On the Sharpness of the Orlicz - Sobolev Imbedding Theorem *

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Abstract

The sharpness of the classical first order Orlicz - Sobolev imbedding theorem is discussed in the case in which the Orlicz function increases essentially more slowly than a power function

1 Introduction

If in the definition of the ordinary first order Sobolev space $W^1L_p(\Omega)$ the role played by the space $L_p(\Omega)$ is assumed instead by a more general Orlicz space $L_A(\Omega)$, the resulting structure is called an Orlicz - Sobolev and is denoted $W^1L_A(\Omega)$.

It seems that the first construction of Orlicz - Sobolev spaces has been made by Donaldson and Trudinger in [DT]; they have shown that Orlicz - Sobolev spaces possess many useful properties of ordinary Sobolev spaces. Also they obtained in [DT] imbeddings and compact imbeddings of Orlicz - Sobolev spaces analogous to those imbeddings and compact imbeddings commonly referred to under the titles "Sobolev Imbedding Theorem" and "Rellich Kondrachov Theorem" (the details can be found in [A2], 5.4 and 6.2). For all their results Donaldson and Trudinger assumed the boundedness of the domain (having the cone property) over which their function spaces are defined.

Adams extended in his paper [A1] the imbedding theorem for Orlicz - Sobolev spaces to every domain (having the cone property); his main result is presented in 2.6.
At the end of [A1] Adams asks if the imbedding result 2.6.1 is best possible if the N-function A increases essentially more slowly than does the N-function t^s for some $s < n$.

The aim of this paper is to give a negative answer to this question in the case that the domain Ω (on which the Orlicz - Sobolev space is defined) is bounded.

2 Preliminaries

Here we list briefly some definitions and well-known facts on Orlicz and Orlicz - Sobolev spaces. Standard references for Orlicz spaces are [KR] and [KJF]. For Orlicz - Sobolev spaces the reader is referred to [DT] and to chapter 8 of [A2].
An N-function (Orlicz function) is a continuous, convex function $A : [0,\infty) \to [0,\infty)$ which satisfies $\lim_{t\to 0} A(t)/t = 0$, $\lim_{t\to\infty} A(t)/t = \infty$ (see [A2], 8.2). The Young complementary of A is the N-function $\tilde{A} : [0,\infty) \to [0,\infty)$ defined by $\tilde{A}(\tau) = \sup\{t\tau - A(t) \mid t \geq 0\}$.

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Let Ω be some domain in $\mathbb{R}^n$ and μ the Lebesgue measure; the Orlicz space on Ω defined by the N-function A is denoted $L_A(\Omega)$ and is the linear hull of the set of all (equivalence classes of) Lebesgue measurable functions $u : \Omega \to \mathbb{C}$ for which $\int_\Omega A(|u(x)|)\, dx < \infty$. The functional

$$\|u\|_A = \inf \left\{ k > 0 \,\middle|\, \int_\Omega A\left(\frac{|u(x)|}{k}\right) dx \leq 1 \right\}$$

is a norm on $L_A(\Omega)$ (Luxemburg's norm) and $(L_A(\Omega)\, , \, \|\cdot\|_A)$ becomes a Banach space.
If A and $\tilde{A}$ are complementary N-functions a generalized version of Hölder's inequality holds; for each $u \in L_A(\Omega)$ and $v \in L_{\tilde{A}}(\Omega)$ we have:

$$\int_\Omega |u(x)v(x)|\, dx \leq 2\, \|u\|_A\, \|v\|_{\tilde{A}}$$

Let $E_A(\Omega)$ be the norm closure in $L_A(\Omega)$ of all measurable bounded functions which have bounded support in $\overline{\Omega}$ (the closure of Ω in $\mathbb{R}^n$). A characterization of the space $E_A(\Omega)$ is given in [KR](8.3):
Theorem 2.1 *If Ω is of finite Lebesgue measure the following assertions are equivalent:*

1. *$u \in E_A(\Omega)$*
2. *for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for every Lebesgue measurable subset M of Ω with $\mu(M) < \delta$ we have $\|u\chi_M\|_A < \varepsilon$*

If A and B are two N-functions we say that B dominates A near infinity if there exist two constants $t_0,\ k > 0$ with $A(t) \leq B(kt)$ for all $t \geq t_0$. The N-functions A and B are equivalent near infinity if each dominates the other near infinity. On domains from $\mathbb{R}^n$ having finite volume, N-functions which are equivalent near infinity determine identical Orlicz spaces.

Definition 2.2 *If A and B are two N-functions and if B dominates A near infinity but A and B are not equivalent near infinity then we say A increases essentially more slowly than B near infinity and write $A \prec\prec B$.*

Remark 2.3 (cf. [A2], 8.5) *If A and B are two N-functions we have*

$$A \prec\prec B \Longleftrightarrow \lim_{t\to\infty} \frac{B^{-1}(t)}{A^{-1}(t)} = 0$$

Definition 2.4 *We say that the normed space X is imbedded in the normed space Y and write $X \to Y$ to designate this imbedding, provided X is a vector subspace of Y and the identity operator I, defined on X into Y by $I(u) = u$ for all $u \in X$, is continuous.*

Since I is linear the second condition is equivalent to the existence of a constant $K > 0$ such that $\|I(u)\|_Y \leq K\|u\|_X$ for all $u \in X$.

For the following result the reader is referred to [A2] (8.16)

Theorem 2.5 *If Ω is a domain from $\mathbb{R}^n$ having finite volume and A and B are two N-functions such that $A \prec\prec B$ then $L_B(\Omega) \to E_A(\Omega)$.*

For an N-function A we define the space $W^1L_A(\Omega)$ consisting of those (equivalence classes of) functions $u \in L_A(\Omega)$ for which all first order distributional derivatives $D_j u$ $(1 \leq j \leq n)$ also belong to $L_A(\Omega)$. The space $W^1E_A(\Omega)$ is defined in analogous fashion.
It's easy to see that putting for every $u \in W^1L_A(\Omega)$

$$\|u\|_{1,A} = \|u\|_A + \sum_{j=1}^{n} \|D_j u\|_A \quad \text{then } \|\cdot\|_A \text{ defines a norm on } W^1L_A(\Omega)\,.$$

The spaces $W^1L_A(\Omega)$ and $W^1E_A(\Omega)$ equipped with the $\|\cdot\|_{1,A}$ norm are called (first order) Orlicz - Sobolev spaces.

We denote $C_B^0(\Omega)$ the space consisting of all continuous bounded functions u defined on Ω; $C_B^0(\Omega)$ is a Banach space with norm $\|u\| = \sup_{x\in\Omega}|u(x)|$. In the case of the imbedding of a normed function space X in $C_B^0(\Omega)$ the first condition from 2.4 is weakend to assert that some element in the equivalence class constituting u belongs to $C_B^0(\Omega)$.

Recall that given a point $x \in \mathbb{R}^n$ an open ball B_1 with center x and an open ball B_2 not containing x the set $C_x = B_1 \cap \{x + \lambda(y-x) \mid y \in B_2, \lambda > 0\}$ is called a finite cone in $\mathbb{R}^n$ having vertex at x. The domain Ω from $\mathbb{R}^n$ has the cone property if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C. Note that C_x need not be obtained from C by parallel translation, just by rigid motion (see [A2], 4.1 and 4.3).

The following imbedding theorem was proved by Donaldson and Trudinger in [DT] for bounded domains (having the cone property) and was extended to every domain (having the cone property) by Adams in [A1]. The proof for bounded domains is also presented in [A2].

Theorem 2.6 (first order Orlicz - Sobolev imbedding theorem)
Let Ω be some domain from $\mathbb{R}^n$ having the cone property and let A be an N-function.

1. *If $\int_1^\infty A^{-1}(t)t^{-(n+1)/n}\,dt = \infty$ then*
   (a) *if $\int_0^1 A^{-1}(t)t^{-(n+1)/n}\,dt < \infty$ then $W^1L_A(\Omega) \to L_{A_*}(\Omega)$ and $W^1E_A(\Omega) \to E_{A_*}(\Omega)$ where $A_*^{-1}(t) = \int_0^t A^{-1}(\tau)\tau^{-(n+1)/n}\,d\tau$ (A_* is called the Sobolev conjugate of A).*
   (b) *if $\int_0^1 A^{-1}(t)t^{-(n+1)/n}\,dt = \infty$ the imbeddings from above remain valid; in this case A_* can be constructed as in* [A1], 4.6, 4.7.

   Moreover, if Ω is bounded the imbedding $W^1L_A(\Omega) \to E_{A_1}(\Omega)$ is compact for each N-function A_1 which increases essentially more slowly than A_* near infinity.
2. *If $\int_1^\infty A^{-1}(t)t^{-(n+1)/n}\,dt < \infty$ then $W^1L_A(\Omega) \to C_B^0(\Omega)$.*

The above result is an extension of the first order Sobolev imbedding theorem for $W^1L_p(\Omega)$ $(1 \le p < \infty)$ which can be found in [A2] (5.4).

However the answer to the question if the imbeddings supplied by 2.6 are the best possible (in the sense that the smallest Orlicz spaces have been used as target spaces) can not be "yes" in general: if $A(t) = t^n/n$ then A_* is equivalent near infinity to the N-function $\exp(t) - t - 1$ which increases essentially more slowly than does the N-function $\exp(t^{n/(n-1)}) - 1$ which defines an Orlicz space in which $W^1L_n(\Omega)$ can be imbedded if Ω is bounded and has the cone property (see [T] and [A2], 8.25). The latter Orlicz space is known to be "best possible" target for $W^1L_n(\Omega)$; the details can be found in [HMT].

At the end of his paper [A1] Adams asks if the conclusion of the first part of theorem 2.6 gives a "best possible" imbedding if the N-function A increases essentially more slowly than does the N-function t^s for some $s < n$.

We would show in the next section that if the domain is bounded the answer is negative. Moreover, we would show that the imbedding 2.6.1 is not best posssible if A increases essentially more slowly than an N-function B with $\int_1^\infty B^{-1}(t)t^{-(n+1)/n}\,dt = \infty$.

3 An imbedding result

Let Ω be some bounded domain from $\mathbb{R}^n$ having the cone property. We would always assume that given an N-function B we have $\int_0^1 B^{-1}(t)t^{-(n+1)/n}\,dt < \infty$ replacing if necessary B by another N-function equivalent to B near infinity; clearly the above condition places no restrictions on B from the point of view of imbedding theorems since N-functions equivalent near infinity determine identical Orlicz spaces. For an N-function B (as above) which satisfies $\int_1^\infty B^{-1}(t)t^{-(n+1)/n}\,dt = \infty$ we denote B_* the Sobolev conjugate of B (see 2.6).

Lemma 3.1 *Let A and B two N-functions such that*

1. $A \prec\prec B$
2. $\int_1^\infty B^{-1}(t)t^{-(n+1)/n}\,dt = \infty$. *Then:*
   (a) $\int_1^\infty A^{-1}(t)t^{-(n+1)/n}\,dt = \infty$
   (b) $A_* \prec\prec B_*$

Proof. It's easy to check using 2.3 that the second condition from above leads to the first conclusion. For the second conclusion we use also 2.3. Since $A \prec\prec B$ it follows $\lim\limits_{t\to\infty} B^{-1}(t)/A^{-1}(t) = 0$. After a simple application of l'Hospital's theorem we obtain

$$\lim_{t\to\infty}\left(\int_0^t B^{-1}(\tau)\,\tau^{-(n+1)/n}\,d\tau\right)\Big/\left(\int_0^t A^{-1}(\tau)\,\tau^{-(n+1)/n}\,d\tau\right) = 0$$

and this is equivalent to $\lim\limits_{t\to\infty} B_*^{-1}(t)/A_*^{-1}(t) = 0$ which is (using again 2.3) in fact $A_* \prec\prec B_*$.

Theorem 3.2 *Let $n \geq 2$ and Ω a bounded domain from $\mathbb{R}^n$ having the cone property and let A and B two N-functions such that*

$$A \prec\prec B \tag{1}$$

$$\int_1^\infty B^{-1}(t)t^{-(n+1)/n}\,dt = \infty \tag{2}$$

Then:

$$W^1L_A(\Omega) \to L_{B_*}(\Omega) \quad \text{and} \quad W^1E_A(\Omega) \to E_{B_*}(\Omega)$$

where

$$B_*^{-1}(t) = \int_0^t B^{-1}(\tau)\tau^{-(n+1)/n}\,d\tau.$$

Moreover, if B_1 is any N-function increasing essentially more slowly near infinity than does B_* then the imbedding $W^1L_A(\Omega) \to L_{B_1}(\Omega)$ is compact.

Proof. The technique is that of Donaldson and Trudinger ([DT]).
We first prove the imbedding $W^1L_A(\Omega) \to L_{B_*}(\Omega)$. Let $u \in W^1L_A(\Omega)$ and suppose for the moment u is real valued, bounded on Ω and not zero in $L_A(\Omega)$. Since Ω is bounded clearly $\|\chi_\Omega\|_{\tilde{A}} < \infty$ and using Hölder's inequality we have

$$\int_\Omega |u(x)|\,dx \leq 2\,\|u\|_A\,\|\chi_\Omega\|_{\tilde{A}} \tag{3}$$

which shows that $L_A(\Omega) \to L_1(\Omega)$ and so $W^1L_A(\Omega) \to W^1L_1(\Omega)$.
Let as in 8.32 from [A2]: $K = \|u\|_{B_*}$ with

$$\int_\Omega B_*\left(\frac{|u(x)|}{K}\right)dx = 1$$

We are looking for a constant $C > 0$, independent of u, such that $K = \|u\|_{B_*} \le C\|u\|_{1,A}$.
Let also $\sigma(t) = (B_*(t))^{(n-1)/n}$ which is a Lipschitz function on the range of $|u|/K$ so that by a similar argument as in 8.32 from [A2] we have:

$$f(x) = \sigma\left(\frac{|u(x)|}{K}\right) \in W^1L_1(\Omega)$$

and

$$|(D_j f)(x)| \le \frac{1}{K}\left|\sigma'\left(\frac{|u(x)|}{K}\right)\right| \cdot |D_j u(x)| \quad \text{for every } 1 \le j \le n.$$

Via the imbedding $W^1L_1(\Omega) \to L_{n/(n-1)}(\Omega)$ (5.4 from [A2]) there exists a constant K_1 such that

$$\begin{aligned} 1 &= \left(\int_\Omega B_*\left(\frac{|u(x)|}{K}\right) dx\right)^{(n-1)/n} = \|f\|_{n/(n-1)} \le \\ &\le K_1\left(\sum_{j=1}^n \frac{1}{K}\int_\Omega \left|\sigma'\left(\frac{|u(x)|}{K}\right)\right| \cdot |D_j u(x)|\, dx + \int_\Omega \sigma\left(\frac{|u(x)|}{K}\right) dx\right) \end{aligned} \tag{4}$$

We denote

$$g(x) = \sigma'\left(\frac{|u(x)|}{K}\right)$$

As in the proof of 8.32 from [A2] we obtain $g \in L_{\tilde{B}}(\Omega)$ and $\|g\|_{\tilde{B}} \le (n-1)/n$
so

$$G : L_B(\Omega) \to \mathbb{C}; \quad G(u) = \int_\Omega u(x)g(x)\, dx \quad \text{if } u \in L_B(\Omega)$$

is a linear bounded functional on $L_B(\Omega)$ with $\|G\| \le 2(n-1)/n$ (cf. 8.17 from [A2]).
From (1) using 2.5 by the Hahn Banach theorem G has a norm preserving linear bounded extension $G_1 : E_A(\Omega) \to \mathbb{C}$ and that is: $\|G_1\| = \|G\| \le 2(n-1)/n$ and $G_1(u) = G(u)$ for all $u \in L_B(\Omega)$.
By [A2] (8.18) there exists a function $g_1 \in L_{\tilde{A}}(\Omega)$ such that $G_1(u) = \int_\Omega u(x)g_1(x)\, dx, \forall u \in L_A(\Omega)$ and $\|g_1\|_{\tilde{A}} \le \|G_1\| \le 2\|g_1\|_{\tilde{A}}$.
In particular for any $\varphi \in C_0^\infty(\Omega)$ we have $G_1(\varphi) = G(\varphi)$ and so $g_1 = g$ a.e on Ω.
Thus $g \in L_{\tilde{A}}(\Omega)$ and

$$\|g\|_{\tilde{A}} = \|g_1\|_{\tilde{A}} \le \|G_1\| = \|G\| \le 2\,\|g\|_{\tilde{B}} \le 2(n-1)/n$$

Using Hölder's inequality we have for each $1 \le j \le n$

$$\int_\Omega \left|\sigma'\left(\frac{|u(x)|}{K}\right)\right| \cdot |D_j u(x)|\, dx = \int_\Omega |g(x)| \cdot |D_j u(x)|\, dx \le 2\,\|D_j u\|_A \cdot \|g\|_{\tilde{A}} \le \frac{4(n-1)}{n}\|D_j u\|_A$$

As in the proof of 8.32 from [A2] we obtain that there exists a constant $K_2 > 0$ such that

$$K_1 \int_\Omega \sigma\left(\frac{|u(x)|}{K}\right) dx \le \frac{1}{2}\int_\Omega B_*\left(\frac{|u(x)|}{K}\right) dx + \frac{K_2}{K}\int_\Omega |u(x)|\, dx = \frac{1}{2} + \frac{K_2}{K}\int_\Omega |u(x)|\, dx$$

So from (4) we obtain:

$$1 \le \frac{K_1}{K}\sum_{j=1}^n \|D_j u\|_A \cdot \frac{4(n-1)}{n} + \frac{1}{2} + \frac{K_2}{K}\int_\Omega |u(x)|\, dx \tag{5}$$

Using (3) clearly (5) becomes:

$$K = \|u\|_{B_*} \le K_3\|u\|_{1,A} \tag{6}$$

Of course K_3 can depend on n, A, B and the cone determining the cone property for Ω.
Let now $u \in W^1L_A(\Omega)$ real valued and not necessary bounded.

We put for each $k \geq 1$

$$u_k(x) = \omega(|u(x)|) = \begin{cases} |u(x)| & : \quad |u(x)| \leq k \\ k \text{ sgn } u(x) & : \quad |u(x)| > k \end{cases}$$

where ω is defined on $\mathbb{R}$ and $\omega(t) = \begin{cases} t & : \quad |t| \leq k \\ k \text{ sgn}\, t & : \quad |t| > k \end{cases}$

Since $|u_k(x)| \leq |u(x)|$ we have $u_k \in L_A(\Omega)$.

Obviously ω is a Lipschitz function so that by lemma 8.31 from [A2] for every $1 \leq j \leq n$ we have $|D_j u_k(x)| \leq |\omega'(|u(x)|)| \cdot |D_j u(x)|$ which shows $D_j u_k \in L_A(\Omega)$ if $1 \leq j \leq n$.

Hence $u_k \in W^1 L_A(\Omega)$, for all $k \geq 1$.

From $|u_k| \leq |u|$, $|D_j u_k| \leq |D_j u|$ for all $1 \leq j \leq n$ we have $\|u_k\|_{1,A} \leq \|u\|_{1,A}$.

Every u_k is bounded so that by (6) we have $\|u_k\|_{B_*} \leq K_3 \|u\|_{1,A}$. The sequence $(\|u_k\|_{B_*})$ increases with k and is bounded by $K_3\|u\|_{1,A}$. Thus $\lim\limits_{k\to\infty} \|u_k\|_{B_*} = K_4$ exists and $K_4 \leq K_3 \|u\|_{1,A}$.

Since $u_k \to u$ a.e. by Fatou's lemma we have:

$$\int_\Omega B_* \left(\frac{|u(x)|}{K_4} \right) dx \;\leq\; \lim_{k\to\infty} \int_\Omega B_* \left(\frac{|u_k(x)|}{K_4} \right) dx \;\leq 1$$

whence $u \in L_{B_*}(\Omega)$ and $\|u\|_{B_*} \;\leq\; K_3 \|u\|_{1,A}$.

The last inequality can be extended to arbitrary complex valued elements of $W^1 L_A(\Omega)$ by separate application to real and imaginary part. So the proof of the imbedding $W^1 L_A(\Omega) \to L_{B_*}(\Omega)$ is finished.

We are going to prove now the imbedding $W^1 E_A(\Omega) \to E_{B_*}(\Omega)$. Let us denote, without loosing generality, K_3 the imbedding constant of $W^1 E_A(\Omega) \to L_{B_*}(\Omega)$ (which certainly exists by the above proof), that is $\|u\|_{B_*} \leq K_3 \|u\|_{1,A} \quad \forall u \in W^1 E_A(\Omega)$.

Let $u \in W^1 E_A(\Omega)$ and $\varepsilon > 0$. By 2.1 there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ such that if M is a measurable subset of Ω with $\mu(M) < \delta_0$ we have $\|u\chi_M\|_A < \varepsilon/(n+1)K_3$. Similarly for each $1 \leq j \leq n$ there exists a $\delta_j = \delta_j(\varepsilon) > 0$ such that if M is a measurable subset of Ω with $\mu(M) < \delta_j$ we have $\|(D_j u)\chi_M\|_A < \varepsilon/(n+1)K_3$.

Taking now $\delta = \delta(\varepsilon) = \min\{\delta_j \mid 0 \leq j \leq n\}$ then for any M with $\mu(M) < \delta$ clearly

$$\|u\chi_M\|_{B_*} \;\leq\; K_3 \|u\chi_M\|_{1,A} \;=\; K_3 \left(\|u\chi_M\|_A + \sum_{j=1}^{n} \|(D_j u)\chi_M\|_A \right) < \varepsilon.$$

The last relation shows by 2.1 (Ω is bounded) that $u \in E_{B_*}(\Omega)$ which completes the proof of the imbedding $W^1 E_A(\Omega) \to E_{B_*}(\Omega)$.

For the compactness assertion, since Ω is bounded we have $W^1 L_A(\Omega) \to W^1 L_1(\Omega) \to L_1(\Omega)$ the latter imbedding being compact by [A2], 6.2. Every bounded subset of $W^1 L_A(\Omega)$ is so precompact in $L_1(\Omega)$; since the inequality $\|u\|_{B_*} \;\leq\; K_3 \|u\|_{1,A}$ holds for every $u \in W^1 L_A(\Omega)$ every bounded subset of $W^1 L_A(\Omega)$ is bounded in $L_{B_*}(\Omega)$. The conclusion follows now by a simple application of 8.23 from [A2].

Remark 3.3 *If Ω is a bounded domain from $\mathbb{R}^n$ having the cone property, A and B are two N-functions such that A increases essentially more slowly than B and $\int_1^\infty B^{-1}(t) t^{-(n+1)/n}\, dt = \infty$ then (accordingly to 2.5) we have the imbeddings*

$$W^1 L_A(\Omega) \;\to L_{B_*}(\Omega) \;\to E_{A_*}(\Omega), \quad W^1 E_A(\Omega) \;\to E_{B_*}(\Omega) \;\to E_{A_*}(\Omega)$$

The above relations show that if the N-functions A and B are as in 3.2 the imbedding result 2.6.1 for Orlicz - Sobolev spaces $W^1 E_A(\Omega)$ and $W^1 L_A(\Omega)$ is not "best possible" on bounded domains.

Corollary 3.4 *Let Ω be a bounded domain from $\mathbb{R}^n$ having the cone property. Let A be an N-function which increases essentially more slowly than the N-function t^s for some $1 < s < n$. Then:*

1. $\int_1^\infty A^{-1}(t)t^{-(n+1)/n}\,dt = \infty$

2. $W^1E_A(\Omega) \rightarrow W^1L_A(\Omega) \rightarrow L_{ns/(n-s)}(\Omega)$

3. $A_* \prec\prec t^{ns/(n-s)}$

Proof. It's a simple application of 3.2 and 3.1 considering the N-function $B(t) = t^s$ and using that B_* is different from $t^{ns/(n-s)}$ by at most a constant.

The above corollary is in fact a negative answer to the question of Adams from [A1] if the result 2.6.1 gives a "best possible" imbedding if A increases essentially more slowly than does the N-function t^s for some $1 < s < n$.

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