The "More for Less"-Paradox in Transportation Problems with Infinite-Dimensional Supply and Demand Vectors

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Abstract

Recently Deineko, Klinz, and Woeginger have shown that a transportation problem is immune against the "more for less"-paradox if and only if the cost matrix $C = (c_{i,j})$ (of dimension $m \times n$) does not contain a bad quadruple. In this note a counter-example with *infinite-dimensional* supply and demand vectors is given.

In the second part we show that the quadruple-characterization of paradox-immune cost matrices remains valid in the infinite-dimensional case in a slightly weaker form. As a side result a smooth inequality is obtained for the situation where a transportation plan is split in two or more arbitrary subplans.

Keywords: transportation problem - infinite transportation problem - transportation paradox - combinatorial optimization

1 Introduction

An instance of the classical transportation problem is specified by an $m \times n$ matrix $C = (c_{ij})$, an m-dimensional vector $a = (a_i)$, and an n-dimensional vector $b = (b_j)$; all numbers c_{ij}, a_i, b_j are nonnegative real numbers. This data has the following meaning: The are m sources and n sinks; at the ith source there is a supply of a_i units, and at the jth sink there is a demand of b_j units. It is assumed, that $\sum_{i=0}^{m-1} a_i = \sum_{j=0}^{n-1} b_j$, i.e., that the total supply equals the total demand. The cost for transporting one unit from the ith source to the jth sink is c_{ij}. The goal is to find a transportation plan that satisfies all the demand and that minimizes the overall transportation cost:

$$\min \quad \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} c_{ij} x_{ij}$$

$$\begin{array}{lll} s.t. & \sum_{j=0}^{n-1} x_{ij} = a_i & \text{for } i = 0, \ldots, m-1 \\ & \sum_{i=0}^{m-1} x_{ij} = b_j & \text{for } j = 0, \ldots, n-1 \\ & x_{ij} \geq 0 & \text{for } i = 0, \ldots, m-1,\ j = 0, \ldots, n-1 \end{array}$$

Here x_{ij} denotes the quantity shipped from source i to sink j. Let X denote the set of all transportation plans $x = (x_{ij})$ that fulfill the transportation constraints above. Moreover,

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let TP(C, a, b) denote the optimal objective value of the transportation instance specified by C, a, and b. We refer the reader to the book by Ahuja, Magnanti & Orlin [1] for more information on the transportation problem and to the survey article of Burkard, Klinz, and Rudolf on generalized transportation problems [2].

Here we investigate the transportation problem with *infinite-dimensional* supply and demand vectors $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$. In this case the cost matrix C has entries c_{ij} for all natural numbers i and j. As before, a transportation plan is given by nonnegative values x_{ij} for all i and j, such that the marginal sum conditions $\sum_{j=0}^{\infty} x_{ij} = a_i$ for all i and $\sum_{i=0}^{\infty} x_{ij} = b_j$ for all j hold. All parameters a_i, b_j, c_{ij}, x_{ij} are nonnegative, so it makes sense to compute the infinite sum $\sum_{ij}(x_{ij}c_{ij})$, hoping for absolute convergence. We call (C, a, b) a transportation problem with infinite-dimensional supply and demand.

Transportation problems with infinite-dimensional supply and demand occur for instance in the study of discrete random processes: Consider two random variables X and Y, both with countable state spaces. X and Y may depend on each other, but instead of the probabilities p_{ij} for all pairs (i, j) we only know the marginal probabilities $a_i = \Pr(X = i)$ and $b_j = \Pr(Y = j)$. So, $\sum_i a_i = \sum_j b_j = 1$ in this case. Given cost parameters c_{ij} for all possible realizations $(X = i, Y = j)$, it is of interest to know how small (or large) the expected total cost $\sum_{ij} p_{ij} \cdot c_{ij}$ may be. Observe that this question is nontrivial only in the case where the cost c_{ij} do not have a sum decomposition (like $c_{ij} = d_i + e_j$ for all i and j).

In the beginning 1970s, Charnes & Klingman [3] and Szwarc [5] independently of each other identified the *transportation paradox*: In certain cases of the transportation problem, an *increase* in the supplies and demands may lead to a *decrease* in the optimal transportation costs. In other words, by moving bigger amounts of goods around, one may save money. This surely sounds paradoxical! Recently Deineko, Klinz, and Woeginger [4] gave a simple characterization of those finite-dimensional cost matrices $C = (c_{ij})$ which are immune against this transportation paradox. Four indices q, r, s, t with $q \neq s$ and $r \neq t$ are said to form a bad quadruple if $c_{qt} + c_{sr} < c_{qr}$.

Theorem 1.1 (Deíneko, Klinz, Woeginger [4]) *An $m \times n$ cost matrix $C = (c_{ij})$ is immune against the transportation paradox, if and only if C does not contain a bad quadruple.*

In the sequel, let us call c_{qr} the anchor of the quadruple (q, r, s, t).

In this note we present the following results. Section 2 gives a counter-example with infinite-dimensional supply and demand: Its cost matrix does not contain a bad quadruple, but nevertheless the problem exhibits the "more for less"-paradox. In Section 3 we prove that the characterization of immune cost matrices by the absence of bad quadruples remains valid in case of finite supply $\sum_{i=0}^{\infty} a_i$. Moreover, the characterization remains valid even in the case of infinite supply $\sum_{i=0}^{\infty} a_i$ when we substitute "minimum cost" by "infimum cost". Furthermore, a smooth inequality is given for the situation where a transportation plan is split in two or more arbitrary subplans. Section 4 contains concluding remarks.

2 A Counter-Example with Infinite-dimensional Supply and Demand Vectors

Define a cost matrix C by $c_{ii} = 2^{-i}$ for $i = 0, 1, 2, \ldots$ and $c_{ij} = 2^{-i} + 2^{-j}$ for all pairs i, j with $i \neq j$. Obviously, C does not contain a bad quadruple.

Define supply vectors $a = (a_i)$ and $a' = (a'_i)$ by

$$\begin{array}{ll} a_0 = 0, a_i = 1 & \text{for all } i \geq 1, \\ a'_i = 1 & \text{for all } i \geq 0, \end{array}$$

and demand vectors $b = (b_j)$ and $b' = (b'_j)$ by

$$b_j = b'_j = 1 \quad \text{for all } j \geq 0.$$

We have $a \geq a'$ component-wise (with inequality only in component $i = 0$), and $b = b'$. The best transportation plan for (C, a', b') is to use the main diagonal only, by setting $x'_{ii} = 1$ for all i. This gives total cost $1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2$.

In (C, a, b) the situation is less smooth, as we have to set $x_{00} = 0$ because of $a_0 = 0$. Let any feasible transportation plan for (C, a, b) be given, and count its cost column-wise. For each $i > 0$, column i will contribute at least c_{ii} to the cost, as $b_i = 1$ and c_{ii} is the cheapest element in this column. Now assume that $b_0 = 1$ is collected via some vector $(x_{00}, x_{10}, x_{20}, \ldots)$ with $x_{00} = 0$, $x_{i0} \geq 0$ for all $i > 0$, and $\sum x_{i0} = 1$. $c_{i0} > c_{00}$ for all $i > 0$, hence column 0 gives cost $> c_{00}$, and thus total cost $> c_{00} + c_{11} + c_{22} + \ldots =$ optimal cost of (C, a', b').

Two generalizations:

- Each element outside of the main diagonal belongs to an "indifferent quadruple", as $c_{ij} = c_{ii} + c_{jj}$ for $i \neq j$. By setting $c'_{ij} = 2^{-i} + 2^{-j} - 3^{-(i+j)}$ for all $i \neq j$, the vectors a, a', $b = b'$ from above still give the paradox, although the cost matrix C' no longer has any bad or indifferent quadruple.
- Instead of $a = (0, 1, 1, 1, ...)$ also any other 0-1-vector a with infinitely many 1-entries and at least one 0-entry leads to the paradox, when put together with a' and $b = b'$.
- Even more generally, any vector $(a_i)_{i \in \mathbb{N}}$ with $a_i \leq 1$ for all i, $a_k < 1$ for at least one k, and $\sum_{i=0}^{\infty} a_i = \infty$ leads to the paradox.

3 Weak Immunity against the Transportation Paradox

3.1 Infinite Sums

The example of Section 2 is not too worse, as (C, a, b) has transportation plans with costs arbitrarily close to the optimal cost of (C, a', b'). Here we show thas this is always the case.

Theorem 3.1 *Let $C = (c_{ij})_{i,j \in \mathbb{N}}$ be a non-negative cost matrix without bad quadruples. Let $a = (a_i)_{i \in \mathbb{N}}$, $b = (b_j)_{j \in \mathbb{N}}$, $a' = (a'_i)_{i \in \mathbb{N}}$, $b' = (b'_j)_{j \in \mathbb{N}}$ be infinite-dimensional supply and demand vectors such that $0 \leq a_i \leq a'_i$, $0 \leq b_j \leq b'_j$ for all i,j and $\sum_{i=0}^{\infty} a_i = \sum_{j=0}^{\infty} b_j$, $\sum_{i=0}^{\infty} a'_i = \sum_{j=0}^{\infty} b'_j$. Let $x' = (x'_{ij})$ be an admissible transportation plan for (C, a', b'), and assume any constant $\varepsilon > 0$. Then there exists an admissible transportation plan x for (C, a, b), such that*

$$\text{cost}(x) \leq \text{cost}(x') + \varepsilon.$$

The proof will make use of the following auxiliary result which is interesting in its own light. Therefore, we call it also "theorem" and not only "lemma". The proof of Theorem 3.1 will be given after that of Theorem 3.2.

Theorem 3.2 *Let $C = (c_{ij})_{i,j\in\mathbb{N}}$ be a non-negative cost matrix without bad quadruples. Let supply and demand vectors a, b, a', b' be given like in Theorem 3.1. Let x and x' be arbitrary admissible transportation plans for (C, a, b) and (C, a', b'), respectively. Let* $\mathrm{cost}(x')$ *be finite. Then*

$$\mathrm{cost}(x) \leq 2\,\mathrm{cost}(x').$$

Proof of Theorem 3.2:
Define $d_i = \inf_j c_{ij}$ for all i and $e_j = \inf_i c_{ij}$ for all j. For each pair (i, j) we have

$$c_{ij} \leq d_i + e_j$$

because otherwise c_{ij} would be the anchor of a bad quadruple. By definition of d_i and e_j we also have:

$$c_{ij} \geq d_i \quad \text{and} \quad c_{ij} \geq e_j. \tag{3.1}$$

The cost of plan x may be bounded from above by the following chain of transformations

$$\begin{aligned}
\mathrm{cost}(x) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij}c_{ij} \\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij}(d_i + e_j) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij}d_i + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij}e_j \\
&= \sum_{i=0}^{\infty} a_i d_i + \sum_{j=0}^{\infty} b_j e_j.
\end{aligned} \tag{3.2}$$

Without loss of generality assume

$$\sum_{i=0}^{\infty} d_i a_i \quad \geq \quad \sum_{j=0}^{\infty} e_j b_j. \tag{3.3}$$

The cost of plan x' may be bounded from below by

$$\begin{aligned}
\mathrm{cost}(x') &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x'_{ij}c_{ij} \\
&\overset{(3.1)}{\geq} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x'_{ij}d_i \\
&= \sum_{i=0}^{\infty} d_i a'_i \\
&\geq \sum_{i=0}^{\infty} d_i a_i \\
&\overset{(3.3)}{\geq} \frac{1}{2} \cdot \left[\sum_{i=0}^{\infty} a_i d_i + \sum_{j=0}^{\infty} b_j e_j\right] \\
&\overset{(3.2)}{\geq} \frac{1}{2}\,\mathrm{cost}(x).
\end{aligned}$$

This completes the proof of Theorem 3.2. ■

Proof of Theorem 3.1:
The key idea is to split, for some appropriate indices $m = m(\varepsilon)$ and $n = n(\varepsilon)$, the supply and demand vectors a and b in two pairs a^1, b^1 and a^2, b^2 of vectors with $a^1 = (a_0, \dots, a_m, 0, 0, \dots)$, $b^1 = (b_0, \dots, b_n, 0, 0, \dots)$ and $a^2 = (0, \dots, 0, a_{m+1}, a_{m+2}, \dots)$, $b^2 = (0, \dots, 0, b_{n+1}, b_{n+2}, \dots)$ and to show that there are transportation plans x^1 for (C, a^1, b^1) and x^2 for (C, a^2, b^2) with $\mathrm{cost}(x^1) \le \mathrm{cost}(x')$ and $\mathrm{cost}(x^2) \le \varepsilon$. The existence of x^1 will follow by an application of Theorem 1.1. x^2 will be a diagonal transportation plan whose cost can be bounded by an application of Theorem 3.2. In detail, the proof works as follows.

Let x' be given for (C, a', b') with finite total cost $\mathrm{cost}(x') = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x'_{ij} c_{ij}$. Select $m = m(\varepsilon) \in \mathbb{N}$, such that

$$\sum_{i=0}^{m} \sum_{j=0}^{m} x'_{ij} c_{ij} \quad \ge \quad \mathrm{cost}(x') - \frac{1}{2}\varepsilon. \tag{3.4}$$

Without loss of generality assume $\sum_{i=0}^{m} a_i \ge \sum_{j=0}^{m} b_j$. Now let be $n \ge m$, such that $\sum_{j=0}^{n-1} b_j < \sum_{i=0}^{m} a_i \le \sum_{j=0}^{n} b_j$. By an artifical doubling of column n in two columns (with identical costs c_{in}) and appropriate splits of b_n and b'_n we can achieve that $\sum_{i=0}^{m} a_i = \sum_{j=0}^{n} b_j$.

The splits of b_n and b'_n can be done such a way that the monotonicity $b_j \le b'_j$ for all j is preserved. Because of $n \ge m$ and inequality (3.4) we also have

$$\sum_{i=0}^{m} \sum_{j=0}^{n} x'_{ij} c_{ij} \ge \mathrm{cost}(x') - \frac{1}{2}\varepsilon. \tag{3.5}$$

Construction of the first subproblem:
The problem is finite-dimensional and has $m+2$ rows and $n+2$ columns. Its cost-matrix C^1 is the $(m+1) \times (n+1)$ starting block of C plus some extra, namely in total

$$c^1_{ij} = \begin{cases} c_{ij} & \text{for } 0 \le i \le m \text{ and } 0 \le j \le n, \\ \inf_{l>n} c_{il} & \text{for } i \le m \text{ and } j = n+1, \\ \inf_{k>m} c_{kj} & \text{for } i = m+1 \text{ and } j \le n, \\ \inf_{k>m, l>n} c_{kl} & \text{for } i = m+1 \text{ and } j = n+1. \end{cases}$$

It is easy to check that C^1 does not contain a bad quadruple. Supply a^1 and demand b^1 are defined by

$$a^1_i = \begin{cases} a_i & \text{if } i \le m, \\ 0 & \text{if } i = m+1, \end{cases}$$

$$b^1_j = \begin{cases} b_j & \text{if } j \le n, \\ 0 & \text{if } j = n+1. \end{cases}$$

The definition of the vectors a'^1 and b'^1 depends on the plan x'.

$$a'^1_i = \begin{cases} a'_i & \text{if } i \le m, \\ \sum_{j=0}^{n} b'_j - \sum_{i=0}^{m} \sum_{j=0}^{n} x'_{ij} & \text{if } i = m+1, \end{cases}$$

$$b'^1_j = \begin{cases} b'_j & \text{if } j \le n, \\ \sum_{i=0}^{m} a'_i - \sum_{i=0}^{m} \sum_{j=0}^{n} x'_{ij} & \text{if } j = n+1. \end{cases}$$

By admissibility of x' we have $a'^1_{m+1} \geq 0$, $b'^1_{n+1} \geq 0$, and equality of $\sum_{i=0}^{m+1} a'^1_i = \sum_{i=0}^{n+1} b'^1_j$. Furthermore, $a^1_i \leq a'^1_i$ for all i and $b^1_j \leq b'^1_j$ for all j. For the transportation problem (C^1, a'^1, b'^1) there is a very natural transportation plan $S^1 = (s^1_{ij})$, namely

$$s^1_{ij} = \begin{cases} x'_{ij} & \text{for } i \leq m \text{ and } j \leq n, \\ a'_i - \sum_{k=0}^{n} x'_{ik} & \text{for } i \leq m \text{ and } j = n+1, \\ b'_j - \sum_{k=0}^{m} x'_{kj} & \text{for } i = m+1 \text{ and } j \leq n, \\ 0 & \text{for } i = m+1 \text{ and } j = n+1. \end{cases}$$

It is $s^1_{m+1,j} = \sum_{i=m+1}^{\infty} x'_{ij}$ for all $j \leq n$ and $s^1_{i,n+1} = \sum_{j=n+1}^{\infty} x'_{ij}$ for all $i \leq m$.

The cost of S^1 is upper bounded by the cost of x' in (C, a', b'), namely

$$\begin{aligned}
\text{cost}(S^1) &= \sum_{i=0}^{m}\sum_{j=0}^{n} x'_{ij}c_{ij} + \sum_{j=0}^{n} s_{m+1,j} \inf_{k>m} c_{kj} + \sum_{i=0}^{m} s_{i,n+1} \inf_{k>n} c_{ik} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} x'_{ij}c_{ij} + \sum_{j=0}^{n}\sum_{i=m+1}^{\infty} x'_{ij} \inf_{k>m} c_{kj} + \sum_{i=0}^{m}\sum_{j=n+1}^{\infty} x'_{ij} \inf_{k>n} c_{ik} \\
&\leq \sum_{i=0}^{m}\sum_{j=0}^{n} x'_{ij}c_{ij} + \sum_{j=0}^{n}\sum_{i=m+1}^{\infty} x'_{ij}c_{ij} + \sum_{i=0}^{m}\sum_{j=n+1}^{\infty} x'_{ij}c_{ij} \\
&\leq \sum_{i=0}^{m}\sum_{j=0}^{n} x'_{ij}c_{ij} + \sum_{i=m+1}^{\infty}\sum_{j=0}^{n} x'_{ij}c_{ij} + \sum_{i=0}^{m}\sum_{j=n+1}^{\infty} x'_{ij}c_{ij} + \sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} x'_{ij}c_{ij} \\
&= \text{cost}(x').
\end{aligned}$$

As $a^1 \leq a'^1$ and $b^1 \leq b'^1$, Theorem 1.1 implies the existence of a transportation plan x^1 for (C^1, a^1, b^1) with $\text{cost}(x^1) \leq \text{cost}(S) \leq \text{cost}(x')$.

Construction of the second subproblem:
This problem has countably many rows and columns. Its cost matrix will be the cost matrix C of the original problem. Hence there is no bad quadruple. The supplies a^2 and demands b^2 are given by

$$\begin{aligned}
a^2_i &= \begin{cases} 0 & \text{if } i \leq m, \\ a_i & \text{if } i \geq m+1, \end{cases} \\
b^2_j &= \begin{cases} 0 & \text{if } j \leq n, \\ b_j & \text{if } j \geq n+1. \end{cases}
\end{aligned}$$

Because of $\sum_{i=0}^{\infty} a_i = \sum_{j=0}^{\infty} b_j$ and $\sum_{i=0}^{m} a_i = \sum_{j=0}^{n} b_j$ we also have the equality

$$\sum_{i=0}^{\infty} a^2_i = \sum_{j=0}^{\infty} b^2_j.$$

The supplies a'^2 and demands b'^2 are given by

$$\begin{aligned}
a'^2_i &= \begin{cases} a'_i - \sum_{j=0}^{n} x'_{ij} & \text{if } i \leq m, \\ a'_i & \text{if } i \geq m+1, \end{cases} \\
b'^2_j &= \begin{cases} b'_j - \sum_{i=0}^{m} x'_{ij} & \text{if } j \leq n, \\ b'_j & \text{if } j \geq n+1. \end{cases}
\end{aligned}$$

Again we have equality, namely

$$\begin{aligned}\sum_{i=0}^{\infty} a_i'^2 &= \sum_{i=0}^{\infty} a_i' - \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}' \\ &= \sum_{j=0}^{\infty} b_j' - \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}' \\ &= \sum_{j=0}^{\infty} b_j'^2\end{aligned}$$

and monotonicity $a_i^2 \leq a_i'^2$ for all i and $b_j^2 \leq b_j'^2$ for all j.

From x' we can derive an admissible transportation plan $S^2 = (s_{ij}^2)$ for (C, a'^2, b'^2) by setting

$$s_{ij}^2 = \begin{cases} 0 & \text{if } i \leq m \text{ and } j \leq n \\ x_{ij}' & \text{otherwise.}\end{cases}$$

This leads to

$$\begin{aligned}\mathrm{cost}(S^2) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij}' c_{ij} - \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}' c_{ij} \\ &\overset{(3.5)}{\leq} \frac{1}{2}\varepsilon.\end{aligned}$$

Application of Theorem 3.2 shows that every admissible transportation plan for (C, a^2, b^2) has cost at most $2 \cdot \mathrm{cost}(S^2) \leq \varepsilon$. There exist admissible transportation plans x^2 for (C, a^2, b^2), namely for instance the simple one constructed by the north-west rule:

$$x_{m+1,n+1}^2 = \min\{a_{m+1}, b_{n+1}\}$$

and so on.

Both plans x^1 and x^2 may be merged to a plan x by setting

$$x_{ij} = \begin{cases} x_{ij}^1 & \text{if } i \leq m \text{ and } j \leq n, \\ x_{ij}^2 & \text{if } i > m \text{ and } j > n, \\ 0 & \text{otherwise.}\end{cases}$$

x is an admissible plan for (C, a, b), with total cost

$$\mathrm{cost}(x) \leq \mathrm{cost}(x^1) + \mathrm{cost}(x^2) \leq \mathrm{cost}(x') + \varepsilon$$

This completes the proof of Theorem 3.1. ■

3.2 Finite Sums

In case of finite sums $\sum_{i=0}^{\infty} a_i = \sum_{j=0}^{\infty} b_j$ Theorem 3.1 may be improved by the following result.

Theorem 3.3 *Let (C, a, b) be a transportation problem with infinite-dimensional supplies and demands and $\sum_{i=0}^{\infty} a_i = \sum_{j=0}^{\infty} b_j < \infty$. When there exists any transportation plan $y = (y_{ij})$ with finite cost, then there exists a transportation plan for (C, a, b) with minimum cost, i.e. the infimum is realized.*

Observe that here except for the existence of plan y no other structure of C is demanded.

Proof of Theorem 3.3:

Let $c^* = \inf \mathrm{cost}(x)$, where the infimum is taken over all admissible transportation plans x for (C, a, b). It is sufficient to prove the existence of a sequence $(x(t))_{t\in\mathbb{N}}$ of admissible plans for (C, a, b) with the following three properties:

i. $\lim_{t\to\infty} \mathrm{cost}(x(t)) = c^*$ (convergence of cost).

ii. $\lim_{t\to\infty} x_{ij}(t) =: x_{ij}$ exists for all (i, j) (convergence of $x(t)$).

iii. $\sum_{j=0}^{\infty} x_{ij} = a_i$ for all i and $\sum_{i=0}^{\infty} x_{ij} = b_j$ for all j (admissibility of the limit plan).

Then $x = (x_{ij})$ is a plan with

$$\mathrm{cost}(x) = c^*. \tag{3.6}$$

Assume to the contrary that $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij}c_{ij} \geq c^* + 3\varepsilon$ for some $\varepsilon > 0$.

Then there exists some index $n \in \mathbb{N}$ such that

$$\sum_{i=0}^{n}\sum_{j=0}^{n} x_{ij}c_{ij} \geq c^* + 2\varepsilon. \tag{3.7}$$

By convergence of the $x_{ij}(t)$ to x_{ij} there exists a $T \in \mathbb{N}$ such that

$$c_{ij}x_{ij}(t) > c_{ij}x_{ij} - \frac{1}{(n+1)^2}\varepsilon \tag{3.8}$$

for all $i \leq n$, $j \leq n$, $t \geq T$. Putting (3.8) into (3.7) gives for all $t \geq T$

$$\begin{aligned}
\sum_{i=0}^{n}\sum_{j=0}^{n} x_{ij}(t)c_{ij} &> \sum_{i=0}^{n}\sum_{j=0}^{n}\left[x_{ij}c_{ij} - \frac{\varepsilon}{(n+1)^2}\right] \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} x_{ij}c_{ij} - \varepsilon \\
&\geq c^* + \varepsilon.
\end{aligned}$$

By non-negativity of all $x_{ij}(t)$ and c_{ij} we have $\mathrm{cost}(x(t)) \geq \sum_{i=0}^{n}\sum_{j=0}^{n} x_{ij}(t)c_{ij} \geq c^* + \varepsilon$, which is a contradiction to the convergence of $\mathrm{cost}(x(t))$ to c^*. So, property (3.6) holds for the limit plan x.

It remains to show the existence of a sequence $(x(t))_{t\in\mathbb{N}}$ with the properties (i) to (iii) from above. The most critical part will be the inconspicuous condition (iii). Namely, for instance, the following sample sequence would not work: $x_{0t}(t) = 1$, $x_{0j}(t) = 0$ for all $j \neq t$. Here $\sum_{j=0}^{\infty} x_{0j}(t) = 1$ for all t, but $\lim_{t\to\infty} x_{0j}(t) =: x_{0j} = 0$ for all $j \in \mathbb{N}$ and thus $\sum_{j=0}^{\infty} x_{0j} = 0 \neq 1$.

But first of all let us check for condition (ii). Let $x(t)$ be an arbitrary sequence satisfying the cost convergence (i). Such a sequence exists by the definition of c^*. It is $0 \leq x_{ij}(t) \leq$

$\min\{a_i, b_j\}$ for all t, hence for each pair (i,j) the sequence $(x_{ij}(t))_{t=0}^{\infty}$ is bounded in a compact interval.

Therefore there exists a subsequence $(t_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{00}(t_k) =: x_{00}$ exists. In this subsequence there exists a subsequence with existing $\lim x_{01}(t) =: x_{01}$. So, running trough all pairs (i,j) by an infinite repetition of this subsequence argument gives the desired limit plan $x = (x_{ij})$. It remains to show (iii).

Have a look at a_0 and assume $\sum_{j=0}^{\infty} x_{0j} \neq a_0$.

Case 1 $\sum_{j=0}^{\infty} x_{0j} \geq a_0 + \delta$ for some $\delta > 0$.

Then there exists an $n \in \mathbb{N}$ with $\sum_{j=0}^{n-1} x_{0j} \geq a_0 + \frac{1}{2}\delta$.

For each $j \in \{0, \ldots, n-1\}$ there exists some T_j such that $x_{0j}(t) > x_{0j} - \frac{\delta}{2n}$ for all $t \geq T_j$. Let $T = \max\{T_0, \ldots, T_{n-1}\}$.

Then for all $t \geq T$

$$\begin{aligned} \sum_{j=0}^{n-1} x_{0j}(t) &> \sum_{j=0}^{n-1} \left[x_{0j} - \frac{\delta}{2n} \right] \\ &= \sum_{j=0}^{n-1} x_{0j} - \frac{1}{2}\delta \\ &\geq a_0 + \frac{1}{2}\delta - \frac{1}{2}\delta \\ &= a_0 \end{aligned}$$

Hence also $\sum_{j=0}^{\infty} x_{0j}(t) > a_0$, and $x(t)$ would not have been admissible. So, Case 1 can not happen.

Case 2 $\sum_{j=0}^{\infty} x_{0j} \leq a_0 - \varepsilon$ for some $\varepsilon > 0$.

By $\sum_{j=0}^{\infty} b_j < \infty$ there exists an $n \in \mathbb{N}$, such that $\sum_{j=n}^{\infty} b_j \leq \frac{1}{2}\varepsilon$.

From $x_{0j}(t) \leq b_j$ for all j we conclude that

$$\sum_{j=n}^{\infty} x_{0j}(t) \leq \frac{1}{2}\varepsilon \quad \text{for all } t \in \mathbb{N}. \tag{3.9}$$

For every $j \in \{0, \ldots, n-1\}$ there exists some T_j such that

$$x_{0j}(t) < x_{0j} + \frac{\varepsilon}{2n} \quad \text{for all } t \geq T_j. \tag{3.10}$$

Let $T = \max\{T_0, \ldots, T_{n-1}\}$. Then for all $t \geq T$ the combination of (3.9) and (3.10) gives

$$\begin{aligned}
\sum_{j=0}^{\infty} x_{0j}(t) &= \sum_{j=0}^{n-1} x_{0j}(t) + \sum_{j=n}^{\infty} x_{0j}(t) \\
&< \sum_{j=0}^{n-1} (x_{0j} + \frac{\varepsilon}{2n}) + \frac{1}{2}\varepsilon \\
&= \sum_{j=0}^{n-1} x_{0j} + \varepsilon \\
&\leq \sum_{j=0}^{\infty} x_{0j} + \varepsilon \\
&\leq (a_0 - \varepsilon) + \varepsilon = a_0.
\end{aligned}$$

This would imply that $x(t)$ were not admissible. So, also Case 2 cannot happen, and we have $\sum_{j=0}^{\infty} x_{0j} = a_0$.

Analogously we get $\sum_{j=0}^{\infty} x_{ij} = a_i$ for all $i \in \mathbb{N}$ and $\sum_{i=0}^{\infty} x_{ij} = b_j$ for all $j \in \mathbb{N}$. Hence x is indeed an optimal plan for (C, a, b), and the proof of Theorem 3.3 is complete. ■

3.3 Arbitrary Splits of Transportation Plans

Theorem 3.2 has an interesting consequence for the situation where a supply/demand pair is split in arbitrary sub-pairs. The general formulation reads as follows.

Corollary 3.4 *Let $C = (C_{ij})_{i,j \in \mathbb{N}}$ be a non-negative cost matrix without bad quadruples. Let $a = (a_i)_{i \in \mathbb{N}}$ and $b = (b_j)_{j \in \mathbb{N}}$ be supply and demand vectors such that there exists a transportation plan x for (C, a, b) with finite* cost(x). *Assume k supply/demand pairs $(a^1, b^1), \ldots, (a^k, b^k)$, such that $\sum_{n=1}^{k} a_i^n \leq a_i$ for all i, $\sum_{n=1}^{k} b_j^n \leq b_j$ for all j, and $\sum_{i=0}^{\infty} a_i^n = \sum_{j=0}^{\infty} b_j^n$ for all n.*
Let x^n be arbitrary admissible transportation plans for (C, a^n, b^n) for all n. Then

$$\sum_{n=1}^{k} \operatorname{cost}(x^n) \leq 2 \operatorname{cost}(x).$$

Proof:
Summing up the plans x^n for $n = 1, \ldots, k$ gives an admissible transportation plan $\bar{x}$ for $(C, \sum_{n=1}^{k} a^n, \sum_{n=1}^{k} b^n)$. Application of Theorem 3.2 gives

$$\sum_{n=1}^{k} \operatorname{cost}(x^n) = \operatorname{cost}(\bar{x}) \leq 2 \operatorname{cost}(x). ■$$

4 Concluding Remarks

- The results of Section 3 show that the characterization of paradox-immune cost matrices remains valid in the weaker "infimum"-sense (instead of "min"-sense) for problems with infinite-dimensional supply and demand vectors. Moreover, for finite quantities $\sum_{i \in \mathbb{N}} a_i = \sum_{j \in \mathbb{N}} b_j$ even the strong min-formulation remains true.

- Good quadruples (q, r, s, t) are defined by $c_{qr} \leq c_{qt} + c_{sr}$. If instead all quadruples satisfy the similar condition $c_{qr} \leq \lambda(c_{qt} + c_{sr})$ for some $\lambda \geq \frac{1}{2}$, then Theorem 3.2 and Corollary 3.4 remain valid with the coefficient "2λ" instead of "2".
- The construction in the proof of Theorem 3.1 shows that there always exist simple good plans for infinite-dimensional transportation problems, when the cost matrix C has no bad quadruples: The basic strategy is to compute an optimal (or good) plan for some starting section $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_n)$ and to complete this to a full plan by simply applying the north-west rule to $(a_{m+1}, a_{m+2}, \ldots)$, $(b_{n+1}, b_{n+2}, \ldots)$.

5 Acknowledgments

Our thanks go to Bettina Klinz for raising our interest in the problem and for giving background information on generalized transportation problems.

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