Omega,p}^{p}$$

holds, and if in addition the condition (A.C.) is satisfied, then for any $f\in(\overset{0}{W}{}_p^{\aleph}(\Omega))^*$ the equation $Pu=f$ has a weak solution belonging to the space $\overset{0}{W}{}_p^{\aleph}(\Omega)$.

Proof. From the conditions of the theorem it follows that P is a monotone and coercive operator. Since all the conditions of Theorem 1.1 are satisfied, the existence of a weak solution is proved.

Remark 1.2. *If we impose some additional conditions on the operator P that ensure its strict monotonicity, then the equation $Pu = f$ will have a unique weak solution $u \in \overset{0}{W}{}_p^{\aleph}(\Omega)$ for any $f \in (\overset{0}{W}{}_p^{\aleph}(\Omega))^*$.*

§ 2. Approximation theorem

Let $\Omega \subset R^n$ be an unbounded domain with a smooth boundary. Further, for any $R \ge R_0 > 0$ let $\Omega_R \subset \Omega$ be a bounded domain with smooth boundary. It is assumed that

a) $\Omega_R \supset (\Omega \cap B_R)$, where B_R is the sphere of radius R;

b) there exist linear continuous operators $\Phi_R : \overset{0}{W}{}_p^{\aleph}(\Omega_R) \to \overset{0}{W}{}_p^{\aleph}(\Omega)$ such that $\Phi_R v\big|_{\Omega_R} = v$ a.e. for all $v \in \overset{0}{W}{}_p^{\aleph}(\Omega_R)$ and $\| \Phi_R \| \le c^*$ for all $R \ge R_0$ ($c^* > 0$ is a constant).

Let $f_\alpha \in L_q(\Omega)$. Consider the functional $f \in (\overset{0}{W}{}_p^{\aleph}(\Omega))^*$ defined by the formula $\langle f, v \rangle = \sum_{\alpha \in \aleph} \int_\Omega f_\alpha D^\alpha v \, dx$.

Remark 2.1. *By Proposition 1.1 for any function $u_R \in \overset{0}{W}{}_p^{\aleph}(\Omega_R)$ there exists a sequence of functions $\omega_{k,R} \in \overset{0}{W}{}_p^{\aleph}(\Omega_R) \cap L_\infty(\Omega_R)$ for which $(\omega_{k,R})$ tends to u weakly in $\overset{0}{W}{}_p^{\aleph}(\Omega_R)$, $D^\alpha \omega_{k,R} \in L_\infty(\Omega_R)$ if $\alpha \in \aleph^{(l)}$ and $\left| D^\alpha \omega_{k,R} \right| \le c_5 \left| D^\alpha u_R(x) \right|$ a.e. in Ω_R.*

Theorem 2.1. *Suppose that the conditions of Theorem 1.1 and conditions a) and b) are satisfied. Then for any $R \ge R_0$ there exists a function $u_R \in \overset{0}{W}{}_p^{\aleph}(\Omega_R)$ satisfying the conditions,*

$$g_\alpha(\cdot, D(u_R)) \in L_1(\Omega_R), \quad g_\alpha(\cdot, D(u_R)) D^\alpha u_R \in L_1(\Omega_R), \ \alpha \in \aleph^{(l)}, \qquad (7)$$

$$\sum_{\alpha \in \aleph} \int_{\Omega_R} A_\alpha(x, D(u_R)) D^\alpha v_R \, dx + \sum_{\alpha \in \aleph^{(l)}} \int_{\Omega_R} g_\alpha(x, D(u_R)) D^\alpha v_R \, dx = \sum_{\alpha \in \aleph} \int_{\Omega_R} f_\alpha D^\alpha v_R dx, \quad (8)$$

for $v_R = u_R$ *and for all* $v_R \in \overset{0}{W}{}_p^{\aleph}(\Omega_R)$ *such that* $D^{\alpha} v_R \in L_\infty(\Omega_R)$ $\forall \alpha \in \aleph^{(l)}$. *Further, suppose that* $\lim_{j\to\infty} R_j = +\infty$, $R_j \ge R_0$ *and* u_{R_j} *is a solution of the problem (7), (8) for* $R = R_j$. *Then there is a subsequence* (R'_j) *of the sequence* (R_j) *for which* $(\Phi_{R'_j} u_{R'_j})$ *converges weakly in* $\overset{0}{W}{}_p^{\aleph}(\Omega)$ *to some solution* u^* *of the problem (3) - (5). If the solution of problem (3) - (5) is unique, then* $(\Phi_{R_j} u_{R_j})$ *also converges weakly to the solution* u *in* $\overset{0}{W}{}_p^{\aleph}(\Omega)$.

Suppose that the conditions of Theorem 2.1 are satisfied. Similar to the proof of Lemma 1.5 we can find the following Lemma.

Lemma 2.1. *For any* $R \ge R_0$, $u \in \overset{0}{W}{}_p^{\aleph}(\Omega_R)$ *the estimate*

$$\sum_{\alpha\in\aleph} \int_{\Omega_R} A_\alpha(x,D(u)) D^\alpha u \, dx + \sum_{\alpha\in\aleph^{(l)}} \int_{\Omega_R} g_\alpha(x,D(u)) D^\alpha u \, dx \ge$$

$$\ge c' \| u \|^p_{\aleph,\Omega_R,p} - c'' - \left(\sum_{\alpha\in\aleph^{(l)}} \| h_\alpha \|_{L_q(\Omega)} \right) \| u \|_{\aleph,\Omega_R,p} .$$

holds.

We set $g_{\alpha,R}(x,\eta) = \begin{cases} g_\alpha(x,\eta), & x \in \Omega_R \\ 0, & x \in \Omega \setminus \Omega_R \end{cases}$. The following lemma is proved in analogy to Lemma 1.6.

Lemma 2.2. *Assume that* (u_j) *converges weakly to* u *in* $\overset{0}{W}{}_p^{\aleph}(\Omega)$, $g_{\alpha,R_j}(\cdot, D(u_j)) D^\alpha u_j \in L_1(\Omega)$, $\alpha \in \aleph^{(l)}$ *and that inequality*

$$\sum_{\alpha\in\aleph^{(l)}} \int_\Omega g_{\alpha,R_j}(x,D(u_j)) D^\alpha u_j \, dx \le c$$

holds with some constant c *which does not depend on* j, $\lim_{j\to\infty} R_j = +\infty$. *Then (4), (5) hold and there exists a subsequence* (u_{j_k}) *of the sequence* (u_j) *for which*

$$\lim_{k\to\infty} D^\beta u_{j_k} = D^\beta u \quad \text{a.e. in } \Omega \quad \text{for } \beta \in \aleph^{(0)},$$

$$\lim_{k\to\infty}\left\| g_{\alpha,R_{j_k}}(\,\cdot\,,D(u_{j_k}))-g_\alpha(\,\cdot\,,D(u))\right\|_{L_1(\Omega)}=0\,.$$

For any $u,\,v\in\overset{0}{W}{}^{\aleph}_{p}(\Omega)$, $R\geq R_0$ let

$$\left\langle T_R(u),v\right\rangle=\sum_{\alpha\in\aleph}\int_{\Omega_R}A_\alpha(x,D(u))D^\alpha v\,dx\ \ .$$

Analogously to Lemma 1.4 the following lemma holds.

Lemma 2.3. *Suppose that for* $j\to\infty$

$$u_j \textit{ tends to } u \textit{ weakly in } \overset{0}{W}{}^{\aleph}_{p}(\Omega)\ ,$$

$$T_{R_j}(u_j) \text{ tends to } u \textit{ weakly in } (\overset{0}{W}{}^{\aleph}_{p}(\Omega))^*\ ,$$

where $\lim_{j\to\infty}R_j=+\infty$. *Moreover, let*

$$\limsup_{j\to\infty}\left\langle T_{R_j}(u_j),u_j-u\right\rangle\leq 0\,.$$

Then $y=T(u)$.

Proof of Theorem 2.1. By Theorem 1.1 and Remark 2.1 for any $R\geq R_0$ there exists a function $u_R\in\overset{0}{W}{}^{\aleph}_{p}(\Omega_R)$ satisfying (7) and (8). Further, if u_{R_j} is a solution of problem (7), (8) for $R=R_j$ ($R_j\geq R_0$), then in view of Lemma 2.1

$$c'\left\|u_{R_j}\right\|^p_{\aleph,\Omega_{R_j},p}-c''-\left(\sum_{\alpha\in\aleph^{(l)}}\|h_\alpha\|_{L_q(\Omega)}\right)\left\|u_{R_j}\right\|_{\aleph,\Omega_{R_j},p}\leq\sum_{\alpha\in\aleph}\int_{\Omega_{R_j}}f_\alpha\,D^\alpha u_{R_j}\,dx\leq$$

$$\leq\sum_{\alpha\in\aleph}\|f_\alpha\|_{L_q(\Omega)}\left\|u_{R_j}\right\|_{\aleph,\Omega_{R_j},p}\,.$$

Hence we obtain the boundedness of the sequence (u_{R_j}) in $\overset{0}{W}{}^{\aleph}_{p}(\Omega_{R_j})$. Therefore, by condition b) the sequence $(\Phi_{R_j}u_{R_j})$ is bounded in $\overset{0}{W}{}^{\aleph}_{p}(\Omega)$. By virtue of conditions A_0) and A_1) the sequence $T_{R_j}(\Phi_{R_j}u_{R_j})$ is also bounded in $(\overset{0}{W}{}^{\aleph}_{p}(\Omega))^*$. Since $W^{\aleph}_{p}(\Omega)$ is a reflexive Banach space, there are a subsequence (R'_j) of the sequence (R_j) and $u\in\overset{0}{W}{}^{\aleph}_{p}(\Omega)$, $y\in(\overset{0}{W}{}^{\aleph}_{p}(\Omega))^*$

for which

$$\Phi_{R'_j} u_{R'_j} \text{ tends to } u^* \text{ weakly in } \overset{\mathbf{0}}{W}{}_p^{\aleph}(\Omega) \,, \tag{9}$$

$$T_{R'_j}(\Phi_{R'_j} u_{R'_j}) \text{ tends to } y \text{ weakly in } (\overset{\mathbf{0}}{W}{}_p^{\aleph}(\Omega))^* \tag{10}$$

when $j \to \infty$. Since $u_{R'_j}$ is a solution of the problem (7), (8) for $R = R'_j$ it follows

$$\left\langle T_{R'_j}(\Phi_{R'_j} u_{R'_j}), \Phi_{R'_j} u_{R'_j} \right\rangle + \sum_{\alpha \in \aleph^{(l)}} \int_{\Omega_{R'_j}} g_\alpha(x, D(u_{R'_j})) D^\alpha u_{R'_j} dx = \sum_{\alpha \in \aleph} \int_{\Omega_{R'_j}} f_\alpha D^\alpha u_{R'_j} dx. \tag{11}$$

Therefore

$$\sum_{\alpha \in \aleph^{(l)}} \int_{\Omega_{R'_j}} g_\alpha(x, D(u_{R'_j})) D^\alpha u_{R'_j} dx \le \sum_{\alpha \in \aleph} \| f_\alpha \|_{L_q(\Omega)} \left\| u_{R'_j} \right\|_{\aleph, \Omega_{R'_j}, p} +$$

$$+ \left\| T_{R'_j}(\Phi_{R'_j} u_{R'_j}) \right\|_{(\aleph,\Omega)^*, p} \left\| \Phi_{R'_j} u_{R'_j} \right\|_{\aleph, \Omega, p} \,, \quad (\| . \|_{(\aleph,\Omega)^*} \text{ is the norm in } (\overset{\mathbf{0}}{W}{}_p^{\aleph}(\Omega))^*).$$

The right-hand side of this inequality is bounded, therefore, taking into account (9), condition $G_3)$, Lemma 2.2 and Lemma 1.3 of [7] we find: $u = u^*$ satisfies conditions (4), (5), there exists a subsequence (R''_j) of the sequence (R'_j) such that

$$\lim_{j \to \infty} D^\beta (\Phi_{R''_j} u_{R''_j}) = D^\beta u^* \text{ a.e. in } \Omega \tag{12}$$

$$\lim_{j \to \infty} \left\| g_{\alpha, R''_j}(\cdot\,, D(u_{R''_j})) - g_\alpha(\cdot\,, D(u^*)) \right\|_{L_1(\Omega)} = 0. \tag{13}$$

if $\beta \in \aleph^{(0)}$. The function $u_{R''_j}$ is a solution of the problem (7), (8) for $R = R''_j$. Thus, for any function $v \in \overset{\mathbf{0}}{W}{}_p^{\aleph}(\Omega)$ satisfying the condition $D^\alpha v \in L_\infty(\Omega) \quad \forall \alpha \in \aleph^{(l)}$ we have

$$\left\langle T_{R''_j}(\Phi_{R''_j} u_{R''_j}), v \right\rangle + \sum_{\alpha \in \aleph^{(l)}} \int_{\Omega_{R''}} g_{\alpha, R''_j}(x, D(u_{R''_j})) D^\alpha v\, dx = \sum_{\alpha \in \aleph} \int_{\Omega_{R''_j}} f_\alpha D^\alpha v\, dx.$$

Therefrom and by virtue of (10) and (13) we obtain

$$\langle y, v \rangle + \sum_{\alpha \in \aleph^{(l)}} \int_{\Omega} g_\alpha(x, D(u^*)) D^\alpha v\, dx = \sum_{\alpha \in \aleph} \int_{\Omega} f_\alpha D^\alpha v\, dx. \tag{14}$$

for $v \in \overset{0}{W}{}_p^{\aleph}(\Omega)$, $D^\alpha v \in L_\infty(\Omega) \quad \forall \alpha \in \aleph^{(\mathbf{0})}$.

Now we prove that

$$y = T(u^*). \tag{15}$$

As a consequence it follows from (14) and Proposition 1.1 that $u = u^*$ satisfies (3). In order to prove (15) it suffices to show the inequality

$$\limsup_{j\to\infty} \left\langle T_{R''_j}(\Phi_{R''_j} u_{R''_j}),\, \Phi_{R''_j} u_{R''_j} - u^* \right\rangle \le 0 . \qquad (16)$$

Then, by Lemma 2.3 (15) follows from (9), (10) and (16). Equality (11) implies

$$\left\langle T_{R_j}(\Phi_{R''_j} u_{R''_j}), \Phi_{R''_j} u_{R''_j} - u^* \right\rangle = \sum_{\alpha\in\aleph} \int_{\Omega_{R''_j}} f_\alpha D^\alpha u_{R''_j} dx - \sum_{\alpha\in\aleph^{(l)}} \int_{\Omega_{R''_j}} g_\alpha(x,D(u_{R''_j}))D^\alpha u_{R''_j} dx -$$

$$- \left\langle T_{R''_j}(\Phi_{R''_j} u_{R''_j}),\, u^* \right\rangle . \qquad (17)$$

Using Fatou's lemma by means of (12) and condition $G_1)$ we obtain

$$\int_\Omega p_\alpha(x,D(u^*))D^\alpha u^* \, dx \le \liminf_{j\to\infty} \int_{\Omega_{R''_j}} p_\alpha(x,D(u_{R''_j}))D^\alpha u_{R''_j} \, dx . \qquad (18)$$

In view of condition $G_1)$ and Hölder 's inequality the inequality

$$\left| \int_E r_\alpha(x, D(u_{R''_j}))D^\alpha u_{R''_j} \, dx \right| \le c \left\{ \int_E |h_\alpha|^q \right\}^{1/q} \cdot \left\| u_{R''_j} \right\|_{\aleph, \Omega_{R''_j}, p} ,$$

holds for any measurable set E. Therefore, by Vitali's theorem (see [8], p. 203)

$$\lim_{j\to\infty} \int_{\Omega_{R''_j}} r_\alpha(x, D(u_{R''_j}))D^\alpha u_{R''_j} \, dx = \int_\Omega r_\alpha(x, D(u^*))D^\alpha u^* \, dx . \qquad (19)$$

Combining (18) and (19) we find that

$$\limsup_{j\to\infty} \left[- \sum_{\alpha\in\aleph^{(l)}} \int_{\Omega_{R''_j}} g_\alpha(x,D(u_{R''_j}))D^\alpha u_{R''_j} \, dx \right] \le - \sum_{\alpha\in\aleph^{(l)}} \int_\Omega g_\alpha(x,D(u^*))D^\alpha u^* \, dx . \qquad (20)$$

The first term on the right-hand side of (17) can be written in the form

$$\sum_{\alpha\in\aleph} \int_{\Omega_{R''_j}} f_\alpha D^\alpha u_{R''_j} \, dx = \sum_{\alpha\in\aleph} \int_\Omega f_\alpha D^\alpha (\Phi_{R''_j} u_{R''_j}) dx - \sum_{\alpha\in\aleph} \int_{\Omega\setminus\Omega_{R''_j}} f_\alpha D^\alpha (\Phi_{R''_j} u_{R''_j}) dx . \qquad (21)$$

Using Hölder's inequality and Vitali's theorem we obtain

$$\lim_{j\to\infty} \int_{\Omega\setminus\Omega_{R''_j}} f_\alpha D^\alpha (\Phi_{R''_j} u_{R''_j}) dx = 0 ,$$

Therefore it follows from (9), and (21) that

$$\lim_{j\to\infty} \sum_{\alpha\in\aleph} \int_{\Omega_{R''_j}} f_\alpha D^\alpha u_{R''_j} \, dx = \left\langle f, u^* \right\rangle . \qquad (22)$$

As a consequence of (10), (17), (20), and (22) we obtain

$$\limsup_{j\to\infty} \left\langle T_{R''_j}(\Phi_{R''_j} u_{R''_j}), \Phi_{R''_j} u_{R''_j} - u^* \right\rangle \le \left\langle f - y, u^* \right\rangle - \sum_{\alpha\in\aleph^{(l)}} \int_\Omega g_\alpha(x, D(u^*)) D^\alpha u^* dx \ .$$

Hence it follows by (4) that

$$\limsup_{j\to\infty} \left\langle T_{R''_j}(\Phi_{R''_j} u_{R''_j}), \Phi_{R''_j} u_{R''_j} - u^* \right\rangle \le \left\langle f - y, u^* - \omega_k \right\rangle +$$

$$+ \sum_{\alpha\in\aleph^{(l)}} \int_\Omega g_\alpha(x, D(u^*))(D^\alpha \omega_k - D^\alpha u^*) dx \ , \tag{23}$$

where ω_k is a function satisfying Proposition 1.1 for $u = u^*$.

Proposition 1.1 yields

$$\lim_{k\to\infty} \left\langle f - y, u^* - \omega_k \right\rangle = 0 \ . \tag{24}$$

We observe that for some subsequence (ω'_k) of the sequence (ω_k) we have

$$\lim_{k\to\infty} D^\alpha \omega'_k = D^\alpha u^* \text{ a.e. in } \Omega \text{ for } \alpha \in \aleph^{(0)} .$$

Further, $u = u^*$ satisfies (4) and (5). Hence, applying Lebesgue's theorem and Proposition 1.1 we obtain

$$\lim_{k\to\infty} \sum_{\alpha\in\aleph^{(l)}} \int_\Omega g_\alpha(x, D(u^*))(D^\alpha \omega'_k - D^\alpha u^*) dx = 0 \ .$$

Therefore, (16) follows from (23) and (24). Thus, by (14) and (15) $u = u^*$ satisfies (3) for all $v \in \overset{0}{W}{}^{\aleph}_p(\Omega)$ for which $D^\alpha v \in L_\infty(\Omega)$ $\forall \alpha \in \aleph^{(l)}$. Since (3) holds for $v = \omega'_k$ we obtain (3) also for $v = u^*$ if $k \to \infty$. Finally, if the solution of problem (3) - (5) is unique then the sequence $(\Phi_{R_j} u_{R_j})$ converges weakly to u in $\overset{0}{W}{}^{\aleph}_p(\Omega)$, otherwise, by the method of the first part of the proof of Theorem 2.1 it is not difficult to obtain a contradiction.

Remark 2.2. *Let* $\aleph = \left\{\alpha \in N_0^n \ ; (\mu, \alpha) \le k \right\}$*, where* μ *is a vector with positive components and* k *is a natural number. Then the above results are true without the condition (A.C.) (see [1],§ 3).*

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