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MODULAR SPECIES AND PRIME IDEALS FOR THE RING OF MONOMIAL REPRESENTATIONS OF A FINITE GROUP#

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The ring of monomial representations of a finite group has been investigated by Dress (1971) and Boltje (1990), among others. It is of interest in connection with induction theorems in representation theory. Its species have recently been determined by Boltje. In this article, we will analyze the block distribution of species. As an application, we will determine the prime ideals of the ring of monomial representations. The results here constitute a slightly modified version of part of the first author's Diplomarbeit (Fotsing, 2003), written under the direction of the second author.

Key Words: Character ring; Monomial representation; Representation ring; Species.

Subject Classification: 19A22; 20C15.

Let G be a finite group. In this article, we are concerned with $D(G)$, the *ring of monomial representations* of G. This ring has proved to be useful in connection with induction theorems in representation theory (cf. Boltje, 1990, to appear; Dress, 1971; Snaith, 1994). It is defined as the Grothendieck ring of the *monomial category* $\mathbf{mon}_{\mathbf{C}G}$. So we start by recalling the definition of and some facts about $\mathbf{mon}_{\mathbf{C}G}$.

The objects of $\mathbf{mon}_{\mathbf{C}G}$ are pairs $(V, \mathscr{L})$, where V is a finitely generated $\mathbf{C}G$-module and $\mathscr{L}$ is a set of one-dimensional subspaces (called *lines*) of V, such that $V = \bigoplus_{L\in\mathscr{L}} L$ and $gL \in \mathscr{L}$ for $g \in G$ and $L \in \mathscr{L}$. A morphism $f : (V, \mathscr{L}) \longrightarrow (W, \mathscr{M})$ in $\mathbf{mon}_{\mathbf{C}G}$ is a homomorphism of $\mathbf{C}G$-modules $f : V \longrightarrow W$ such that, for $L \in \mathscr{L}$, there exists $M \in \mathscr{M}$ with $f(L) \subseteq M$. Composition in $\mathbf{mon}_{\mathbf{C}G}$ is defined in the obvious way. In contrast to $\mathbf{mod}_{\mathbf{C}G}$, the category of finitely generated $\mathbf{C}G$-modules, the category $\mathbf{mon}_{\mathbf{C}G}$ is not additive, in general. (In Boltje, 2001, a different (i.e., non-equivalent) definition of the morphisms in $\mathbf{mon}_{\mathbf{C}G}$ was given. However, this change does not affect the results below.)

For objects $(V, \mathscr{L})$ and $(W, \mathscr{M})$ in $\mathbf{mon}_{\mathbf{C}G}$, there are a *direct sum*

$$(V, \mathscr{L}) \oplus (W, \mathscr{M}) := (V \oplus W, \mathscr{L} \cup \mathscr{M})$$

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and a *tensor product*

$$(V, \mathscr{L}) \otimes (W, \mathscr{M}) := (V \otimes_{\mathbf{C}} W, \{L \otimes M : L \in \mathscr{L}, M \in \mathscr{M}\})$$

in $\mathbf{mon}_{\mathbf{C}G}$, with the usual properties. Moreover, for a subgroup H of G, one has *restriction* and *induction functors*

$$\mathrm{Res}^G_H : \mathbf{mon}_{\mathbf{C}G} \longrightarrow \mathbf{mon}_{\mathbf{C}H}, \qquad \mathrm{Ind}^G_H : \mathbf{mon}_{\mathbf{C}H} \longrightarrow \mathbf{mon}_{\mathbf{C}G}$$

given as follows: for $(V, \mathscr{L}) \in \mathbf{mon}_{\mathbf{C}G}$, one has $\mathrm{Res}^G_H(V, \mathscr{L}) = (\mathrm{Res}^G_H(V), \mathscr{L})$, where $\mathrm{Res}^G_H(V)$ denotes the $\mathbf{C}H$-module obtained by restriction from $\mathbf{C}G$. For $(W, \mathscr{M}) \in \mathbf{mon}_{\mathbf{C}H}$, one has

$$\mathrm{Ind}^G_H(W, \mathscr{M}) = (\mathrm{Ind}^G_H(W), \{g \otimes M : g \in G, M \in \mathscr{M}\}),$$

where $\mathrm{Ind}^G_H(W) = \mathbf{C}G \otimes_{\mathbf{C}H} W$ denotes the $\mathbf{C}G$-module obtained by induction from $\mathbf{C}H$. The effect of the functors Res^G_H and Ind^G_H on morphisms is the obvious one.

An object $(V, \mathscr{L})$ in $\mathbf{mon}_{\mathbf{C}G}$ with $V \neq 0$ is called *indecomposable* if $(V, \mathscr{L}) \cong (V_1, \mathscr{L}_1) \oplus (V_2, \mathscr{L}_2)$ in $\mathbf{mon}_{\mathbf{C}G}$ implies that $V_1 = 0$ or $V_2 = 0$. Thus $(V, \mathscr{L})$ is indecomposable if and only if the action of G on $\mathscr{L}$ is transitive.

Every group homomorphism $\phi : G \longrightarrow \mathbf{C}^\times$ defines an indecomposable object $(\mathbf{C}_\phi, \{\mathbf{C}_\phi\})$ in $\mathbf{mon}_{\mathbf{C}G}$, where $\mathbf{C}_\phi$ denotes the $\mathbf{C}G$-module $\mathbf{C}$, on which G acts via

$$g\alpha = \phi(g)\alpha \qquad (g \in G, \alpha \in \mathbf{C}).$$

More generally, the objects $\mathrm{Ind}^G_H(\mathbf{C}_\psi, \{\mathbf{C}_\psi\})$, where H is a subgroup of G and $\psi : H \longrightarrow \mathbf{C}^\times$ is a group homomorphism, are indecomposable in $\mathbf{mon}_{\mathbf{C}G}$. Moreover, every indecomposable object of $\mathbf{mon}_{\mathbf{C}G}$ is isomorphic to one of this form. For subgroups H, K of G and group homomorphisms $\psi : H \longrightarrow \mathbf{C}^\times$, $\omega : K \longrightarrow \mathbf{C}^\times$, one has $\mathrm{Ind}^G_H(\mathbf{C}_\psi, \{\mathbf{C}_\psi\}) \cong \mathrm{Ind}^G_K(\mathbf{C}_\omega, \{\mathbf{C}_\omega\})$ in $\mathbf{mon}_{\mathbf{C}G}$ if and only if $K = {}^xH$ and $\omega = {}^x\psi$ for some $x \in G$; here ${}^xH = xHx^{-1}$, ${}^xh = xhx^{-1}$ and $({}^x\psi)({}^xh) = \psi(h)$ for $h \in H$.

In this way one is led to consider

$$\mathscr{M}(G) := \{(H, \psi) : H \leq G, \psi \in \mathrm{Hom}(H, \mathbf{C}^\times)\},$$

the set of *monomial pairs* of G. The group G acts on $\mathscr{M}(G)$ via the conjugation

$${}^x(H, \psi) = ({}^xH, {}^x\psi) \qquad (x \in G, (H, \psi) \in \mathscr{M}(G)).$$

For $(H, \psi) \in \mathscr{M}(G)$, we denote the stabilizer of (H, ψ) in G by

$$N_G(H, \psi) := \{x \in G : {}^x(H, \psi) = (H, \psi)\},$$

so that $H \subseteq N_G(H, \psi) \subseteq N_G(H)$, and we denote the orbit of (H, ψ) by $[H, \psi]_G$. Also we denote the set of orbits by

$$\mathscr{M}(G)/G = \{[H, \psi]_G : (H, \psi) \in \mathscr{M}(G)\}.$$

Then, as we saw above, the isomorphism classes $[V, \mathcal{L}]_G$ of indecomposable objects $(V, \mathcal{L})$ in $\mathbf{mon}_{\mathbf{C}G}$ are in bijection with $\mathcal{M}(G)/G$. The bijection maps $[H, \psi]_G \in \mathcal{M}(G)/G$ to $[\mathrm{Ind}_H^G(\mathbf{C}_\psi, \{\mathbf{C}_\psi\})]_G$.

The Grothendieck ring $D(G)$ of the category $\mathbf{mon}_{\mathbf{C}G}$ is defined in the usual way, with

$$[V, \mathcal{L}]_G + [W, \mathcal{M}]_G = [(V, \mathcal{L}) \oplus (W, \mathcal{M})]_G$$

and

$$[V, \mathcal{L}]_G \cdot [W, \mathcal{M}]_G := [(V, \mathcal{L}) \otimes (W, \mathcal{M})]_G,$$

for $(V, \mathcal{L}), (W, \mathcal{M}) \in \mathbf{mon}_{\mathbf{C}G}$. The isomorphism classes of indecomposable objects in $\mathbf{mon}_{\mathbf{C}G}$ form a $\mathbf{Z}$-basis for the additive group of $D(G)$; in particular, the additive group of $D(G)$ is free abelian of rank $|\mathcal{M}(G)/G|$. If we identify an element $[H, \psi]_G \in \mathcal{M}(G)/G$ with the isomorphism class $[\mathrm{Ind}_H^G(\mathbf{C}_\psi, \{\mathbf{C}_\psi\})]_G$ in $\mathbf{mon}_{\mathbf{C}G}$ then we can also view $\mathcal{M}(G)/G$ as a $\mathbf{Z}$-basis of the additive group of $D(G)$. In this notation, the product on $D(G)$ is given by

$$[H, \psi]_G \cdot [K, \omega]_G = \sum_{HgK \in H\backslash G/K} [H \cap {}^gK, \psi({}^g\omega)]_G,$$

where $\psi({}^g\omega) : H \cap {}^gK \longrightarrow \mathbf{C}^\times$ is defined by $(\psi({}^g\omega))(x) = \psi(x)\omega(g^{-1}xg)$ for $x \in H \cap {}^gK$.

For any subgroup H of G, the functors $\mathrm{Res}_H^G : \mathbf{mon}_{\mathbf{C}G} \longrightarrow \mathbf{mon}_{\mathbf{C}H}$ and $\mathrm{Ind}_H^G : \mathbf{mon}_{\mathbf{C}H} \longrightarrow \mathbf{mon}_{\mathbf{C}G}$ give rise to group homomorphisms

$$\mathrm{res}_H^G : D(G) \longrightarrow D(H), \qquad [V, \mathcal{L}]_G \longmapsto [\mathrm{Res}_H^G(V, \mathcal{L})]_H,$$

$$\mathrm{ind}_H^G : D(H) \longrightarrow D(G), \qquad [W, \mathcal{M}]_H \longmapsto [\mathrm{Ind}_H^G(W, \mathcal{M})]_G.$$

These group homomorphisms have the usual properties. In particular, res_H^G is even a ring homomorphism, and the image of ind_H^G is an ideal in $D(G)$. Moreover, in the notation introduced above, we have

$$\mathrm{ind}_H^G([K, \omega]_H) = [K, \omega]_G \qquad ((K, \omega) \in \mathcal{M}(H)),$$

$$\mathrm{res}_H^G([K, \omega]_G) = \sum_{HgK \in H\backslash G/K} [H \cap {}^gK, {}^g\omega|H \cap {}^gK]_H \qquad ((K, \omega) \in \mathcal{M}(G)).$$

In the following, we will denote the character ring of G by $R(G)$. There is a ring homomorphism $\pi_G : D(G) \longrightarrow R(G/G')$ defined by $\pi_G([H, \psi]_G) = 0$ if $H < G$ and $\pi_G([H, \psi]_G) = \overline{\psi}$ if $H = G$; here G' denotes the commutator group of G, and $\overline{\psi} : G/G' \longrightarrow \mathbf{C}^\times$ is the group homomorphism given by $\overline{\psi}(gG') = \psi(g)$ for $g \in G$.

For $g \in G$, there is a ring homomorphism $t_g : R(G) \longrightarrow \mathbf{C}$ defined by $t_g(\chi) = \chi(g)$ for $\chi \in R(G)$.

The *species* of $D(G)$, i.e. the ring homomorphisms $D(G) \longrightarrow \mathbf{C}$, have recently been determined by Boltje (to appear). In order to construct them, it is useful to introduce the set

$$\mathcal{D}(G) := \{(H, hH') : H \leq G, h \in H\}.$$

Then G acts on $\mathcal{D}(G)$ via the conjugation

$${}^{g}(H, hH') = ({}^{g}H, {}^{g}h{}^{g}H') \qquad (g \in G, (H, hH') \in \mathcal{D}(G)).$$

For $(H, hH') \in \mathcal{D}(G)$, we denote its orbit by $[H, hH']_G$ and its stabilizer by

$$N_G(H, hH') = \{g \in G : {}^{g}(H, hH') = (H, hH')\}.$$

Then

$$H' \subseteq H \subseteq N_G(H, hH') \subseteq N_G(H) \subseteq N_G(H')$$

and

$$N_G(H, hH')/H' = C_{N_G(H)/H'}(hH').$$

In the following, we will denote the set of orbits of G on $\mathcal{D}(G)$ by

$$\mathcal{D}(G)/G = \{[H, hH']_G : (H, hH') \in \mathcal{D}(G)\}.$$

Each $(H, hH') \in \mathcal{D}(G)$ defines a species of $D(G)$ as a composition of the following maps:

$$s_{(H,hH')} : D(G) \xrightarrow{\mathrm{res}^G_H} D(H) \xrightarrow{\pi_H} R(H/H') \xrightarrow{t_{hH'}} \mathbf{C}.$$

Boltje (to appear) has shown that every species of $D(G)$ arises in this way. Moreover, one has $s_{(H,hH')} = s_{(K,kK')}$ if and only if $(K, kK') = {}^{g}(H, hH')$ for some $g \in G$. This means that the species of $D(G)$ are in bijection with $\mathcal{D}(G)/G$.

We note that the species of $D(G)$ all take their values in $\mathbf{Z}[\zeta]$, where ζ is a primitive $|G|$-th root of unity in $\mathbf{C}$. We fix a maximal ideal P of $\mathbf{Z}[\zeta]$. Then $\mathbf{Z}[\zeta]/P$ is a finite field, and we denote its characteristic by p. We call the species $s_{(H,hH')}$ and $s_{(K,kK')}$ of $D(G)$ *P-equivalent* if

$$s_{(H,hH')}(x) \equiv s_{(K,kK')}(x) \qquad (\mathrm{mod}\ P)$$

for all $x \in D(G)$. In this case, we also call the pairs $(H, hH'), (K, kK') \in \mathcal{D}(G)$ *P-equivalent* and write $(H, hH') \equiv_P (K, kK')$. Our aim is to determine the P-equivalence classes of $\mathcal{D}(G)$.

In order to state our first (easy and well-known) lemma, we recall that every element $g \in G$ can be written uniquely in the form $g = g_p g_{p'} = g_{p'} g_p$, where g_p is a p-element and $g_{p'}$ is a p'-element in G. Then g_p is called the *p-factor*, and $g_{p'}$ is called the *p'-factor* of g.

Lemma 1. *Let $(H, hH') \in \mathcal{D}(G)$. Then $(H, hH') \equiv_P (H, h_{p'}H')$.*

Proof. By the definition of $s_{(H,hH')}$ and $s_{(H,h_{p'}H')}$, it suffices to show that $t_{hH'}(\psi) \equiv t_{h_{p'}H'}(\psi) \pmod{P}$ for $\psi \in \mathrm{Hom}(H/H', \mathbf{C}^\times)$. But

$$t_{hH'}(\psi) - t_{h_{p'}H'}(\psi) = \psi(hH') - \psi(h_{p'}H') = (\psi(h_pH') - 1)\psi(h_{p'}H'),$$

and $\psi(h_pH') + P$ is a root of unity of p-power order in the field $\mathbf{Z}[\zeta]/P$ of characteristic p, so $\psi(h_pH') + P = 1 + P$, i.e. $\psi(h_pH') - 1 \in P$, and the result follows.

Lemma 1 gives an easy method to construct P-equivalent species. The following result gives another such method.

Lemma 2. *Let $(H, hH') \in \mathcal{D}(G)$, and let K/H be a p-subgroup of $N_G(H, hH')/H$. Then $(H, hH') \equiv_P (K, hK')$.*

Proof. By the definition of $s_{(H,hH')}$ and $s_{(K,hK')}$, we may assume that $G = K$ and $H' = 1$. Then H is a normal subgroup of G, G/H is a p-group, and $h \in Z(G)$. Let $(L, \lambda) \in \mathcal{M}(G)$. We need to show that

$$s_{(G,hG')}([L, \lambda]_G) \equiv s_{(H,hH')}([L, \lambda]_G) \qquad \pmod{P}.$$

If $L = G$, then $s_{(G,hG')}([L, \lambda]_G) = \lambda(h) = s_{(H,hH')}([L, \lambda]_G)$. So we may assume that $L < G$. Then $s_{(G,hG')}([L, \lambda]_G) = 0$. If $H \not\subseteq L$, then $s_{(H,hH')}([L, \lambda]_G) = 0$ since

$$\pi_H(\mathrm{res}^G_H([L, \lambda]_G)) = \sum_{HgL \in H\backslash G/L} \pi_H([H \cap {}^gL, {}^g\lambda|H \cap {}^gL]_H) = 0.$$

It remains to consider the case $H \leq L < G$. But then

$$s_{(H,hH')}([L, \lambda]_G) = \sum_{gL \in G/L} \lambda(ghg^{-1}) = |G : L|\lambda(h) \equiv 0 \qquad \pmod{P},$$

and the result is proven.

We are going to show that a combination of the two methods given by Lemmas 1 and 2 determines the P-equivalence classes of $\mathcal{D}(G)$. In order to explain this, let $(H, hH') \in \mathcal{D}(G)$. Then $(H, hH') \equiv_P (H, h_{p'}H')$ by Lemma 1. Next, let H_1/H be a Sylow p-subgroup of $N_G(H, h_{p'}H')/H$. Then $(H, h_{p'}H') \equiv_P (H_1, h_{p'}H_1')$ by Lemma 2. Similarly, let H_2/H_1 be a Sylow p-subgroup of $N_G(H_1, h_{p'}H_1')/H_1$. Then $(H_1, h_{p'}H_1') \equiv_P (H_2, h_{p'}H_2')$ by Lemma 2. We continue in this way until we reach a pair $(H_n, h_{p'}H_n') = (H_{n+1}, h_{p'}H_{n+1}') = \ldots$.

Setting

$$\mathcal{D}_p(G) := \{(K, kK') \in \mathcal{D}(G) : |\langle k\rangle| \not\equiv 0 \not\equiv |N_G(K, kK') : K| (\mathrm{mod}\, p)\}$$

we have $(H_n, h_{p'}H_n') \in \mathcal{D}_p(G)$. We call the elements in $\mathcal{D}_p(G)$ *p-regular* and the pair $(H_n, h_{p'}H_n')$ a *p-regularization* of (H, hH'). Note that $O^p(H_n) \subseteq H$, that $(H_n, h_{p'}H_n')$ is uniquely determined by (H, hH'), up to conjugation, and that (H, hH') is

P-equivalent to each of its p-regularizations. Our next result shows that two pairs in $\mathcal{D}_p(G)$ are P-equivalent if and only if they are conjugate.

Proposition 3. *Let $(H, hH'), (K, kK') \in \mathcal{D}_p(G)$, and suppose that $(H, hH') \equiv_P (K, kK')$. Then $(K, kK') = {}^g(H, hH')$ for some $g \in G$.*

Proof. We write $H/H' = (A/H') \times (B/H')$ with a p'-group A/H' and a p-group B/H'. Moreover, we denote by $\lambda_1, \dots, \lambda_r$ the group homomorphisms $H \longrightarrow \mathbf{C}^\times$ containing B in their kernel, so that $r = |H : B| \not\equiv 0 \pmod p$. Furthermore, we set

$$y := \sum_{i=1}^{r} \lambda_i(h^{-1}) s_{(H,hH')}([H, \lambda_i]_G), \qquad z := \sum_{i=1}^{r} \lambda_i(h^{-1}) s_{(K,kK')}([H, \lambda_i]_G) \in \mathbf{Z}[\zeta],$$

so that $y \equiv z \pmod P$ by our hypothesis. For $i = 1, \dots, r$, we have

$$\mathrm{res}_H^G([H, \lambda_i]_G) = \sum_{HgH \in H\backslash G/H} [H \cap {}^gH, {}^g\lambda_i | H \cap {}^gH]_H,$$

so $\pi_H(\mathrm{res}_H^G([H, \lambda_i]_G)) = \sum_{gH \in N_G(H)/H} \overline{{}^g\lambda_i}$ where $\overline{{}^g\lambda_i} : H/H' \longrightarrow \mathbf{C}^\times$ is defined by $\overline{{}^g\lambda_i}(xH') = \lambda_i(g^{-1}xg)$ for $x \in H$. Thus

$$s_{(H,hH')}([H, \lambda_i]_G) = \sum_{gH \in N_G(H)/H} \lambda_i(g^{-1}hg)$$

and $y = \sum_{gH \in N_G(H)/H} \sum_{i=1}^{r} \lambda_i(h^{-1})\lambda_i(g^{-1}hg)$. By the orthogonality relations for H/B, we have

$$\sum_{i=1}^{r} \lambda_i(h^{-1})\lambda_i(g^{-1}hg) = 0$$

unless $g^{-1}hgB = hB$. But $h \in A$ since $(H, hH') \in \mathcal{D}_p(G)$, so $g^{-1}hgB = hB$ is equivalent to $g^{-1}hgH' = hH'$, and in this case we have $\sum_{i=1}^{r} \lambda_i(h^{-1})\lambda_i(g^{-1}hg) = r$. We conclude that $y = |N_G(H, hH') : H| \cdot r \not\equiv 0 \pmod P$. This implies that $0 \not\equiv s_{(H,hH')}([H, \lambda_i]_G) \equiv s_{(K,kK')}([H, \lambda_i]_G) \pmod P$ for some $i \in \{1, \dots, r\}$. In particular, we have $0 \neq s_{(K,kK')}([H, \lambda_i]_G)$ and

$$0 \neq \pi_K(\mathrm{res}_K^G([H, \lambda_i]_G)) = \sum_{KgH \in K\backslash G/H} \pi_K([K \cap {}^gH, {}^g\lambda_i | K \cap {}^gH]_K).$$

This means that $K = K \cap {}^gH \subseteq {}^gH$ for some $g \in G$. By symmetry, we have $H \subseteq {}^{g'}K$ for some $g' \in G$. Thus H and K are conjugate in G. We may therefore assume that $H = K$, and it remains to show that hH' and kH' are conjugate in $N_G(H)/H'$. We assume that this is not the case. Then hB and kB are not conjugate in $N_G(H)/B$, and

$$0 \not\equiv y \equiv z \equiv \sum_{gH \in N_G(H)/H} \sum_{i=1}^{r} \lambda_i(h^{-1})\lambda_i(g^{-1}kg) \qquad \pmod P,$$

by a computation similar to the one above. By the orthogonality relations for H/B, we have

$$\sum_{i=1}^{r} \lambda_i(h^{-1})\lambda_i(g^{-1}kg) = 0$$

for $gH \in N_G(H)/H$, so $z = 0$. This is a contradiction, so the result follows.

This leads us to the main result of this article.

Theorem 4. *Let ζ be a primitive $|G|$th root of unity in $\mathbf{C}$, let P be a maximal ideal of $\mathbf{Z}[\zeta]$, and let p denote the characteristic of the field $\mathbf{Z}[\zeta]/P$. Then each P-equivalence class of $\mathscr{D}(G)$ contains a unique conjugacy class of $\mathscr{D}_p(G)$. This way, the P-equivalence classes of species of $D(G)$ are in bijection with*

$$\mathscr{D}_p(G)/G = \{[H, hH']_G : (H, hH') \in \mathscr{D}_p(G)\}.$$

As an application, we will determine the prime spectrum $\mathrm{Spec}(D(G))$ of $D(G)$. We begin by investigating the prime spectrum $\mathrm{Spec}(D_{\mathbf{Z}[\zeta]}(G))$ of $D_{\mathbf{Z}[\zeta]}(G) := \mathbf{Z}[\zeta] \otimes_{\mathbf{Z}} D(G)$. For $(H, hH') \in \mathscr{D}(G)$, the map

$$D_{\mathbf{Z}[\zeta]}(G) \longrightarrow \mathbf{Z}[\zeta], \qquad a \otimes x \longmapsto as_{(H,hH')}(x) \qquad (a \in \mathbf{Z}[\zeta], x \in D(G)),$$

is a homomorphism of rings, which we denote by $s_{(H,hH')}$ again. Then, for $P \in \mathrm{Spec}(\mathbf{Z}[\zeta])$,

$$\mathscr{P}(H, hH', P) := \{y \in D_{\mathbf{Z}[\zeta]}(G) : s_{(H,hH')}(y) \in P\}$$

is a prime ideal of $D_{\mathbf{Z}[\zeta]}(G)$. Moreover, $s_{(H,hH')}$ induces an isomorphism of rings

$$D_{\mathbf{Z}[\zeta]}(G)/\mathscr{P}(H, hH', P) \longrightarrow \mathbf{Z}[\zeta]/P, \qquad y + \mathscr{P}(H, hH', P) \longmapsto s_{(H,hH')}(y) + P;$$

in particular, we have

$$\mathrm{char}\ D_{\mathbf{Z}[\zeta]}(G)/\mathscr{P}(H, hH', P) = \mathrm{char}\ \mathbf{Z}[\zeta]/P.$$

We prove

Proposition 5. *Every prime ideal of $D_{\mathbf{Z}[\zeta]}(G)$ has the form $\mathscr{P}(H, hH', P)$ for some $(H, hH') \in \mathscr{D}(G)$ and some $P \in \mathrm{Spec}(\mathbf{Z}[\zeta])$.*

Proof. It is known (cf. Boltje, to appear) that the ring homomorphisms $s_{(H,hH')}$: $D_{\mathbf{Z}[\zeta]}(G) \longrightarrow \mathbf{Z}[\zeta]$, where (H, hH') runs through $\mathscr{D}(G)$, induce a monomorphism of rings

$$D_{\mathbf{Z}[\zeta]}(G) \longrightarrow \mathbf{Z}[\zeta]^{|\mathscr{D}(G)/G|}.$$

Hence

$$\prod_{(H,hH')\in\mathscr{D}(G)} \mathrm{Ker}(s_{(H,hH')}) \subseteq \bigcap_{(H,hH')\in\mathscr{D}(G)} \mathrm{Ker}(s_{(H,hH')}) = 0.$$

Now let $\mathscr{P}$ be a prime ideal of $D_{\mathbf{Z}[\zeta]}(G)$. Then $\mathscr{P}$ contains $\mathrm{Ker}(s_{(H,hH')})$ for some $(H, hH') \in \mathscr{D}(G)$. Thus $P := s_{(H,hH')}(\mathscr{P})$ is a prime ideal of $\mathbf{Z}[\zeta]$. We conclude that

$$\mathscr{P} = \{y \in D_{\mathbf{Z}[\zeta]}(G) : s_{(H,hH')}(y) \in P\}.$$

It remains to determine the fibres of the map

$$\Phi : \mathscr{D}(G) \times \mathrm{Spec}(\mathbf{Z}[\zeta]) \longrightarrow \mathrm{Spec}(D_{\mathbf{Z}[\zeta]}(G)), \qquad (H, hH', P) \longmapsto \mathscr{P}(H, hH', P).$$

We certainly have $\mathscr{P}(H, hH', P) = \mathscr{P}(K, kK', P)$, whenever $(H, hH'), (K, kK') \in \mathscr{D}(G)$ are conjugate in G. Also, we have $\mathscr{P}(H, hH', P) = \mathscr{P}(K, kK', P)$, whenever $(H, hH'), (K, kK') \in \mathscr{D}(G)$ are P-equivalent. We conclude by

Corollary 6. (i) *Every prime ideal of $D_{\mathbf{Z}[\zeta]}(G)$ of residue characteristic zero has the form $\mathscr{P}(H, hH', 0)$ for some $(H, hH') \in \mathscr{D}(G)$.*

(ii) *Every prime ideal of $D_{\mathbf{Z}[\zeta]}(G)$ of residue characteristic $p > 0$ has the form $\mathscr{P}(H, hH', P)$ for some $(H, hH') \in \mathscr{D}_p(G)$ and some $P \in \mathrm{Spec}(\mathbf{Z}[\zeta])$ containing p.*

We are now in a position to determine the fibres of Φ.

Proposition 7. (i) *Let $(H, hH'), (K, kK') \in \mathscr{D}(G)$ such that $\mathscr{P}(H, hH', 0) = \mathscr{P}(K, kK', 0)$. Then (H, hH') and (K, kK') are conjugate in G.*

(ii) *Let $P, Q \in \mathrm{Spec}(\mathbf{Z}[\zeta])$ such that $\mathrm{char}\,\mathbf{Z}[\zeta]/P = \mathrm{char}\,\mathbf{Z}[\zeta]/Q = p > 0$, and let $(H, hH'), (K, kK') \in \mathscr{D}_p(G)$ such that $\mathscr{P}(H, hH', P) = \mathscr{P}(K, kK', Q)$. Then $P = Q$, and $(H, hH'), (K, kK')$ are conjugate in G.*

Proof. (i) It is easy to see that $s_{(H,hH')}([H, 1]_G) = |N_G(H) : H| \neq 0$. Thus $[H, 1]_G \notin \mathscr{P}(H, hH', 0) = \mathscr{P}(K, kK', 0)$, so $s_{(K,kK')}([H, 1]_G) \neq 0$; in particular, we have

$$0 \neq \pi_K(\mathrm{res}^G_K([H, 1]_G)) = \sum_{KgH\in K\backslash G/H} \pi_K([K \cap {}^gH, 1_{K\cap {}^gH}]_K).$$

Hence $K = K \cap {}^gH \subseteq {}^gH$ for some $g \in G$. By symmetry, it follows that K and H are conjugate in G. So we may assume that $H = K$.

Let $\lambda_1, \ldots, \lambda_r$ denote the group homomorphisms $H \longrightarrow \mathbf{C}^\times$, and let

$$y := \sum_{i=1}^{r} \lambda_i(h^{-1})[H, \lambda_i]_G \in D_{\mathbf{Z}[\zeta]}(G).$$

It is easy to see that $s_{(H,hH')}(y) = |N_G(H, hH') : H'| \neq 0$. Thus $y \notin \mathscr{P}(H, hH', 0) = \mathscr{P}(H, kH', 0)$, and $s_{(H,kH')}(y) \neq 0$. The orthogonality relations for H/H' imply that kH' and hH' are conjugate in $N_G(H)/H'$, and (i) follows.

(ii) In the proof of Proposition 3, we had found a group homomorphism $\lambda : H \longrightarrow \mathbf{C}^\times$ such that $s_{(H,hH')}([H, \lambda]_G) \not\equiv 0 \pmod{P}$. Thus $[H, \lambda]_G \notin \mathscr{P}(H, hH', P) = \mathscr{P}(K, kK', Q)$; in particular, we have $s_{(K,kK')}([H, \lambda]_G) \neq 0$. As above, it follows easily that $K \subseteq {}^gH$ for some $g \in G$. By symmetry, H and K are conjugate in G. We may assume that $H = K$.

As in the proof of Proposition 3, we write $H/H' = A/H' \times B/H'$ with a p'-group A/H' and a p-group B/H'. Moreover, we denote the group homomorphisms $H \longrightarrow \mathbf{C}^\times$ containing B in their kernel by $\lambda_1, \ldots, \lambda_r$. We set

$$y := \sum_{i=1}^{r} \lambda_i(h^{-1})[H, \lambda_i]_G \in D_{\mathbf{Z}[\zeta]}(G).$$

The proof of Proposition 3 shows that

$$s_{(H,hH')}(y) = |N_G(H, hH') : B| \not\equiv 0 \pmod{P}.$$

Thus $y \notin \mathscr{P}(H, hH', P) = \mathscr{P}(H, kH', Q)$, and $s_{(H,kH')}(y) \not\equiv 0 \pmod{Q}$. As in the proof of Proposition 3, we conclude that kH' is conjugate to hH' in $N_G(H)/H'$. Finally, we obtain that

$$P = s_{(H,hH')}(\mathscr{P}(H, hH', P)) = s_{(K,kK')}(\mathscr{P}(K, kK', Q)) = Q,$$

and the result is proven.

Next, we indicate how to descend from $\mathrm{Spec}(D_{\mathbf{Z}[\zeta]}(G))$ to $\mathrm{Spec}(D(G))$, by using some Galois theory. In the following, we denote the Galois group of $\mathbf{Q}(\zeta)$ over $\mathbf{Q}$ by Γ. There is an isomorphism of groups

$$(\mathbf{Z}/|G|\mathbf{Z})^\times \longrightarrow \Gamma, \qquad k + |G|\mathbf{Z} \longmapsto \sigma_k,$$

such that $\sigma_k(\zeta) = \zeta^k$. Moreover, Γ acts on G (considered just as a set) by

$$\sigma_k(g) := g^k \qquad (g \in G, k + |G|\mathbf{Z} \in (\mathbf{Z}/|G|\mathbf{Z})^\times).$$

For $(H, hH') \in \mathscr{D}(G)$, $k + |G|\mathbf{Z} \in (\mathbf{Z}/|G|\mathbf{Z})^\times$ and $x \in D(G)$, we have

$$\sigma_k(s_{(H,hH')}(x)) = \sigma_k(t_{hH'}(\pi_H(\mathrm{res}^G_H(x)))) = t_{h^kH'}(\pi_H(\mathrm{res}^G_H(x))) = s_{(H,\sigma_k(h)H')}(x),$$

so

$$\sigma \circ s_{(H,hH')} = s_{(H,\sigma(h)H')} : D(G) \longrightarrow \mathbf{C}$$

for $\sigma \in \Gamma$ and $(H, hH') \in \mathscr{D}(G)$. Also, Γ acts on $D_{\mathbf{Z}[\zeta]}(G)$ in such a way that

$$\sigma(a \otimes x) = \sigma(a) \otimes x \qquad (\sigma \in \Gamma, a \in \mathbf{Z}[\zeta], x \in D(G)).$$

It is easy to verify that

$$D_{\mathbf{Z}[\zeta]}(G)^\Gamma := \{y \in D_{\mathbf{Z}[\zeta]}(G) : \sigma(y) = y \text{ for } \sigma \in \Gamma\} = 1 \otimes D(G).$$

For $\sigma \in \Gamma$, $(H, hH') \in \mathscr{D}(G)$, $a \in \mathbf{Z}[\zeta]$ and $x \in D(G)$, we have

$$\begin{aligned}\sigma(s_{(H,hH')}(\sigma^{-1}(a \otimes x))) &= \sigma(s_{(H,hH')}(\sigma^{-1}(a) \otimes x)) = \sigma(\sigma^{-1}(a)s_{(H,hH')}(x)) \\ &= a\sigma(s_{(H,hH')}(x)) = as_{(H,\sigma(h)H')}(x) = s_{(H,\sigma(h)H')}(a \otimes x).\end{aligned}$$

We conclude that

$$\sigma \circ s_{(H,hH')} \circ \sigma^{-1} = s_{(H,\sigma(h)H')} : D_{\mathbf{Z}[\zeta]}(G) \longrightarrow \mathbf{C}$$

for $\sigma \in \Gamma$ and $(H, hH') \in \mathscr{D}(G)$. The action of Γ on $D_{\mathbf{Z}[\zeta]}(G)$ induces an action of Γ on $\mathrm{Spec}(D_{\mathbf{Z}[\zeta]}(G))$. It is easy to check that

$$\sigma(\mathscr{P}(H, hH', P)) = \mathscr{P}(H, \sigma(h)H', \sigma(P))$$

for $\sigma \in \Gamma$, $(H, hH') \in \mathscr{D}(G)$ and $P \in \mathrm{Spec}(\mathbf{Z}[\zeta])$. In the following, we will regard $D(G)$ as a subring of $D_{\mathbf{Z}[\zeta]}(G)$ via the monomorphism of rings

$$D(G) \longrightarrow D_{\mathbf{Z}[\zeta]}(G), \qquad x \longmapsto 1 \otimes x.$$

Let us consider the map

$$\Psi : \mathrm{Spec}(D_{\mathbf{Z}[\zeta]}(G)) \longrightarrow \mathrm{Spec}(D(G)), \qquad \mathscr{P} \longmapsto \mathscr{P} \cap D(G).$$

Since $D_{\mathbf{Z}[\zeta]}(G)$ is an integral extension of $D(G)$, Ψ is certainly surjective.

Proposition 8. *The fibres of the map Ψ above coincide with the Γ-orbits on* $\mathrm{Spec}(D_{\mathbf{Z}[\zeta]}(G))$.

Proof. It is clear that

$$\Psi(\sigma(\mathscr{P})) = \sigma(\mathscr{P}) \cap D(G) = \sigma(\mathscr{P} \cap D(G)) = \mathscr{P} \cap D(G) = \Psi(\mathscr{P})$$

for $\sigma \in \Gamma$ and $\mathscr{P} \in \mathrm{Spec}(D_{\mathbf{Z}[\zeta]}(G))$.

Conversely, let $\mathscr{P}, \mathscr{Q} \in \mathrm{Spec}(D_{\mathbf{Z}[\zeta]}(G))$ such that $\mathscr{P} \cap D(G) = \mathscr{Q} \cap D(G)$. We assume that $\mathscr{Q} \notin \{\sigma(\mathscr{P}) : \sigma \in \Gamma\}$. Then $\mathscr{Q}$ is not contained in $\bigcup_{\sigma\in\Gamma} \sigma(\mathscr{P})$, so we may choose an element $y \in \mathscr{Q} \setminus \bigcup_{\sigma\in\Gamma} \sigma(\mathscr{P})$. But now

$$\prod_{\sigma\in\Gamma} \sigma(y) \in \mathscr{Q} \cap D_{\mathbf{Z}[\zeta]}(G)^{\Gamma} = \mathscr{Q} \cap D(G) = \mathscr{P} \cap D(G) \subseteq \mathscr{P},$$

so $\tau(y) \in \mathscr{P}$ for some $\tau \in \Gamma$. Thus $y \in \tau^{-1}(\mathscr{P})$, a contradiction which proves the result.

We are now in a position to determine the prime spectrum of $D(G)$. In the following, we set

$$\wp(H, hH', P) := \{x \in D(G) : s_{(H,hH')} \in P\} = \mathscr{P}(H, hH', P) \cap D(G),$$

for $(H, hH') \in \mathscr{D}(G)$ and $P \in \mathrm{Spec}(\mathbf{Z}[\zeta])$. We summarize our results above in the following theorem.

Theorem 9. *For $(H, hH') \in \mathscr{D}(G)$ and $P \in \mathrm{Spec}(\mathbf{Z}[\zeta])$, $\wp(H, hH', P)$ is a prime ideal of $D(G)$. Moreover, every prime ideal of $D(G)$ arises in this way. More precisely, we have:*

(i) *Every prime ideal of $D(G)$ of residue characteristic zero has the form $\wp(H, hH', 0)$ for some $(H, hH') \in \mathscr{D}(G)$. If $\wp(H, hH', 0) = \wp(K, kK', 0)$ for some $(K, kK') \in \mathscr{D}(G)$ then there are $g \in G$ and $\sigma \in \Gamma$ such that $(K, kK') = {}^g(H, \sigma(h)H')$.*

(ii) *Every prime ideal of $D(G)$ of residue characteristic $p > 0$ has the form $\wp(H, hH', P)$ for some $(H, hH') \in \mathscr{D}_p(G)$ and some $P \in \mathrm{Spec}(\mathbf{Z}[\zeta])$ containing p. If $\wp(H, hH', P) = \wp(K, kK', Q)$ for some $(K, kK') \in \mathscr{D}_p(G)$ and some $Q \in \mathrm{Spec}(\mathbf{Z}[\zeta])$ containing p, then there are $g \in G$ and $\sigma \in \Gamma$ such that $(K, kK') = {}^g(H, \sigma(h)H')$ and $Q = \sigma(P)$.*

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