

The complexity of primal logic with disjunction

Marco Magirius^a, Martin Mundhenk^{a,*}, Raphaela Palenta^b

^a*Institut für Informatik, Universität Jena, 07737 Jena, Germany*

^b*Institut für Informatik, Technische Universität München, 85748 Garching, Germany*

Abstract

We investigate the complexity of primal logic with disjunction according to the Kripke semantics as defined in [1] and the quasi-boolean semantics as defined in [2]. We show that the validity problem is **coNP**-complete, even for variable-free sequents. For quasi-boolean semantics, the satisfiability problem is shown to be **NP**-complete (even for variable-free sequents), whereas for Kripke semantics it is shown to be **coNP**-complete for variable-free sequents and Σ_2^P -complete in the general case. The evaluation problem is in **P** for quasi-boolean semantics, but **coNP**-complete for Kripke semantics.

Keywords: computational complexity, non-classical logics

1. Introduction

We investigate the complexity of primal propositional logic [1, 2], called $\mathbf{P}[\vee, \perp]$ in [2]. It belongs to the family of intuitionistic logics and serves as a basis of primal infon logic [1], the latter being used as a core concept of Distributed Knowledge Authorization Language DKAL [3].

The primal implication “ \rightarrow ” differs from the intuitionistic implication in a way that e.g. $\perp \rightarrow \perp$ is not valid. Its semantics is dealt differently in [1] and in [2]. A Kripke semantics with a nondeterministic interpretation of primal implication is developed in [1], under which $\perp \rightarrow \perp$ is even not satisfiable. In [2] the quasi-boolean semantics for primal logic is defined. It has the property that a model does not only give the interpretation of the propositional atoms of a formula, but additionally of all subformulas that are implications. Informally speaking, whether a primal formula $\alpha \rightarrow \beta$ is satisfied by a quasi-boolean model can be read directly from the model like the satisfaction of an atom x_i . As a consequence, $\perp \rightarrow \perp$ is satisfiable and not valid under quasi-boolean semantics.

Both semantics are complete [1, 2] and thus do not distinguish w.r.t. the validity problem (given a sequent; is it satisfied by every model?), that was shown to be **coNP**-complete in [2]. We strengthen this result and show that the validity problem is **coNP**-complete already for primal sequents that do not contain any atoms x_i but only \perp . It is clear that each semantics has its own evaluation problem (given a model and a sequent; does the model satisfy the sequent?), since the instances—pairs of models and sequents—are different for both semantics. It is rather easy and efficiently solvable for quasi-boolean semantics. In contrast, we show that it is **coNP**-complete for Kripke semantics (Theorem 13). As mentioned above, the satisfiability problem (given a sequent; is there some model that satisfies it?) is different for both semantics, even though the instances (i.e. sequents) are the same for both. We show that the satisfiability problem is **NP**-complete for quasi-boolean semantics (Theorem 15), independent on whether the sequent contains atoms or not. In contrast, for Kripke semantics the satisfiability problem is shown to be **coNP**-complete for sequents without atoms (Theorem 13) and even Σ_2^P -complete for arbitrary sequents (Theorem 14).

*Corresponding author

Email addresses: marco.magirius@uni-jena.de (Marco Magirius), martin.mundhenk@uni-jena.de (Martin Mundhenk), raphaela.palenta@tum.de (Raphaela Palenta)

In some sense, our results show a relatively usual complexity—NP-completeness of satisfiability, coNP-completeness of validity, and efficient solvability of evaluation—of primal logic under quasi-boolean semantics. Actually, the “costs” for this are paid by the user who is responsible for solving the nondeterminism of primal implication in terms of writing the interpretations of implicational subformulas into the model. In contrast, the complexities obtained for primal logics under Kripke semantics look somewhat weird since evaluation is as hard as validity, and satisfiability is even harder. This is only due to the coNP-hardness of evaluating sequents under Kripke semantics and exactly reflects the price of letting nondeterminism be solved by machines. It is also interesting to notice that the nondeterministic semantics of primal logic has an effect on the complexity of the fragments without atoms. Whereas the complexity of validity and evaluation for primal logic is maximal already for atom-free formulas, for intuitionistic logic the maximal complexity is reached with two atoms, and the cases without atoms resp. with one atom have lower complexity [4, 5, 6].

2. Preliminaries

The complexity theoretic basics we use can be found e.g. in [7]. We use the complexity classes P, NP, coNP, and $\Sigma_2^P (= \text{NP}^{\text{NP}})$. The complete problems we use are the NP-complete problem 3SAT (the satisfiability problem for propositional formulas in conjunctive normal form with three literals in each clause), the coNP-complete problem 3DNF-TAUT (the validity problem for propositional formulas in disjunctive normal form with three literals in each monomial (3DNF)), and the Σ_2^P -complete problem QSAT₂ (the validity problem for quantified propositional formulas of the form $\exists x_1, \dots, x_\ell \forall y_1, \dots, y_m \psi'$ where ψ' is in 3DNF and contains atoms $x_1, \dots, x_\ell, y_1, \dots, y_m$).

The language of primal logic is the set of formulas FORM defined as

$$\varphi ::= \perp \mid \top \mid x_i \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi),$$

where x_i is an atom from a countable set PROP of propositional atoms (also called variables). A *sequent* $[\Gamma \vdash \alpha]$ consists of a finite set of formulas Γ and a formula α .

A *Kripke model for primal logic* is a triple $\mathcal{M} = (U, \leq, \xi)$, where U is a nonempty and finite set of *states*, \leq is a reflexive and transitive binary relation on U , and the *assignment* $\xi : \text{PROP} \rightarrow \mathfrak{P}(U)$ is a function with the property that $w \in \xi(x_i)$ and $w \leq v$ implies $v \in \xi(x_i)$. A set having this property is called a *cone*. We will use $C_w^{\mathcal{M}}$ to denote the cone $C_w^{\mathcal{M}} = \{u \in U \mid w \leq u\}$ with base $w \in U$.

An *extension* of assignment ξ is a function $\hat{\xi} : \text{FORM} \rightarrow \mathfrak{P}(U)$ that satisfies the following conditions K0–K5 for all $\alpha, \beta \in \text{FORM}$.

- K0. $\hat{\xi}(\alpha)$ is a cone.
- K1. $\hat{\xi}(\perp) = \emptyset$ and $\hat{\xi}(\top) = U$
- K2. $\hat{\xi}(\alpha) = \xi(\alpha)$, if $\alpha \in \text{PROP}$
- K3. $\hat{\xi}(\alpha \wedge \beta) = \hat{\xi}(\alpha) \cap \hat{\xi}(\beta)$
- K4. $\hat{\xi}(\alpha \vee \beta) = \hat{\xi}(\alpha) \cup \hat{\xi}(\beta)$
- K5. $\hat{\xi}(\beta) \subseteq \hat{\xi}(\alpha \rightarrow \beta) \subseteq \{u \in U \mid \hat{\xi}(\alpha) \cap C_u^{\mathcal{M}} \subseteq \hat{\xi}(\beta)\}$

For a set Γ of formulas let $\hat{\xi}(\Gamma) = \hat{\xi}(\bigwedge_{\gamma \in \Gamma} \gamma) (= \bigcap_{\gamma \in \Gamma} \hat{\xi}(\gamma))$, in particular $\hat{\xi}(\emptyset) = U$.

K5 makes primal logic interesting. It has a flavour of nondeterminism and allows ξ to have an arbitrary number of extensions. Intuitively speaking, a primal formula φ is satisfied by a model $\mathcal{M} = (U, \leq, \xi)$, if, for $w \in \xi(\text{varphi})$, $w \in \hat{\xi}(\varphi)$ holds for *all* extensions $\hat{\xi}$ of ξ . Unfortunately, the satisfiability problem for primal formulas tends to get meaningless under this intuition, since formulas with negations like $x_i \rightarrow \perp$ get unsatisfiable too easily. With regard to infon logic, it is necessary to consider not only formulas but sequents. The satisfaction relation for primal logic sequents is defined as follows (cf. [1]).

Definition 1. Let $\mathcal{M} = (U, \leq, \xi)$ be a primal Kripke model and $[\Gamma \vdash \varphi]$ be a sequent. \mathcal{M} satisfies $[\Gamma \vdash \varphi]$ w.r.t. Kripke semantics (denoted as $\mathcal{M} \stackrel{\text{Kr}}{\models} [\Gamma \vdash \varphi]$), if $\hat{\xi}(\Gamma) \subseteq \hat{\xi}(\varphi)$ holds for all extensions $\hat{\xi}$ of ξ .

We will see that the evaluation problem for primal logic under Kripke semantics is coNP-complete. A reason for this hardness is that *all* extensions of a given assignment must be checked. In [2], a semantics for primal logic is investigated that avoids this. It is called *quasi-boolean semantics*. Similar to boolean semantics for propositional logics, it can be seen as to use only one state, e.g. the universe is $U = \{0\}$. Instead of boolean values 0 and 1 we use \emptyset and $\{0\}$, in order to get a definition comparable to the Kripke semantics. Unlike in boolean semantics, a boolean value is not only assigned to atoms but also to implications. Such a valuation $\check{\xi}$ determines a unique extension $\hat{\xi}$ with $\hat{\xi}(\alpha \rightarrow \beta) = \check{\xi}(\alpha \rightarrow \beta)$ for all $\alpha, \beta \in \text{FORM}$.

Definition 2. Let $\mathcal{M} = (U, \leq, \check{\xi})$ be a quasi-boolean model with $U = \{0\}$ and valuation $\check{\xi} : \text{PROP} \cup \{\alpha \rightarrow \beta \mid \alpha, \beta \in \text{FORM}\} \rightarrow \{\emptyset, \{0\}\}$, and let $[\Gamma \vdash \varphi]$ be a sequent. \mathcal{M} satisfies $[\Gamma \vdash \varphi]$ w.r.t. quasi-boolean semantics (denoted as $\mathcal{M} \models_{qb} [\Gamma \vdash \varphi]$) if $\hat{\xi}(\Gamma) \subseteq \hat{\xi}(\varphi)$ holds for the extension $\hat{\xi}$ of $\check{\xi}$.

We are now ready to define the decision problems that use the satisfaction relations.

Definition 3. For semantics $X \in \{Kr, qb\}$, the decision problems EVAL_X-PL, SAT_X-PL, and VALID_X-PL are defined as follows.

<p>EVAL_X-PL — the evaluation problem for primal logic w.r.t. semantics X</p> <p><i>Input:</i> an X-model \mathcal{M} and a sequent $[\Gamma \vdash \varphi]$</p> <p><i>Question:</i> does $\mathcal{M} \models_X [\Gamma \vdash \varphi]$ hold ?</p>
<p>SAT_X-PL — the satisfiability problem for primal logic w.r.t. semantics X</p> <p><i>Input:</i> a sequent $[\Gamma \vdash \varphi]$</p> <p><i>Question:</i> does $\mathcal{M} \models_X [\Gamma \vdash \varphi]$ hold for some X-model \mathcal{M} ?</p>
<p>VALID_X-PL — the validity problem for primal logic w.r.t. semantics X</p> <p><i>Input:</i> a sequent $[\Gamma \vdash \varphi]$</p> <p><i>Question:</i> does $\mathcal{M} \models_X [\Gamma \vdash \varphi]$ hold for all X-models \mathcal{M} ?</p>

The respective decision problems for sequents without propositional atoms are EVAL_X-PL₀, SAT_X-PL₀, and VALID_X-PL₀.

The complexity of the validity problem for primal logic was investigated in [2]. Notice that it is independent of the semantics, i.e. VALID_{qb}-PL = VALID_{Kr}-PL.

Theorem 4. [2] VALID_{qb}-PL and VALID_{Kr}-PL are coNP-complete.

3. Basic Results

In logics over Kripke models, the upper bounds for the complexity of the satisfiability resp. validity problem rely on a small model property—if some satisfying resp. unsatisfying model exists, there is some which has a small set of states. For primal logic, we can bound the size of the set of states even to one. Moreover, for sequents without propositional atoms, the assignment ξ does not contribute to the satisfaction of the sequent. Thus, for sequents without propositional atoms only the Kripke model $\mathcal{M}_0 = (\{0\}, \{(0, 0)\}, \xi_0)$ with $\xi_0(x_i) = \emptyset$ for every atom x_i has to be considered in order to decide validity, satisfiability, or evaluation. This also shows that the difference between Kripke semantics and quasi-boolean semantics for primal logic is essentially the difference between assignments and valuations.

Theorem 5. 1. Let $[\Gamma \vdash \alpha]$ be a sequent. $[\Gamma \vdash \alpha] \in \text{SAT}_{K_r}\text{-PL}$ if and only if $\mathcal{M} \models_{K_r} [\Gamma \vdash \alpha]$ for some primal Kripke model $\mathcal{M} = (U_1, \leq, \xi)$ with one state $U_1 = \{v\}$.

2. Let $[\Gamma \vdash \alpha]$ be a sequent without atoms and \mathcal{M}_0 be the primal Kripke model $\mathcal{M}_0 = (\{0\}, \{(0, 0)\}, \xi_0)$ with $\xi_0(x_i) = \emptyset$ for every atom x_i . Then $\mathcal{M} \models_{K_r} [\Gamma \vdash \alpha]$ if and only if $\mathcal{M}_0 \models_{K_r} [\Gamma \vdash \alpha]$ holds for all primal Kripke models \mathcal{M} .

PROOF. Firstly, we consider 1. The “ \Leftarrow ” direction is straightforward. To prove the “ \Rightarrow ” direction, assume that $[\Gamma \vdash \alpha]$ is satisfiable and thus $\mathcal{M} \models_{K_r} [\Gamma \vdash \alpha]$ for some primal model $\mathcal{M} = (U, \leq, \xi)$. There exists some maximal state $v \in U$ in the sense that for every atom x_i appearing in $[\Gamma \vdash \alpha]$ the assignment to x_i does not change in any successor of v , as can be seen as follows. Let $w \in U$ be arbitrarily chosen. If $\xi(x_i) \cap C_w^{\mathcal{M}} = \emptyset$ for all atoms x_i entailed by $[\Gamma \vdash \alpha]$, we already found a maximal state and set v to w . If $\xi(x_i) \cap C_w^{\mathcal{M}} \neq \emptyset$ for some x_i , there is some $w' \in U$ with $w \leq w'$ and $\xi(x_i) \cap C_{w'}^{\mathcal{M}} = C_w^{\mathcal{M}}$, since $\xi(x_i)$ is a cone. After repeating this step for at most all x_i appearing in $[\Gamma \vdash \alpha]$, we found a maximal state $v = w'$. Note that this also works for the more general case where U may be infinite.

Let the v -restriction of \mathcal{M} be $\mathcal{M}_v = (\{v\}, \{(v, v)\}, \vartheta)$ with $\vartheta(x_i) = \{v\} \cap \xi(x_i)$. In order to show $\mathcal{M}_v \models_{K_r} [\Gamma \vdash \alpha]$, we prove the following two claims, where $\hat{\vartheta}_\xi(\varphi) = \{v\} \cap \hat{\xi}(\varphi)$.

Claim 1. Let $\hat{\xi}$ be an extension of ξ . Then $\hat{\vartheta}_\xi$ is an extension of ϑ .

PROOF OF CLAIM. The verification of K0, ..., K4 is straightforward. To verify K5, there are two statements to be shown for every formula $\beta \rightarrow \gamma$: (a) $\hat{\vartheta}_\xi(\gamma) \subseteq \hat{\vartheta}_\xi(\beta \rightarrow \gamma)$ and (b) $\hat{\vartheta}_\xi(\beta \rightarrow \gamma) \subseteq \{u \mid \hat{\vartheta}_\xi(\beta) \cap C_u^{\mathcal{M}_v} \subseteq \hat{\vartheta}_\xi(\gamma)\}$. Since $\hat{\vartheta}_\xi(\gamma) = \hat{\xi}(\gamma) \cap \{v\} \subseteq \hat{\xi}(\beta \rightarrow \gamma) \cap \{v\} = \hat{\vartheta}_\xi(\beta \rightarrow \gamma)$. To get (b), consider the following chain of equations and inclusions. holds due to the definition of $\hat{\vartheta}_\xi$ and $\hat{\xi}$ being an extension of ξ , we got (a).

$$\begin{aligned} & \hat{\vartheta}_\xi(\beta \rightarrow \gamma) \\ &= \{v\} \cap \hat{\xi}(\beta \rightarrow \gamma) && \text{(definition of } \hat{\vartheta}_\xi) \\ &\subseteq \left\{ \begin{array}{ll} \{v\}, & \text{if } \hat{\xi}(\beta) \cap C_v^{\mathcal{M}} \subseteq \hat{\xi}(\gamma) \\ \emptyset, & \text{otherwise} \end{array} \right\} && (\hat{\xi} \text{ satisfies K5)} \\ &\subseteq \left\{ \begin{array}{ll} \{v\}, & \text{if } \hat{\vartheta}_\xi(\beta) \cap \{v\} \subseteq \hat{\vartheta}_\xi(\gamma) \\ \emptyset, & \text{otherwise} \end{array} \right\} && \text{(see below)} \\ &= \{u \mid \hat{\xi}(\beta) \cap C_u^{\mathcal{M}_v} \subseteq \hat{\vartheta}_\xi(\gamma)\} \end{aligned}$$

The second inclusion uses that $\hat{\xi}(\beta) \cap C_v^{\mathcal{M}} \subseteq \hat{\xi}(\gamma)$ implies $\hat{\vartheta}_\xi(\beta) \cap \{v\} \subseteq \hat{\xi}(\gamma)$. This can be seen as follows. Assume $\hat{\xi}(\beta) \cap C_v^{\mathcal{M}} \subseteq \hat{\xi}(\gamma)$. If $v \in \hat{\xi}(\beta)$, then $v \in \hat{\xi}(\gamma)$ and therefore, $\hat{\vartheta}_\xi(\beta) \cap \{v\} = \hat{\vartheta}_\xi(\gamma)$. If otherwise $v \notin \hat{\xi}(\beta)$, then $\hat{\vartheta}_\xi(\beta) \cap \{v\} = \emptyset$ and consequently $\hat{\vartheta}_\xi(\beta) \cap \{v\} \subseteq \hat{\vartheta}_\xi(\gamma)$. ■

Claim 2. Let $\hat{\vartheta}$ be an extension of ϑ . Then there is some extension $\hat{\xi}$ of ξ such that $\hat{\vartheta} = \hat{\vartheta}_\xi$, i.e., $\hat{\xi} \mapsto \hat{\vartheta}_\xi$ is surjective.

PROOF OF CLAIM. We exhibit

$$\hat{\xi}(\varphi) = \begin{cases} C_v^{\mathcal{M}}, & \text{if } \hat{\vartheta}(\varphi) = \{v\} \\ \emptyset, & \text{otherwise} \end{cases} \quad (1)$$

and only elaborate on the verification of K5 for $\hat{\xi}$, as the rest is straightforward. We do so by showing (a) $\hat{\xi}(\gamma) \subseteq \hat{\xi}(\beta \rightarrow \gamma)$ and (b) $\hat{\xi}(\beta \rightarrow \gamma) \subseteq \{u \mid \hat{\xi}(\beta) \cap C_u^{\mathcal{M}} \subseteq \hat{\xi}(\gamma)\}$ for every primal implication $\beta \rightarrow \gamma$.

At first, let $\hat{\xi}(\gamma) = C_v^{\mathcal{M}}$. Then $\{v\} \stackrel{(1)}{=} \hat{\vartheta}(\gamma) \subseteq \hat{\vartheta}(\beta \rightarrow \gamma) = \{v\}$, since K5 holds for $\hat{\vartheta}$ by Claim 1. Consequently, $\hat{\xi}(\gamma) \subseteq C_v^{\mathcal{M}} \stackrel{(1)}{=} \hat{\xi}(\beta \rightarrow \gamma)$ and $\hat{\xi}(\beta \rightarrow \gamma) \subseteq U = \{u \mid \hat{\xi}(\beta) \cap C_u^{\mathcal{M}} \subseteq C_v^{\mathcal{M}}\}$ for each case of $\hat{\xi}(\beta)$. Thus, (a) and (b) follow.

On the other hand, let $\hat{\xi}(\gamma) = \emptyset$. Note that (a) holds trivially. If $\hat{\xi}(\beta) = \emptyset$, then $\hat{\xi}(\beta \rightarrow \gamma) \subseteq U = \{u \mid \emptyset \cap C_u^{\mathcal{M}} \subseteq \hat{\xi}(\gamma)\}$ and thus (b) follows, too. If $\hat{\xi}(\beta) = C_v^{\mathcal{M}}$, then $\hat{\vartheta}(\beta) \stackrel{(1)}{=} \{v\}$. In addition, we know $\hat{\vartheta}(\gamma) \stackrel{(1)}{=} \emptyset$ and therefore, $\hat{\vartheta}(\beta \rightarrow \gamma) = \emptyset$. Since $\hat{\xi}(\beta \rightarrow \gamma) = \emptyset$ follows from (1), we finished showing (b). ■

Now we are able to show $\forall \hat{\vartheta} : \hat{\vartheta}(\Gamma) \subseteq \hat{\vartheta}(\alpha)$. Let $\hat{\vartheta}$ be an extension of ϑ . According to Claim 2 there is an extension $\hat{\xi}$ with $\hat{\vartheta} = \hat{\vartheta}_{\hat{\xi}}$. Since $\mathcal{M} \stackrel{\text{Kripke}}{\models} [\Gamma \vdash \alpha]$ and thus $\hat{\xi}(\Gamma) \subseteq \hat{\xi}(\alpha)$, it is clear that $\hat{\vartheta}(\Gamma) \subseteq \hat{\vartheta}(\alpha)$ follows from the definition of $\hat{\vartheta}$.

We continue with the proof of part 2 of the theorem. To prove the “ \Rightarrow ” direction, assume $\mathcal{M}' \stackrel{\text{Kripke}}{\models} [\Gamma \vdash \alpha]$. As above, we can conclude that $\mathcal{M} \stackrel{\text{Kripke}}{\models} [\Gamma \vdash \alpha]$ for some Kripke model \mathcal{M} with state set $\{0\}$. Since the sequent $[\Gamma \vdash \alpha]$ has no atoms, the assignment to the atoms is irrelevant, and we can conclude $\mathcal{M}_0 \stackrel{\text{Kripke}}{\models} [\Gamma \vdash \alpha]$. For the “ \Leftarrow ” direction, assume $\mathcal{M} \not\stackrel{\text{Kripke}}{\models} [\Gamma \vdash \alpha]$ for $\mathcal{M} = (U, \leq, \xi)$. Then there is some extension $\hat{\xi}$ of ξ and some $w \in U$ with $w \in \hat{\xi}(\Gamma)$ and $w \notin \hat{\xi}(\alpha)$. Let the w -restriction of \mathcal{M} be $\mathcal{M}_w = (\{w\}, \{(w, w)\}, \xi_0)$ with $\xi_0(x_i) = \emptyset$. In order to show $\mathcal{M}_0 \not\stackrel{\text{Kripke}}{\models} [\Gamma \vdash \alpha]$ it suffices to show $\mathcal{M}_w \not\stackrel{\text{Kripke}}{\models} [\Gamma \vdash \alpha]$, since the only difference between \mathcal{M}_0 and \mathcal{M}_w is the name of the unique state. Let $\hat{\xi}_w(x_i) = \{w\} \cap \hat{\xi}(x_i)$. Since $w \in \hat{\xi}_w(\Gamma)$ and $w \notin \hat{\xi}_w(\alpha)$, we only have to show that $\hat{\xi}_w$ is an extension. That follows from Claim 1 and we conclude $\mathcal{M}_w \not\stackrel{\text{Kripke}}{\models} [\Gamma \vdash \alpha]$. □

It follows that $\text{SAT}_{K_r\text{-PL}} \subsetneq \text{SAT}_{qb\text{-PL}}$, since from a single state Kripke model that satisfies a sequent we can take an arbitrary extension of its assignment and regard it as the extension of a quasi-boolean assignment, that also satisfies the sequent. The properness of the inclusion follows from $\perp \rightarrow \perp \in \text{SAT}_{qb\text{-PL}}$ together with $\perp \rightarrow \perp \notin \text{SAT}_{K_r\text{-PL}}$, what even implies $\text{SAT}_{K_r\text{-PL}_0} \subsetneq \text{SAT}_{qb\text{-PL}_0}$. Since for $x \in \text{PROP}$, $x \rightarrow \perp \in \text{SAT}_{qb\text{-PL}} \setminus \text{SAT}_{K_r\text{-PL}_0}$ and $x \rightarrow \perp \notin \text{SAT}_{K_r\text{-PL}} \setminus \text{SAT}_{K_r\text{-PL}_0}$, we also obtain $\text{SAT}_{K_r\text{-PL}} \setminus \text{SAT}_{K_r\text{-PL}_0} \subsetneq \text{SAT}_{qb\text{-PL}} \setminus \text{SAT}_{qb\text{-PL}_0}$. Concluding, we have the following inclusions between the considered decision problems.

- Theorem 6.**
1. $\text{SAT}_{K_r\text{-PL}_0} = \text{VALID}_{K_r\text{-PL}_0} = \text{VALID}_{qb\text{-PL}_0}$
 2. $\text{EVAL}_{K_r\text{-PL}_0} = \{\langle \mathcal{M}, \sigma \rangle \mid \mathcal{M} \text{ is a primal Kripke model and } \sigma \in \text{SAT}_{K_r\text{-PL}_0}\}$
 3. $\text{SAT}_{K_r\text{-PL}_0} \subsetneq \text{SAT}_{qb\text{-PL}_0}$ and $\text{SAT}_{K_r\text{-PL}} \setminus \text{SAT}_{K_r\text{-PL}_0} \subsetneq \text{SAT}_{qb\text{-PL}} \setminus \text{SAT}_{qb\text{-PL}_0}$

As in intuitionistic logic, $\alpha \rightarrow \perp$ can be seen as negation of α . Since the rule of the excluded third does not hold in intuitionistic or primal logic, $\alpha \vee (\alpha \rightarrow \perp)$ is not valid. For our reductions from satisfiability or validity problems in classical propositional logic, we need to translate classical negations into primal logic. Due to the small model theorem (Theorem 5), we can use the left-hand side of a sequent to obtain this.

Lemma 7. *Let $\mathcal{M} = (U, \leq, \xi)$ be a model with one state $U = \{0\}$, let $\hat{\xi}$ be an extension of ξ , and let*

$$\Gamma = \{ \alpha_\ell \vee (\alpha_\ell \rightarrow \perp) \mid \alpha_\ell \in \text{FORM}, 1 \leq \ell \leq m \}.$$

Then the following holds.

$$\hat{\xi}(\Gamma) = \{0\} \quad \text{if and only if} \quad \text{for all } \ell = 1, 2, \dots, m : \hat{\xi}(\alpha_\ell) = \overline{\hat{\xi}(\alpha_\ell \rightarrow \perp)}$$

where $\overline{\hat{\xi}(\alpha_\ell \rightarrow \perp)}$ is the complement of the set $\hat{\xi}(\alpha_\ell \rightarrow \perp)$.

PROOF. “ \Rightarrow ”: Assume $\hat{\xi}(\Gamma) = \{0\}$. For each ℓ there are two cases to consider. First, consider $\hat{\xi}(\alpha_\ell) = \{0\}$, then $\hat{\xi}(\alpha_\ell \rightarrow \perp) = \emptyset$ follows from K5. Second, consider $\hat{\xi}(\alpha_\ell) = \emptyset$. Since $\hat{\xi}(\alpha_\ell \vee (\alpha_\ell \rightarrow \perp)) = \{0\}$, we deduce $\hat{\xi}(\alpha_\ell \rightarrow \perp) = \{0\}$, which implies $\overline{\hat{\xi}(\alpha_\ell \rightarrow \perp)} = \emptyset$.

“ \Leftarrow ”: Assume $\hat{\xi}(\Gamma) \neq \{0\}$, i.e. for some ℓ holds $\hat{\xi}(\alpha_\ell \vee (\alpha_\ell \rightarrow \perp)) = \emptyset$. Then $\hat{\xi}(\alpha_\ell) = \hat{\xi}(\alpha_\ell \rightarrow \perp) = \emptyset$ and thus $\hat{\xi}(\alpha_\ell) \neq \overline{\hat{\xi}(\alpha_\ell \rightarrow \perp)}$. □

Later on, we will show NP hardness resp. coNP hardness of primal logic problems by reductions from 3SAT resp. 3DNF-TAUT. The main idea of this reductions is to translate classical normal form formulas to primal formulas without atoms such that truth assignments to classical formulas can be simulated by extensions of assignments to the resulting primal formulas. The translation changes the literals of the classical normal form formulas to primal implications without atoms as follows.

Definition 8. For $\ell \geq 1$, let the primal formulas \mathfrak{Y}_ℓ and $\overline{\mathfrak{Y}}_\ell$ be as follows.

$$\mathfrak{Y}_\ell = \perp \rightarrow (\bigwedge_\ell \perp) \quad \text{and} \quad \overline{\mathfrak{Y}}_\ell = \mathfrak{Y}_\ell \rightarrow \perp, \quad \text{with} \quad \bigwedge_\ell \perp = \underbrace{\perp \wedge \cdots \wedge \perp}_{\ell \text{ times}} .$$

For $m \geq 1$, the set $\Gamma(m)$ is defined as

$$\Gamma(m) = \{ \mathfrak{Y}_\ell \vee \overline{\mathfrak{Y}}_\ell \mid \ell = 1, 2, \dots, m \} .$$

All \mathfrak{Y}_ℓ and $\overline{\mathfrak{Y}}_\ell$ are unsatisfiable in Kripke semantics. Nevertheless, they are “independent” in the sense that for every set $S \subseteq \{1, 2, \dots, m\}$ there exists an extension $\hat{\xi}$ of the empty assignment ξ_0 such that $\hat{\xi}(\mathfrak{Y}_\ell) = \{0\}$ if and only if $\ell \in S$. This can be used to simulate classical truth assignments T to atoms x_ℓ by the assignment of an extension $\hat{\xi}$ to \mathfrak{Y}_ℓ .

Definition 9. An extension $\hat{\xi}$ of ξ_0 is called m -classical, if $\hat{\xi}(\Gamma(m)) = \{0\}$.

A truth assignment T to y_1, \dots, y_m and an extension $\hat{\xi}_T$ of ξ_0 are called m -similar, if $\hat{\xi}_T$ is m -classical and for all $\ell = 1, 2, \dots, m$ the following holds.

1. If $T(y_\ell) = 1$, then $\hat{\xi}_T(\mathfrak{Y}_\ell) = \{0\}$, and
2. if $T(y_\ell) = 0$, then $\hat{\xi}_T(\overline{\mathfrak{Y}}_\ell) = \{0\}$.

Classical extensions of ξ_0 are essentially the same as assignments in classical propositional logic.

Lemma 10. 1. For every truth assignment T to y_1, \dots, y_m , there is an m -similar extension of ξ_0 .
2. For every m -classical extension of ξ_0 , there is an m -similar truth assignment.

Proof. 1. Let T be a truth assignment to y_1, \dots, y_m . K5 allows to choose $\hat{\xi}_T(\mathfrak{Y}_\ell)$ independent of $\hat{\xi}_T(\overline{\mathfrak{Y}}_\ell)$ for $i \neq \ell$. Moreover, K5 guarantees that $\hat{\xi}_T(\overline{\mathfrak{Y}}_\ell) \subseteq \hat{\xi}_T(\mathfrak{Y}_i)$. Thus, for given T , $\hat{\xi}_T$ with

$$\hat{\xi}_T(\mathfrak{Y}_\ell) = \begin{cases} \{0\}, & \text{if } T(y_\ell) = 1 \\ \emptyset, & \text{if } T(y_\ell) = 0 \end{cases} \quad \text{and}$$

$$\hat{\xi}_T(\overline{\mathfrak{Y}}_\ell) = \begin{cases} \{0\}, & \text{if } T(y_\ell) = 0 \\ \emptyset, & \text{if } T(y_\ell) = 1 \end{cases}$$

is an m -classical extension of ξ_0 and is m -similar to T .

2. Let $\hat{\xi}$ be an m -classical extension of ξ_0 . Then either $\hat{\xi}(\mathfrak{Y}_\ell) = \{0\}$ or $\hat{\xi}(\overline{\mathfrak{Y}}_\ell) = \{0\}$. Thus, T with $T(y_\ell) = 1 \Leftrightarrow \hat{\xi}(\mathfrak{Y}_\ell) = \{0\}$ is m -similar to $\hat{\xi}$. \square

Now we are ready to give the polynomial time reduction from 3DNF-TAUT to VALID_{Kr}-PL₀.

Lemma 11. Let $\psi = \bigvee_{i=1}^n \bigwedge_{j=1}^3 L_{i,j}$ be a propositional logic formula with atoms y_1, y_2, \dots, y_m and optional with constant symbols \perp and \top . Define the primal formula $\varphi_\psi = \bigvee_{i=1}^n \bigwedge_{j=1}^3 \mathfrak{L}_{i,j}$ with

$$\mathfrak{L}_{i,j} = \begin{cases} \mathfrak{Y}_\ell, & \text{if } L_{i,j} = y_\ell \\ \overline{\mathfrak{Y}}_\ell, & \text{if } L_{i,j} = \neg y_\ell \\ \perp, & \text{if } L_{i,j} = \perp \\ \top, & \text{if } L_{i,j} = \top. \end{cases} \quad (2)$$

ψ is a tautology if and only if $\mathcal{M} \models_{Kr} [\Gamma(m) \vdash \varphi_\psi]$ holds for every primal Kripke model \mathcal{M} .

PROOF. “ \Rightarrow ”: We begin by assuming that ψ is a tautology. By virtue of Theorem 5.2, it suffices to show $\hat{\xi}(\Gamma(m)) \subseteq \hat{\xi}(\varphi_\psi)$ for all extensions $\hat{\xi}$ of ξ_0 . If $\hat{\xi}(\Gamma(m)) = \emptyset$, then $\hat{\xi}(\Gamma(m)) \subseteq \hat{\xi}(\varphi_\psi)$ holds trivially. Now, consider $\hat{\xi}(\Gamma(m)) = \{0\}$. Then $\hat{\xi}$ is m -classical and Lemma 10.2 provides us with a truth assignment $T_{\hat{\xi}}$ being m -similar to $\hat{\xi}$. Since ψ is a tautology, there is some $c \in \{1, 2, \dots, n\}$ with $T_{\hat{\xi}}(L_{c,1}) = T_{\hat{\xi}}(L_{c,2}) = T_{\hat{\xi}}(L_{c,3}) = 1$. Due to the m -similarity of $T_{\hat{\xi}}$ and $\hat{\xi}$ we obtain $\hat{\xi}(\mathcal{L}_{c,1}) = \hat{\xi}(\mathcal{L}_{c,2}) = \hat{\xi}(\mathcal{L}_{c,3}) = \{0\}$. Now we can conclude $\hat{\xi}(\varphi_\psi) = \{0\}$ and therefore, $\hat{\xi}(\Gamma(m)) \subseteq \hat{\xi}(\varphi_\psi)$.

“ \Leftarrow ”: Conversely, we assume ψ is no tautology. Then there is some truth assignment T with $\forall i \in \{1, 2, \dots, n\}$

$\exists j \in \{1, 2, 3\} : T(L_{i,j}) = 0$. Due to Lemma 10.1 there is an extension $\hat{\xi}_T$ being m -similar to T . Thus, $\forall i \in \{1, 2, \dots, n\} \exists j \in \{1, 2, 3\} : \hat{\xi}_T(\mathcal{L}_{i,j}) = \{0\}$, and we conclude $\hat{\xi}_T(\varphi_\psi) = \emptyset$. Since $\hat{\xi}_T$ is m -classical, we have $\hat{\xi}_T(\Gamma(m))$

$= \{0\}$, what eventually yields $\hat{\xi}_T(\Gamma(m)) \not\subseteq \hat{\xi}_T(\varphi_\psi)$.

Note that in both directions of the proof, we can extend m -similarity to the cases $L_{i,j} = \perp$ and $L_{i,j} = \top$ naturally, since K1 holds for all $\hat{\xi}$. \square

4. Complexity Results

We begin by investigating complexity bounds for decision problems w.r.t. Kripke semantics.

Lemma 12. $\text{EVAL}_{K^r\text{-PL}} \in \text{coNP}$.

PROOF. $\langle (U, \leq, \xi), [\Gamma \vdash \varphi] \rangle \in \text{EVAL}_{K^r\text{-PL}}$ holds if and only if $\hat{\xi}(\Gamma) \subseteq \hat{\xi}(\varphi)$ holds for all extensions $\hat{\xi}$ of ξ . Note that $\hat{\xi}(\Gamma) \subseteq \hat{\xi}(\varphi)$ can be verified in polynomial time. \square

Theorem 13. $\text{EVAL}_{K^r\text{-PL}}, \text{EVAL}_{K^r\text{-PL}_0}, \text{SAT}_{K^r\text{-PL}_0}$, and $\text{VALID}_{K^r\text{-PL}_0}$ are coNP -complete.

PROOF. Since 3DNF-TAUT is coNP -hard and $\text{EVAL}_{K^r\text{-PL}}$ is in coNP as stated in Lemma 12, it suffices to show the following chain of relations:

$$\begin{aligned} 3\text{DNF-TAUT} &\stackrel{(i)}{\leq_m^P} \text{VALID}_{K^r\text{-PL}_0} \stackrel{(ii)}{=} \text{SAT}_{K^r\text{-PL}_0} \\ &\stackrel{(iii)}{\leq_m^P} \text{EVAL}_{K^r\text{-PL}_0} \stackrel{(iv)}{\leq_m^P} \text{EVAL}_{K^r\text{-PL}}. \end{aligned}$$

We show (i) by $\psi \mapsto [\Gamma(m) \vdash \varphi_\psi]$ as reduction, where φ_ψ is defined as in Lemma 11, from which the soundness and completeness of the reduction follows. Next, we recall Theorem 6.1, which states (ii). In order to show (iii), we reduce from $\text{SAT}_{K^r\text{-PL}_0}$ to $\text{EVAL}_{K^r\text{-PL}_0}$ with $[\Gamma \vdash \varphi] \mapsto \langle \mathcal{M}_0, [\Gamma \vdash \varphi] \rangle$. The soundness and completeness of this reduction follows from Theorem 5. Only (iv) remains to be shown, but follows easily by reduction with the identity function. \square

Theorem 14. $\text{SAT}_{K^r\text{-PL}}$ is Σ_2^P -complete.

PROOF. First, we show that $\text{SAT}_{K^r\text{-PL}}$ is Σ_2^P -hard by reducing QSAT_2 to $\text{SAT}_{K^r\text{-PL}}$. In the following logspace reduction we construct a sequent $[\Gamma(\ell, m) \vdash \varphi_\psi]$ from a QSAT_2 instance $\psi = \exists x_1, \dots, x_\ell \forall y_1, \dots, y_m \psi'$ where $\psi' = \bigvee_{i=1}^n \bigwedge_{j=1}^3 L_{i,j}$. We define the formula $\varphi_\psi = \bigvee_{i=1}^n \bigwedge_{j=1}^3 \mathfrak{R}_{i,j}$ with

$$\mathfrak{R}_{i,j} = \begin{cases} x_k, & \text{if } L_{i,j} = x_k \\ x_k \multimap \perp, & \text{if } L_{i,j} = \neg x_k \\ \mathfrak{Y}_l, & \text{if } L_{i,j} = y_l \\ \overline{\mathfrak{Y}_l}, & \text{if } L_{i,j} = \neg y_l. \end{cases} \quad (3)$$

We let $\Gamma(\ell, m) = \Gamma(m) \cup \{x_1 \vee (x_1 \rightarrow \perp), \dots, x_\ell \vee (x_\ell \rightarrow \perp)\}$ for all x_1, \dots, x_ℓ contained in ψ . This completes the construction of the sequent $[\Gamma(\ell, m) \vdash \varphi_\psi]$. We will show now that

$$\psi \in \text{QSAT}_2 \Leftrightarrow \exists \mathcal{M} : \mathcal{M} \stackrel{\text{Kr}}{\models} [\Gamma(\ell, m) \vdash \varphi_\psi]. \quad (4)$$

We know that $\psi \in \text{QSAT}_2$ if and only if there is an interpretation T for the variables $x_i, i = 1, \dots, \ell$ in ψ such that $\psi'[x_1/T(x_1), \dots, x_\ell/T(x_\ell)]$ is a tautology. Based on T we define a Kripke model $\mathcal{M}_T = (\{0\}, \{(0, 0)\}, \xi_T)$ with

$$\xi_T(x_k) = \begin{cases} \{0\}, & \text{if } T(x_k) = 1 \\ \emptyset, & \text{if } T(x_k) = 0. \end{cases} \quad (5)$$

Claim 3. $\mathcal{M}_T \stackrel{\text{Kr}}{\models} [\Gamma(\ell, m) \vdash \varphi_\psi]$ if and only if $\psi'[x_1/T(x_1), \dots, x_\ell/T(x_\ell)]$ is a tautology.

PROOF OF CLAIM. From Lemma 7 we know that for every extension $\hat{\xi}_T$ of ξ_T condition (6) holds if $\hat{\xi}(\Gamma(\ell, m)) = \{0\}$.

$$\hat{\xi}_T(x_k) = \{0\} \text{ if and only if } \hat{\xi}_T(x_k \rightarrow \perp) = \emptyset. \quad (6)$$

Therefore we can apply the following substitution without changing the satisfaction of $[\Gamma(\ell, m) \vdash \varphi_\psi]$ by \mathcal{M}_T . For all x_1, \dots, x_ℓ contained in φ_ψ is

$$x_k \rightarrow \perp \text{ replaced by } \begin{cases} \top, & \text{if } \xi_T(x_k \rightarrow \perp) = \{0\} \\ \perp, & \text{otherwise} \end{cases} \quad (7)$$

$$x_k \text{ replaced by } \begin{cases} \top, & \text{if } \xi_T(x_k) = \{0\} \\ \perp, & \text{otherwise.} \end{cases} \quad (8)$$

We call the formula constructed through this substitution by φ_{ψ_T} . Thus we have $\mathcal{M}_T \stackrel{\text{Kr}}{\models} [\Gamma(\ell, m) \vdash \varphi_\psi]$ if and only if $\mathcal{M}_T \stackrel{\text{Kr}}{\models} [\Gamma(\ell, m) \vdash \varphi_{\psi_T}]$ and as no x_i appears in φ_{ψ_T} this is if and only if $\mathcal{M}_T \stackrel{\text{Kr}}{\models} [\Gamma(m) \vdash \varphi_{\psi_T}]$.

As $[\Gamma(m) \vdash \varphi_{\psi_T}]$ is a sequent without atoms we get using Theorem 5.2 that $\mathcal{M}_T \stackrel{\text{Kr}}{\models} [\Gamma(m) \vdash \varphi_{\psi_T}] \Leftrightarrow \mathcal{M}_0 \stackrel{\text{Kr}}{\models} [\Gamma(m) \vdash \varphi_{\psi_T}]$, that is if and only if $\mathcal{M} \stackrel{\text{Kr}}{\models} [\Gamma(m) \vdash \varphi_{\psi_T}]$ for every Kripke model \mathcal{M} . Using Lemma 11, the latter is equivalent to ψ_T being a tautology. Since actually $\psi_T = \psi'[x_1/T(x_1), \dots, x_\ell/T(x_\ell)]$, the proof of the Claim is concluded.

Thus we showed the \Rightarrow -direction of (4). For the \Leftarrow -direction suppose that $\psi \notin \text{QSAT}_2$. Then for every interpretation T for the variables x_i in ψ , $\psi'[x_1/T(x_1), \dots, x_\ell/T(x_\ell)]$ is no tautology. Using Claim 3 we get that $\mathcal{M}_T \not\stackrel{\text{Kr}}{\models} [\Gamma(\ell, m) \vdash \varphi_\psi]$ for every interpretation T . We will show now that for every Kripke model \mathcal{M} there is an interpretation T such that \mathcal{M} is equivalent to \mathcal{M}_T . From Theorem 5.1 we know that it is enough to consider Kripke models with one state. Let $\mathcal{M} = (\{0\}, \{(0, 0)\}, \xi)$ be a Kripke model. As the only variables in φ_ψ are x_i we can assume that ξ does not contain any other variable. The corresponding T for \mathcal{M} is given by using (5) vice versa. Then \mathcal{M} equals \mathcal{M}_T . This implies that for every Kripke model \mathcal{M} , $\mathcal{M} \not\stackrel{\text{Kr}}{\models} [\Gamma(\ell, m) \vdash \varphi_\psi]$, which ends the proof of (4).

Second, the following description of $\text{SAT}_{\text{Kr}}\text{-PL}$ shows that it is in Σ_2^P .

$$\begin{aligned} [\Gamma \vdash \varphi] \in \text{SAT}_{\text{Kr}}\text{-PL} & \text{ if and only if} \\ \exists \mathcal{M} = (\{0\}, \{(0, 0)\}, \xi) \forall \text{ extensions } \hat{\xi} \text{ of } \xi : & \hat{\xi}(\Gamma) \subseteq \hat{\xi}(\varphi). \end{aligned}$$

The extensions $\hat{\xi}$ of ξ are obviously polynomial in the size of the sequent $[\Gamma \vdash \varphi]$. W.l.o.g. we only have to consider Kripke models with one state as we showed in Theorem 5.1. Thus the quantifiers are both polynomially bounded and the predicate “ $\hat{\xi}(\Gamma) \subseteq \hat{\xi}(\varphi)$ ” can be checked in polynomial time. \square

Now we turn attention to complexity bounds for decision problems w.r.t. quasi-boolean semantics.

Theorem 15. 1. $\text{EVAL}_{qb}\text{-PL}$ is in P .
2. $\text{SAT}_{qb}\text{-PL}_0$ and $\text{SAT}_{qb}\text{-PL}$ are NP-complete.

PROOF. The proofs of 1. and $\text{SAT}_{qb}\text{-PL} \in \text{NP}$ are straightforward. We show that $\text{SAT}_{qb}\text{-PL}_0$ is NP-hard by reducing from 3SAT to $\text{SAT}_{qb}\text{-PL}_0$ with $\psi \mapsto [\emptyset \vdash ((\bigwedge_{\gamma \in \Gamma(m)} \gamma) \wedge \eta_\psi)]$ where ψ entails m atoms, $\eta_\psi = \bigwedge_{i=1}^n \bigvee_{j=1}^3 \mathfrak{L}_{i,j}$, and $\mathfrak{L}_{i,j}$ is defined as in (2).

Claim 4. ψ is satisfiable if and only if
$$\exists \mathcal{M} : \mathcal{M} \models_{qb} [\emptyset \vdash ((\bigwedge_{\gamma \in \Gamma(m)} \gamma) \wedge \eta_\psi)].$$

PROOF OF CLAIM. “ \Rightarrow ”: We assume that ψ is satisfiable and exhibit a quasi-boolean model $\mathcal{M}' = (\{0\}, \{(0, 0)\}, \check{\xi})$ which satisfies $[\emptyset \vdash ((\bigwedge_{\gamma \in \Gamma(m)} \gamma) \wedge \eta_\psi)]$. Since ψ is satisfiable, there is some assignment T appropriate to ψ with $\forall i \in \{1, 2, \dots, n\} \exists j \in \{1, 2, 3\} : T(L_{i,j}) = 1$. Different from Kripke semantics, we do not even need K5 to establish a kind of m -similarity to T . Since $\check{\xi}$ is part of the model \mathcal{M}' and the extension $\hat{\xi}$ of $\check{\xi}$ is uniquely determined by $\check{\xi}(\mathfrak{L}_{i,j})$ for each $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, 3\}$, we are able to choose $\check{\xi}$ accordingly to 1. and 2. in Definition 9. Thus $\hat{\xi}_T(\Gamma(m)) = \{0\}$ is easy to see and $\forall i \in \{1, 2, \dots, n\} \exists j \in \{1, 2, 3\} : \hat{\xi}_T(\mathfrak{L}_{i,j}) = \{0\}$ yields $\hat{\xi}_T(\eta_\psi) = \{0\}$, which implies $\hat{\xi}_T(\emptyset) \subseteq \hat{\xi}_T(\Gamma(m)) \cap \hat{\xi}_T(\eta_\psi)$.

“ \Leftarrow ”: Conversely, we assume that there is some quasi-boolean model $\mathcal{M}' = (\{0\}, \{(0, 0)\}, \check{\xi})$ satisfying $[\emptyset \vdash ((\bigwedge_{\gamma \in \Gamma(m)} \gamma) \wedge \eta_\psi)]$. Since $\hat{\xi}(\emptyset) = \{0\}$ holds for the extension $\hat{\xi}$ of $\check{\xi}$, $\hat{\xi}(\Gamma(m)) = \hat{\xi}(\eta_\psi) = \{0\}$ follows from $\hat{\xi}(\emptyset) \subseteq \hat{\xi}(\Gamma(m)) \cap \hat{\xi}(\eta_\psi)$. Either $\hat{\xi}(\mathfrak{Y}_\ell) = \{0\}$ or $\hat{\xi}(\overline{\mathfrak{Y}_\ell}) = \{0\}$ holds for all $\ell \in \{1, 2, \dots, m\}$ by definition of $\Gamma(m)$ and \mathfrak{Y}_ℓ . Hence it suffices to choose $T_{\hat{\xi}}$ with $T_{\hat{\xi}}(y_\ell) = 1 \Leftrightarrow \hat{\xi}(\mathfrak{Y}_\ell) = \{0\}$ and apply the arguments above *vice versa* to conclude that $T(\psi)_{\hat{\xi}} = 1$ follows.

To complete the proof of the theorem, we reduce from $\text{SAT}_{qb}\text{-PL}_0$ to $\text{SAT}_{qb}\text{-PL}$ trivially. \square

5. Conclusion

We characterized the complexity of primal logic with disjunction by proving complexity bounds for the classical decision problems – evaluation, validity, satisfiability – for both quasi-boolean and Kripke semantics. Our results show a huge difference between both the semantics.

The complexity results for quasi-boolean semantics look quite usual. But unlike in classical propositional logic, we showed that satisfiability and validity reach their maximal hardness already for variable-free sequents. This means that sequents with variables essentially can bear not more information than sequents without variables in quasi-boolean semantics. The technique that we use to simulate propositional variables by variable-free primal terms can also be used to show that $\text{EVAL}_{qb}\text{-PL} \leq_m^{\log} \text{EVAL}_{qb}\text{-PL}_0$, what means that also the evaluation problem reaches its maximal complexity for variable-free sequents.

Under Kripke semantics, we got more surprising results. Evaluation and validity (both even for variable-free sequents) turned out to be as hard as satisfiability for variable-free sequents. All these problems are coNP-complete, whereas in general primal satisfiability proved to be Σ_2^P -complete. This means that sequents with variables are more expressive and bear more information than sequents without variables in primal logic under Kripke semantics. If we fix the number of variables to any constant k , we can use our small model theorem to exhibit a coNP upper bound for $\text{SAT}_{Kr}\text{-PL}_k$.

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