2-Blocks with minimal nonabelian defect groups

Benjamin Sambale
Mathematisches Institut
Friedrich-Schiller-Universität
07743 Jena
Germany
benjamin.sambale@uni-jena.de

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Abstract

We study numerical invariants of 2-blocks with minimal nonabelian defect groups. These groups were classified by Rédei (see [41]). If the defect group is also metacyclic, then the block invariants are known (see [43]). In the remaining cases there are only two (infinite) families of "interesting" defect groups. In all other cases the blocks are nilpotent. We prove Brauer's $k(B)$-conjecture and the Olsson-conjecture for all 2-blocks with minimal nonabelian defect groups. For one of the two families we also show that Alperin's weight conjecture and Dade's conjecture is satisfied. This paper is a part of the author's PhD thesis.

Keywords: blocks of finite groups, minimal nonabelian defect groups, Alperin's conjecture, Dade's conjecture.

Contents

1 Introduction

Let R be a discrete complete valuation ring with quotient field K of characteristic 0. Moreover, let (π) be the maximal ideal of R and $F := R/(\pi)$. We assume that F is algebraically closed of characteristic 2. We fix a finite group G, and assume that K contains all $|G|$-th roots of unity. Let B be a block of RG with defect group D. We denote the number of irreducible ordinary characters of B by $k(B)$. These characters split in $k_i(B)$ characters of height $i \in \mathbb{N}_0$. Similarly, let $k^i(B)$ be the number of characters of defect $i \in \mathbb{N}_0$. Finally, let $l(B)$ be the number of irreducible Brauer characters of B. The defect group D is called *minimal nonabelian* if every proper subgroup of D is abelian, but not D itself. Rédei has shown that D is isomorphic to one of the following groups (see [41]):

(i) $\langle x, y \mid x^{2^r} = y^{2^s} = 1,\ xyx^{-1} = y^{1+2^{s-1}}\rangle$, where $r \ge 1$ and $s \ge 2$,

(ii) $\langle x, y \mid x^{2^r} = y^{2^s} = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$, where $r \ge s \ge 1$, $[x, y] := xyx^{-1}y^{-1}$ and $[x, x, y] := [x, [x, y]]$,

(iii) Q_8.

In the first and last case D is also metacyclic. In this case B is well understood (see [43]). Thus, we may assume that D has the form (ii).

2 Fusion systems

To analyse the possible fusion systems on D we start with a group theoretical lemma.

Lemma 2.1. *Let $z := [x, y]$. Then the following hold:*

(i) $|D| = 2^{r+s+1}$.

(ii) $\Phi(D) = \mathrm{Z}(D) = \langle x^2, y^2, z\rangle \cong C_{2^{r-1}} \times C_{2^{s-1}} \times C_2$.

(iii) $D' = \langle z\rangle \cong C_2$.

(iv) $|\operatorname{Irr}(D)| = 5 \cdot 2^{r+s-2}$.

(v) If $r = s = 1$, then $D \cong D_8$. For $r \ge 2$ the maximal subgroups of D are given by

$$\langle x^2, y, z\rangle \cong C_{2^{r-1}} \times C_{2^s} \times C_2,$$
$$\langle x, y^2, z\rangle \cong C_{2^r} \times C_{2^{s-1}} \times C_2,$$
$$\langle xy, x^2, z\rangle \cong C_{2^r} \times C_{2^{s-1}} \times C_2.$$

We omit the (elementary) proof of this lemma. However, notice that $|P'| = 2$ and $|P : \Phi(P)| = |P : \mathrm{Z}(P)| = p^2$ hold for every minimal nonabelian p-group P. Rédei has also shown that for different pairs (r, s) one gets nonisomorphic groups. This gives precisely $\left[\frac{n-1}{2}\right]$ isomorphism classes of these groups of order 2^n. For $r \ne 1$ (that is $|D| \ge 16$) the structure of the maximal subgroups shows that all these groups are nonmetacyclic.

Now we investigate the automorphism groups.

Lemma 2.2. *The automorphism group $\operatorname{Aut}(D)$ is a 2-group, if and only if $r \ne s$ or $r = s = 1$.*

Proof. If $r \ne s$ or $r = s = 1$, then there exists a characteristic maximal subgroup of D by Lemma 2.1(v). In these cases $\operatorname{Aut}(D)$ must be a 2-group. Thus, we may assume $r = s \ge 2$. Then one can show that the map $x \mapsto y$, $y \mapsto x^{-1}y^{-1}$ is an automorphism of order 3. □

Lemma 2.3. *Let $P \cong C_{2^{n_1}} \times \ldots \times C_{2^{n_k}}$ with $n_1, \ldots, n_k, k \in \mathbb{N}$. Then $\operatorname{Aut}(P)$ is a 2-group, if and only if the n_i are pairwise distinct.*

Proof. See for example Lemma 2.7 in [34]. □

Now we are able to decide, when a fusion system on D is nilpotent.

Theorem 2.4. *Let $\mathcal{F}$ be a fusion system on D. Then $\mathcal{F}$ is nilpotent or $s = 1$ or $r = s$. If $r = s \ge 2$, then $\mathcal{F}$ is controlled by D.*

Proof. We assume $s \neq 1$. Let $Q < D$ be an $\mathcal{F}$-essential subgroup. Since Q is also $\mathcal{F}$-centric, we get $\mathrm{C}_P(Q) = Q$. This shows that Q is a maximal subgroup of D. By Lemma 2.1(v) and Lemma 2.3, one of the following holds:

(i) $r = 2$ $(= s)$ and $Q \in \{\langle x^2, y, z\rangle, \langle x, y^2, z\rangle, \langle xy, x^2, z\rangle\}$,

(ii) $r > s = 2$ and $Q \in \{\langle x, y^2, z\rangle, \langle xy, x^2, z\rangle\}$,

(iii) $r = s + 1$ and $Q = \langle x^2, y, z\rangle$.

In all cases $\Omega(Q) \subseteq \mathrm{Z}(P)$. Let us consider the action of $\mathrm{Aut}_{\mathcal{F}}(Q)$ on $\Omega(Q)$. The subgroup $1 \neq P/Q = \mathrm{N}_P(Q)/\mathrm{C}_P(Q) \cong \mathrm{Aut}_P(Q) \leq \mathrm{Aut}_{\mathcal{F}}(Q)$ acts trivially on $\Omega(Q)$. On the other hand every nontrivial automorphism of odd order acts nontrivially on $\Omega(Q)$ (see for example 8.4.3 in [19]). Hence, the kernel of this action is a nontrivial normal 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. In particular $\mathrm{O}_2(\mathrm{Aut}_{\mathcal{F}}(Q)) \neq 1$. But then $\mathrm{Aut}_{\mathcal{F}}(Q)$ cannot contain a strongly 2-embedded subgroup.

This shows that there are no $\mathcal{F}$-essential subgroups. Now the claim follows from Lemma 2.2 and Alperin's fusion theorem. □

Now we consider a kind of converse. If $r = s = 1$, then there are nonnilpotent fusion systems on D. In the case $r = s \geq 2$ one can construct a nonnilpotent fusion system with a suitable semidirect product (see Lemma 2.2). We show that there is also a nonnilpotent fusion system in the case $r > s = 1$.

Proposition 2.5. *If $s = 1$, then there exists a nonnilpotent fusion system on D.*

Proof. We may assume $r \geq 2$. Let A_4 be the alternating group of degree 4, and let $H := \langle \widetilde{x}\rangle \cong C_{2^r}$. Moreover, let $\varphi : H \to \mathrm{Aut}(A_4) \cong S_4$ such that $\varphi_{\widetilde{x}} \in \mathrm{Aut}(A_4)$ has order 4. Write $\widetilde{y} := (12)(34) \in A_4$ and choose φ such that $\varphi_{\widetilde{x}}(\widetilde{y}) := (13)(24)$. Finally, let $G := A_4 \rtimes_\varphi H$. Since all 4-cycles in S_4 are conjugate, G is uniquely determined up to isomorphism. Because $[\widetilde{x}, \widetilde{y}] = (13)(24)(12)(34) = (14)(23)$, we get $\langle \widetilde{x}, \widetilde{y}\rangle \cong D$. The fusion system $\mathcal{F}_G(D)$ is nonnilpotent, since A_4 (and therefore G) is not 2-nilpotent. □

3 The case $r > s = 1$

Now we concentrate on the case $r > s = 1$, i.e.

$$D := \langle x, y \mid x^{2^r} = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$$

with $r \geq 2$. As before $z := [x, y]$. We also assume that B is a nonnilpotent block. By Lemma 2.2, $\mathrm{Aut}(D)$ is a 2-group, and the inertial index $t(B)$ of B equals 1.

3.1 The B-subsections

Olsson has already obtained the conjugacy classes of so called B-subsections (see [34]). However, his results contain errors. For example he missed the necessary relations $[x, x, y]$ and $[y, x, y]$ in the definition of D.

In the next lemma we denote by $\mathrm{Bl}(RH)$ the set of blocks of a finite group H. If $H \leq G$ and $b \in \mathrm{Bl}(RH)$, then b^G is the Brauer correspondent of b (if exists). Moreover, we use the notion of subpairs and subsections (see [36]).

Lemma 3.1. *Let $b \in \mathrm{Bl}(RD\,\mathrm{C}_G(D))$ be a Brauer correspondent of B. For $Q \leq D$ let $b_Q \in \mathrm{Bl}(RQ\,\mathrm{C}_G(Q))$ such that $(Q, b_Q) \leq (D, b)$. Set $\mathcal{T} := \mathrm{Z}(D) \cup \{x^i y^j : i, j \in \mathbb{Z},\ i \text{ odd}\}$. Then*

$$\bigcup_{a \in \mathcal{T}} \left\{\left(a, b_{\mathrm{C}_D(a)}^{\mathrm{C}_G(a)}\right)\right\}$$

is a system of representatives for the conjugacy classes of B-subsections. Moreover, $|\mathcal{T}| = 2^{r+1}$.

Proof. If $r = 2$, then the claim follows from Proposition 2.14 in [34]. For $r \geq 3$ the same argument works. However, Olsson refers wrongly to Proposition 2.11 (the origin of this mistake already lies in Lemma 2.8). □

From now on we write $b_a := b_{C_D(a)}^{C_G(a)}$ for $a \in \mathcal{T}$.

Lemma 3.2. *Let $P \cong C_{2^s} \times C_2^2$ with $s \in \mathbb{N}$, and let α be an automorphism of P of order 3. Then $\mathrm{C}_P(\alpha) := \{b \in P : \alpha(b) = b\} \cong C_{2^s}$.*

Proof. We write $P = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ with $|\langle a \rangle| = 2^s$. It is well known that the kernel of the restriction map $\mathrm{Aut}(P) \to \mathrm{Aut}(P/\Phi(P))$ is a 2-group. Since $|\mathrm{Aut}(P/\Phi(P))| = |\mathrm{GL}(3,2)| = 168 = 2^3 \cdot 3 \cdot 7$, it follows that $|\mathrm{Aut}(P)|$ is divisible by 3 only once. In particular every automorphism of P of order 3 is conjugate to α or α^{-1}. Thus, we may assume $\alpha(a) = a$, $\alpha(b) = c$ and $\alpha(c) = bc$. Then $\mathrm{C}_P(\alpha) = \langle a \rangle \cong C_{2^s}$. $\square$

3.2 The numbers $k(B)$, $k_i(B)$ and $l(B)$

The next step is to determine the numbers $l(b_a)$. The case $r = 2$ needs special attention, because in this case D contains an elementary abelian maximal subgroup of order 8. We denote the inertial group of a block $b \in \mathrm{Bl}(RH)$ with $H \trianglelefteq G$ by $\mathrm{T}_G(b)$.

Lemma 3.3. *There is an element $c \in \mathrm{Z}(D)$ of order 2^{r-1} such that $l(b_a) = 1$ for all $a \in \mathcal{T} \setminus \langle c \rangle$.*

Proof.
Case 1: $a \in \mathrm{Z}(D)$.
Then $b_a = b_D^{\mathrm{C}_G(a)}$ is a block with defect group D and Brauer correspondent $b_D \in \mathrm{Bl}(RD\,\mathrm{C}_{\mathrm{C}_G(a)}(D))$. Let $M := \langle x^2, y, z \rangle \cong C_{2^{r-1}} \times C_2^2$. Since B is nonnilpotent, there exists an element $\alpha \in \mathrm{T}_{\mathrm{N}_G(M)}(b_M)$ such that $\alpha\,\mathrm{C}_G(M) \in \mathrm{T}_{\mathrm{N}_G(M)}(b_M)/\,\mathrm{C}_G(M)$ has order $q \in \{3, 7\}$. We will exclude the case $q = 7$. In this case $r = 2$ and $\mathrm{T}_{\mathrm{N}_G(M)}(b_M)/\,\mathrm{C}_G(M)$ is isomorphic to a subgroup of $\mathrm{Aut}(M) \cong \mathrm{GL}(3,2)$. Since

$$(M, {}^d b_M) = {}^d(M, b_M) \leq {}^d(D, b_D) = (D, b_D)$$

for all $d \in D$, we have $D \subseteq \mathrm{T}_{\mathrm{N}_G(M)}(b_M)$. This implies $\mathrm{T}_{\mathrm{N}_G(M)}(b_M)/\,\mathrm{C}_G(M) \cong \mathrm{GL}(3,2)$, because $\mathrm{GL}(3,2)$ is simple. By Satz 1 in [2], this contradicts the fact that $\mathrm{T}_{\mathrm{N}_G(M)}(b_M)/\,\mathrm{C}_G(M)$ contains a strongly 2-embedded subgroup (of course this can be shown "by hand" without invoking [2]). Thus, we have shown $q = 3$. Now

$$\mathrm{T}_{\mathrm{N}_G(M)}(b_M)/\,\mathrm{C}_G(M) \cong S_3$$

follows easily. By Lemma 3.2 there is an element $c := x^{2i}y^jz^k \in \mathrm{C}_M(\alpha)$ $(i, j, k \in \mathbb{Z})$ of order 2^{r-1}. Let us assume that j is odd. Since $x\alpha x \equiv x\alpha x^{-1} \equiv \alpha^{-1} \pmod{C_G(M)}$ we get

$$\begin{aligned}\alpha(x^{2i}y^jz^{k+1})\alpha^{-1} &= \alpha x(x^{2i}y^jz^k)x^{-1}\alpha^{-1} = x\alpha^{-1}(x^{2i}y^jz^k)\alpha x^{-1} \\ &= x(x^{2i}y^jz^k)x^{-1} = x^{2i}y^jz^{k+1}.\end{aligned}$$

But this contradicts Lemma 3.2. Hence, we have proved that j is even. In particular $c \in \mathrm{Z}(D)$. For $a \notin \langle c \rangle$ we have $\alpha \notin \mathrm{C}_G(a)$ and $l(b_a) = 1$. While in the case $a \in \langle c \rangle$ we get $\alpha \in \mathrm{C}_G(a)$, and b_a is nonnilpotent. Thus, in this case $l(b_a)$ remains unknown.

Case 2: $a \notin \mathrm{Z}(D)$.
Let $\mathrm{C}_D(a) = \langle \mathrm{Z}(D), a \rangle =: M$. Since (M, b_M) is a Brauer subpair, b_M has defect group M. It follows from $(M, b_M) \trianglelefteq (D, b_D)$ that also b_a has defect group M and Brauer correspondent b_M. In case $M \cong C_{2^r} \times C_2$ we get $l(b_a) = 1$. Now let us assume $M \cong C_{2^{r-1}} \times C_2^2$. As in the first case, we choose $\alpha \in \mathrm{T}_{\mathrm{N}_G(M)}(b_M)$ such that $\alpha\,\mathrm{C}_G(M) \in \mathrm{T}_{\mathrm{N}_G(M)}(b_M)/\,\mathrm{C}_G(M)$ has order 3. Since $a \notin \mathrm{Z}(D)$, we derive $\alpha \notin \mathrm{C}_G(a)$ and $t(b_a) = l(b_a) = 1$. $\square$

We denote by $\mathrm{IBr}(b_u) := \{\varphi_u\}$ for $u \in \mathcal{T} \setminus \langle c \rangle$ the irreducible Brauer character of b_u. Then the generalized decomposition numbers $d^u_{\chi\varphi_u}$ for $\chi \in \mathrm{Irr}(B)$ form a column $d(u)$. Let 2^k be the order of u, and let $\zeta := \zeta_{2^k}$ be a primitive 2^k-th root of unity. Then the entries of $d(u)$ lie in the ring of integers $\mathbb{Z}[\zeta]$. Hence, there exist integers $a^u_i(\chi) \in \mathbb{Z}$ such that

$$d^u_{\chi\varphi_u} = \sum_{i=0}^{2^{k-1}-1} a^u_i(\chi)\zeta^i.$$

We expand this by
$$a^u_{i+2^{k-1}} := -a^u_i$$
for all $i \in \mathbb{Z}$.

Let $|G| = 2^a m$ where $2 \nmid m$. We may assume $\mathbb{Q}(\zeta_{|G|}) \subseteq K$. Then $\mathbb{Q}(\zeta_{|G|})|\mathbb{Q}(\zeta_m)$ is a Galois extension, and we denote the corresponding Galois group by
$$\mathcal{G} := \mathrm{Gal}(\mathbb{Q}(\zeta_{|G|})|\mathbb{Q}(\zeta_m)).$$
Restriction gives an isomorphism
$$\mathcal{G} \cong \mathrm{Gal}(\mathbb{Q}(\zeta_{2^a})|\mathbb{Q}).$$
In particular $|\mathcal{G}| = 2^{a-1}$. For every $\gamma \in \mathcal{G}$ there is a number $\widetilde{\gamma} \in \mathbb{N}$ such that $\gcd(\widetilde{\gamma}, |G|) = 1$, $\widetilde{\gamma} \equiv 1 \pmod{m}$, and $\gamma(\zeta_{|G|}) = \zeta_{|G|}^{\widetilde{\gamma}}$ hold. Then $\mathcal{G}$ acts on the set of subsections by
$${}^{\gamma}(u, b) := (u^{\widetilde{\gamma}}, b).$$
For every $\gamma \in \mathcal{G}$ we get
$$d(u^{\widetilde{\gamma}}) = \sum_{s \in \mathcal{S}} a^u_s \zeta_{2^k}^{s\widetilde{\gamma}}$$
for every system $\mathcal{S}$ of representatives of the cosets of $2^{k-1}\mathbb{Z}$ in $\mathbb{Z}$. It follows that
$$a^u_s = 2^{1-a} \sum_{\gamma \in \mathcal{G}} d\big(u^{\widetilde{\gamma}}\big) \zeta_{2^k}^{-\widetilde{\gamma}s} \tag{1}$$
for $s \in \mathcal{S}$.

Now let $u \in \mathcal{T} \setminus \mathrm{Z}(D)$ and $M := \mathrm{C}_D(u)$. Then b_u and $b_M^{\mathrm{T}_{\mathrm{N}_G(M)}(b_M) \cap \mathrm{N}_G(\langle u \rangle)}$ have M as defect group, because $D \nsubseteq \mathrm{N}_G(\langle u \rangle)$. By (6B) in [6] it follows that the 2^{r-1} distinct B-subsections of the form ${}^{\gamma}(u, b_u)$ with $\gamma \in \mathcal{G}$ are pairwise nonconjugate. The same holds for $u \in \mathrm{Z}(D) \setminus \{1\}$. Using this and equation (1) we can adapt Lemma 3.9 in [33]:

Lemma 3.4. *Let $c \in \mathrm{Z}(D)$ as in Lemma 3.3, and let $u, v \in \mathcal{T} \setminus \langle c \rangle$ with $|\langle u \rangle| = 2^k$ and $|\langle v \rangle| = 2^l$. Moreover, let $i \in \{0, 1, \ldots, 2^{k-1} - 1\}$ and $j \in \{0, 1, \ldots, 2^{l-1} - 1\}$. If there exist $\gamma \in \mathcal{G}$ and $g \in G$ such that ${}^g(u, b_u) = {}^{\gamma}(v, b_v)$, then*
$$(a^u_i, a^v_j) = \begin{cases} 2^{d(B)-k+1} & \text{if } u \in \mathrm{Z}(D) \text{ and } j\widetilde{\gamma} - i \equiv 0 \pmod{2^k} \\ -2^{d(B)-k+1} & \text{if } u \in \mathrm{Z}(D) \text{ and } j\widetilde{\gamma} - i \equiv 2^{k-1} \pmod{2^k} \\ 2^{d(B)-k} & \text{if } u \notin \mathrm{Z}(D) \text{ and } j\widetilde{\gamma} - i \equiv 0 \pmod{2^k} \\ -2^{d(B)-k} & \text{if } u \notin \mathrm{Z}(D) \text{ and } j\widetilde{\gamma} - i \equiv 2^{k-1} \pmod{2^k} \\ 0 & \text{otherwise} \end{cases}.$$
Otherwise $(a^u_i, a^v_j) = 0$. In particular $(a^u_i, a^v_j) = 0$ if $k \neq l$.

Using the theory of contributions we can also carry over Lemma (6.E) in [20]:

Lemma 3.5. *Let $u \in \mathrm{Z}(D)$ with $l(b_u) = 1$. If u has order 2^k, then for every $\chi \in \mathrm{Irr}(B)$ holds:*

(i) $2^{h(\chi)} \mid a^u_i(\chi)$ *for* $i = 0, \ldots, 2^{k-1} - 1$,

(ii) $\sum_{i=0}^{2^{k-1}-1} a^u_i(\chi) \equiv 2^{h(\chi)} \pmod{2^{h(\chi)+1}}$.

By Lemma 1.1 in [39] we have
$$k(B) \le \sum_{i=0}^{\infty} 2^{2i} k_i(B) \le |D|. \tag{2}$$
In particular Brauer's $k(B)$-conjecture holds. Olsson's conjecture
$$k_0(B) \le |D : D'| = 2^{r+1} \tag{3}$$
follows by Theorem 3.1 in [39]. Now we are able to calculate the numbers $k(B)$, $k_i(B)$ and $l(B)$.

Theorem 3.6. *We have*

$$k(B) = 5 \cdot 2^{r-1} = |\operatorname{Irr}(D)|, \qquad k_0(B) = 2^{r+1} = |D : D'|, \qquad k_1(B) = 2^{r-1}, \qquad l(B) = 2.$$

Proof. We argue by induction on r. Let $r = 2$, and let $c \in \mathrm{Z}(D)$ as in Lemma 3.3. By way of contradiction we assume $c = z$. If α and M are defined as in the proof of Lemma 3.3, then α acts nontrivially on $M/\langle z\rangle \cong C_2^2$. On the other hand x acts trivially on $M/\langle z\rangle$. This contradicts $x\alpha x^{-1}\alpha \in \mathrm{C}_G(M)$.

This shows $c \in \{x^2, x^2 z\}$ and $D/\langle c\rangle \cong D_8$. Thus, we can apply Theorem 2 in [8]. For this let

$$M_1 := \begin{cases} \langle x, z\rangle & \text{if } c = x^2 \\ \langle xy, z\rangle & \text{if } c = x^2 z \end{cases}.$$

Then $M \neq M_1 \cong C_4 \times C_2$ and $\overline{M} := M/\langle c\rangle \cong C_2^2 \cong M_1/\langle c\rangle =: \overline{M_1}$. Let β be the block of $R\overline{\mathrm{C}_G(c)} := R[\mathrm{C}_G(c)/\langle c\rangle]$ which is dominated by b_c. By Theorem 1.5 in [33] we have

$$3 \bigm| |\operatorname{T}_{\mathrm{N}_{\overline{\mathrm{C}_G(c)}}(\overline{M})}(\beta_{\overline{M}}) / \mathrm{C}_{\overline{\mathrm{C}_G(c)}}(\overline{M})|$$

and

$$3 \nmid |\operatorname{T}_{\mathrm{N}_{\overline{\mathrm{C}_G(c)}}(\overline{M_1})}(\beta_{\overline{M_1}}) / \mathrm{C}_{\overline{\mathrm{C}_G(c)}}(\overline{M_1})|,$$

where $(\overline{M}, \beta_{\overline{M}})$ and $(\overline{M_1}, \beta_{\overline{M_1}})$ are β-subpairs. This shows that case (ab) in Theorem 2 in [8] occurs. Hence, $l(b_c) = l(\beta) = 2$. Now Lemma 3.3 yields

$$k(B) \geq 1 + k(B) - l(B) = 9.$$

It is well known that $k_0(B)$ is divisible by 4. Thus, the equations (2) and (3) imply $k_0(B) = 8$. Moreover,

$$d^z_{\chi\varphi_z} = a^z_0(\chi) = \pm 1$$

holds for every $\chi \in \operatorname{Irr}(B)$ with $h(\chi) = 0$. This shows $4k_1(B) \leq |D| - k_0(B) = 8$. It follows that $k_1(B) = l(B) = 2$.

Now we consider the case $r \geq 3$. Since z is not a square in D, we have $z \notin \langle c\rangle$. Let $a \in \langle c\rangle$ such that $|\langle a\rangle| = 2^k$. If $k = r - 1$, then $l(b_a) = 2$ as before. Now let $k < r - 1$. Then $D/\langle a\rangle$ has the same isomorphism type as D, but one has to replace r by $r - k$. By induction we get $l(b_a) = 2$ for $k \geq 1$. This shows

$$k(B) \geq 1 + k(B) - l(B) = 2^{r+1} + 2^{r-1} - 1.$$

Equation (2) yields

$$\begin{aligned} 2^{r+2} - 4 = 2^{r+1} + 4(2^{r-1} - 1) &\leq k_0(B) + 4(k(B) - k_0(B)) \\ &\leq \sum_{i=0}^{\infty} 2^{2i} k_i(B) \leq |D| = 2^{r+2}. \end{aligned}$$

Now the conclusion follows easily. $\square$

As a consequence, Brauer's height zero conjecture and the Alperin-McKay-conjecture hold for B.

3.3 Generalized decomposition numbers

Now we will determine some of the generalized decomposition numbers. Again let $c \in \mathrm{Z}(D)$ as in Lemma 3.3, and let $u \in \mathrm{Z}(D) \setminus \langle c\rangle$ with $|\langle u\rangle| = 2^k$. Then $(a^u_i, a^u_i) = 2^{r+3-k}$ and $2 \mid a^u_i(\chi)$ for $h(\chi) = 1$ and $i = 0, \ldots, 2^{k-1} - 1$. This gives

$$|\{\chi \in \operatorname{Irr}(B) : a^u_i(\chi) \neq 0\}| \leq 2^{r+3-k} - 3|\{\chi \in \operatorname{Irr}(B) : h(\chi) = 1,\ a^u_i(\chi) \neq 0\}|.$$

Moreover, for every character $\chi \in \operatorname{Irr}(B)$ there exists $i \in \{0, \ldots, 2^{k-1}-1\}$ such that $a_i^u(\chi) \neq 0$. Hence,

$$\begin{aligned}
k(B) &\le \sum_{i=0}^{2^{k-1}-1} \sum_{\substack{\chi\in\operatorname{Irr}(B),\\ a_i^u(\chi)\neq 0}} 1 \le \sum_{i=0}^{2^{k-1}-1} \Bigg(2^{r+3-k} - 3 \sum_{\substack{\chi\in\operatorname{Irr}(B),\\ h(\chi)=1,\\ a_i^u(\chi)\neq 0}} 1\Bigg) = |D| - 3\sum_{i=0}^{2^{k-1}-1} \sum_{\substack{\chi\in\operatorname{Irr}(B),\\ h(\chi)=1,\\ a_i^u(\chi)\neq 0}} 1 \\
&\le |D| - 3k_1(B) = k(B).
\end{aligned}$$

This shows that for every $\chi \in \operatorname{Irr}(B)$ there exists $i(\chi) \in \{0, \ldots, 2^{k-1}-1\}$ such that

$$d^u_{\chi\varphi_u} = \begin{cases} \pm\zeta_{2^k}^{i(\chi)} & \text{if } h(\chi) = 0 \\ \pm 2\zeta_{2^k}^{i(\chi)} & \text{if } h(\chi) = 1 \end{cases}.$$

In particular

$$d^u_{\chi\varphi_u} = a_0^u(\chi) = \begin{cases} \pm 1 & \text{if } h(\chi) = 0 \\ \pm 2 & \text{if } h(\chi) = 1 \end{cases}$$

for $k = 1$.

By Lemma 3.4 we have $(a_i^u, a_i^u) = 4$ for $u \in \mathcal{T} \setminus \mathrm{Z}(D)$ and $i = 0, \ldots, 2^{r-1}-1$. If a_i^u has only one nonvanishing entry, then a_i^u would not be orthogonal to a_0^z. Hence, a_i^u has up to ordering the form

$$(\pm 1, \pm 1, \pm 1, \pm 1, 0, \ldots, 0)^{\mathrm{T}},$$

where the signs are independent of each other. The proof of Theorem 3.1 in [39] gives

$$|d^u_{\chi\varphi_u}| = 1$$

for $u \in \mathcal{T} \setminus \mathrm{Z}(D)$ and $\chi \in \operatorname{Irr}(B)$ with $h(\chi) = 0$. In particular $d^u_{\chi\varphi_u} = 0$ for characters $\chi \in \operatorname{Irr}(B)$ of height 1. By suitable ordering we get

$$a_i^u(\chi_j) = \begin{cases} \pm 1 & \text{if } j - 4i \in \{1, \ldots, 4\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } d^u_{\chi_j\varphi_u} = \begin{cases} \pm\zeta_{2^r}^{[\frac{j-1}{4}]} & \text{if } 1 \le j \le k_0(B) \\ 0 & \text{if } k_0(B) < j \le k(B) \end{cases}$$

for $i = 0, \ldots, 2^{r-1}-1$, where $\chi_1, \ldots, \chi_{k_0(B)}$ are the characters of height 0.

Now let $\operatorname{IBr}(b_c) := \{\varphi_1, \varphi_2\}$. We determine the numbers $d^c_{\chi\varphi_1}, d^c_{\chi\varphi_2} \in \mathbb{Z}[\zeta_{2^{r-1}}]$. By (4C) in [6] we have $d^c_{\chi\varphi_1} \neq 0$ or $d^c_{\chi\varphi_2} \neq 0$ for all $\chi \in \operatorname{Irr}(B)$. As in the proof of Theorem 3.6, b_c dominates a block $\overline{b_c} \in \operatorname{Bl}(R[\mathrm{C}_G(c)/\langle c\rangle])$ with defect group D_8. The table at the end of [14] shows that the Cartan matrix of $\overline{b_c}$ has the form

$$\begin{pmatrix} 8 & 4 \\ 4 & 3 \end{pmatrix} \text{ or } \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}.$$

We label these possibilities as the "first" and the "second" case. The Cartan matrix of b_c is

$$2^{r-1}\begin{pmatrix} 8 & 4 \\ 4 & 3 \end{pmatrix} \text{ or } 2^{r-1}\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$$

respectively. The inverses of these matrices are

$$2^{-r-2}\begin{pmatrix} 3 & -4 \\ -4 & 8 \end{pmatrix} \text{ and } 2^{-r-2}\begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}.$$

Let $m^{(c,b_c)}_{\chi\psi}$ be the contribution of $\chi, \psi \in \operatorname{Irr}(B)$ with respect to the subsection (c, b_c) (see [6]). Then we have

$$\begin{aligned}
|D| m^{(c,b_c)}_{\chi\psi} &= 3d^c_{\chi\varphi_1}\overline{d^c_{\psi\varphi_1}} - 4(d^c_{\chi\varphi_1}\overline{d^c_{\psi\varphi_2}} + d^c_{\chi\varphi_2}\overline{d^c_{\psi\varphi_1}}) + 8d^c_{\chi\varphi_2}\overline{d^c_{\psi\varphi_2}} \\
&\text{or} \\
|D| m^{(c,b_c)}_{\chi\psi} &= 3d^c_{\chi\varphi_1}\overline{d^c_{\psi\varphi_1}} - 2(d^c_{\chi\varphi_1}\overline{d^c_{\psi\varphi_2}} + d^c_{\chi\varphi_2}\overline{d^c_{\psi\varphi_1}}) + 4d^c_{\chi\varphi_2}\overline{d^c_{\psi\varphi_2}} \qquad (4)
\end{aligned}$$

respectively. For a character $\chi \in \mathrm{Irr}(B)$ with height 0 we get

$$0 = h(\chi) = \nu\big(|D| m_{\chi\chi}^{(c,b_c)}\big) = \nu(3 d_{\chi\varphi_1}^c \overline{d_{\chi\varphi_1}^c}) = \nu(d_{\chi\varphi_1}^c)$$

by (5H) in [6]. In particular $d_{\chi\varphi_1}^c \neq 0$. We define $c_i^j \in \mathbb{Z}^{k(B)}$ by

$$d_{\chi\varphi_j}^c = \sum_{i=0}^{2^{r-2}-1} c_i^j(\chi)\zeta_{2^{r-1}}^i$$

for $j = 1, 2$. Then

$$(c_i^1, c_j^1) = \begin{cases} \delta_{ij}16 & \text{first case} \\ \delta_{ij}8 & \text{second case} \end{cases}, \ (c_i^1, c_j^2) = \begin{cases} \delta_{ij}8 & \text{first case} \\ \delta_{ij}4 & \text{second case} \end{cases}, \ (c_i^2, c_j^2) = \delta_{ij}6$$

as in Lemma 3.4. (Since the 2^{r-2} B-subsections of the form ${}^{\gamma}(c, b_c)$ for $\gamma \in \mathcal{G}$ are pairwise nonconjugate, one can argue like in Lemma 3.4.) Hence, in the second case

$$d_{\chi_i\varphi_1}^c = \begin{cases} \pm\zeta_{2^{r-1}}^{[\frac{i-1}{8}]} & \text{if } 1 \le i \le k_0(B) \\ 0 & \text{if } k_0(B) < i \le k(B) \end{cases} \quad \text{(second case)}$$

holds for a suitable arrangement. Again $\chi_1, \ldots, \chi_{k_0(B)}$ are the characters of height 0. In the first case

$$1 = h(\psi) = \nu\big(|D| m_{\chi\psi}^{(c,b_c)}\big) = \nu(3 d_{\chi\varphi_1}^c \overline{d_{\psi\varphi_1}^c}) = \nu(d_{\psi\varphi_1}^c)$$

by (5G) in [6] for $h(\psi) = 1$ and $h(\chi) = 0$. As in Lemma 3.5 we also have $2 \mid c_i^1(\psi)$ for $h(\psi) = 1$ and $i = 0, \ldots, 2^{r-2} - 1$. Analogously as in the case $u \in \mathrm{Z}(D) \setminus \langle c \rangle$ we conclude

$$d_{\chi\varphi_1}^c = \begin{cases} \pm\zeta_{2^{r-1}}^{i(\chi)} & \text{if } h(\chi) = 0 \\ \pm 2\zeta_{2^{r-1}}^{i(\chi)} & \text{if } h(\chi) = 1 \end{cases} \quad \text{(first case)} \tag{5}$$

for suitable indices $i(\chi) \in \{0, \ldots, 2^{r-2} - 1\}$. Since $(c_i^2, c_j^2) = \delta_{ij}6$, in both cases c_i^2 has the form

$$(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, 0, \ldots, 0)^{\mathrm{T}} \text{ or } (\pm 2, \pm 1, \pm 1, 0, \ldots, 0)^{\mathrm{T}}.$$

We show that the latter possibility does not occur. In the second case for every character $\chi \in \mathrm{Irr}(B)$ with height 1 there exists $i \in \{0, \ldots, 2^{r-2} - 1\}$ such that $c_i^2(\chi) \neq 0$. In this case we get

$$d_{\chi_i\varphi_2}^c = \begin{cases} \pm\zeta_{2^{r-1}}^{[\frac{i-1}{4}]} & \text{if } 1 \le i \le 2^r \\ 0 & \text{if } 2^r < i \le k_0(B) \\ \pm\zeta_{2^{r-1}}^{[\frac{i-k_0(B)-1}{2}]} & \text{if } k_0(B) < i \le k(B) \end{cases} \quad \text{(second case)},$$

where $\chi_1, \ldots, \chi_{k_0(B)}$ are again the characters of height 0. Now let us consider the first case. Since $(c_i^1, c_j^2) = \delta_{ij}8$, the value ± 2 must occur in every column c_i^1 for $i = 0, \ldots, 2^{r-2} - 1$ at least twice. Obviously exactly two entries have to be ± 2. Thus, one can improve equation (5) to

$$d_{\chi_i\varphi_1}^c = \begin{cases} \pm\zeta_{2^{r-1}}^{[\frac{i-1}{8}]} & \text{if } 1 \le i \le k_0(B) \\ \pm 2\zeta_{2^{r-1}}^{[\frac{i-k_0(B)-1}{2}]} & \text{if } k_0(B) < i \le k(B) \end{cases} \quad \text{(first case)}.$$

It follows

$$d_{\chi_i\varphi_2}^c = \begin{cases} \pm\zeta_{2^{r-1}}^{[\frac{i-1}{4}]} & \text{if } 1 \le i \le 2^r \\ 0 & \text{if } 2^r < i \le k_0(B) \\ \pm\zeta_{2^{r-1}}^{[\frac{i-k_0(B)-1}{2}]} & \text{if } k_0(B) < i \le k(B) \end{cases} \quad \text{(first case)}.$$

Hence, the numbers $d_{\chi\varphi_2}^c$ are independent of the case. Of course, one gets similar results for $d_{\chi\varphi_i}^u$ with $\langle u \rangle = \langle c \rangle$.

3.4 The Cartan matrix

Now we investigate the Cartan matrix of B.

Lemma 3.7. *The elementary divisors of the Cartan matrix of B are 2^{r-1} and $|D|$.*

Proof. Let C be the Cartan matrix of B. Since $l(B) = 2$, it suffices to show that 2^{r-1} occurs as elementary divisor of C at least once. In order to proof this, we use the notion of lower defect groups (see [35]). Let (u, b) be a B-subsection with $|\langle u\rangle| = 2^{r-1}$ and $l(b) = 2$. Let $b_1 := b^{\mathrm{N}_G(\langle u\rangle)}$. Then b_1 has also defect group D, and $l(b_1) = 2$ holds. Moreover, $u^{2^{r-2}} \in \mathrm{Z}(\mathrm{N}_G(\langle u\rangle))$. Let $\overline{b_1} \in \mathrm{Bl}(R[\mathrm{N}_G(u)/\langle u^{2^{r-2}}\rangle])$ be the block which is covered by b_1. Then $\overline{b_1}$ has defect group $D/\langle u^{2^{r-2}}\rangle$. We argue by induction on r. Thus, let $r = 2$. Then $b = b_1$ and $D/\langle u^{2^{r-2}}\rangle = D/\langle u\rangle \cong D_8$. By Proposition (5G) in [8] the Cartan matrix of $\overline{b}$ has the elementary divisors 1 and 8. Hence, $2 = 2^{r-1}$ and $16 = |D|$ are the elementary divisors of the Cartan matrix of b. Hence, the claim follows from Theorem 7.2 in [35].

Now assume that the claim already holds for $r - 1 \geq 2$. By induction the elementary divisors of the Cartan matrix of $\overline{b_1}$ are 2^{r-2} and $|D|/2$. The claim follows easily as before. □

Now we are in a position to calculate the Cartan matrix C up to equivalence of quadratic forms. Here we call two matrices $M_1, M_2 \in \mathbb{Z}^{l\times l}$ equivalent if there exists a matrix $S \in \mathrm{GL}(l, \mathbb{Z})$ such that $A = SBS^{\mathrm{T}}$, where S^{T} denotes the transpose of S.

By Lemma 3.7 all entries of C are divisible by 2^{r-1}. Thus, we can consider $\widetilde{C} := 2^{1-r}C \in \mathbb{Z}^{2\times 2}$. Then $\det \widetilde{C} = 8$ and the elementary divisors of $\widetilde{C}$ are 1 and 8. If we write

$$\widetilde{C} = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix},$$

then $\widetilde{C}$ corresponds to the positive definite binary quadratic form $q(x_1, x_2) := c_1x_1^2 + 2c_2x_1x_2 + c_3x_2^2$. Obviously $\gcd(c_1, c_2, c_3) = 1$. If one reduces the entries of $\widetilde{C}$ modulo 2, then one gets a matrix of rank 1 (this is just the multiplicity of the elementary divisor 1). This shows that c_1 or c_3 must be odd. Hence, $\gcd(c_1, 2c_2, c_3) = 1$, i. e. q is primitive (see [10] for example). Moreover, $\Delta := -4\det \widetilde{C} = -32$ is the discriminant of q. Now it is easy to see that q (and $\widetilde{C}$) is equivalent to exactly one of the following matrices (see page 20 in [10]):

$$\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

The Cartan matrices for the block $\overline{b_c}$ with defect group D_8 (used before) satisfy

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Hence, only the second matrix occurs up to equivalence. We show that this holds also for the block B.

Theorem 3.8. *The Cartan matrix of B is equivalent to*

$$2^{r-1}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Proof. We argue by induction on r. The smallest case was already considered by b_c (this would correspond to $r = 1$). Thus, we may assume $r \geq 2$ (as usual). First, we determine the generalized decomposition numbers $d^u_{\chi\varphi}$ for $u \in \langle c\rangle \setminus \{1\}$ with $|\langle u\rangle| = 2^k < 2^{r-1}$. As in the proof of Theorem 3.6, the group $D/\langle u\rangle$ has the same isomorphism type as D, but one has to replace r by $r - k$. Hence, by induction we may assume that b_u has a Cartan matrix which is equivalent to the matrix given in the statement of the theorem. Let C_u be the Cartan matrix of b_u, and let $S_u \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$C_u = 2^{r-1}S_u^{\mathrm{T}}\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}S_u,$$

i. e. with the notations of the previous section, we assume that the "second case" occurs. (This is allowed, since we can only compute the generalized decomposition numbers up to multiplication with S_u anyway.) As before we write $\mathrm{IBr}(b_u) = \{\varphi_1, \varphi_2\}$, $D_u := (d^u_{\chi\varphi_i})$ and $(\widetilde{d}^u_{\chi\varphi_i}) := D_u S_u^{-1}$. The consideration in the previous section carries over, and one gets

$$\widetilde{d}^u_{\chi\varphi_1} = \begin{cases} \pm\zeta_{2^k}^{[\frac{i-1}{2^{r+2-k}}]} & \text{if } 1 \le i \le k_0(B) \\ 0 & \text{if } k_0(B) < i \le k(B) \end{cases}$$

and

$$\widetilde{d}^u_{\chi\varphi_2} = \begin{cases} \pm\zeta_{2^k}^{[\frac{i-1}{2^{r-k+1}}]} & \text{if } 1 \le i \le 2^r \\ 0 & \text{if } 2^r < i \le k_0(B) \\ \pm\zeta_{2^k}^{[\frac{i-k_0(B)-1}{2^{r-k}}]} & \text{if } k_0(B) < i \le k(B) \end{cases},$$

where $\chi_1, \ldots, \chi_{k_0(B)}$ are the characters of height 0. But notice that the ordering of those characters for φ_1 and φ_2 is different.

Now assume that there is a matrix $S \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$C = 2^{r-1} S^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} S.$$

If Q denotes the decomposition matrix of B, we set $(\widetilde{d}_{\chi\varphi_i}) := QS^{-1}$ for $\mathrm{IBr}(B) = \{\varphi_1, \varphi_2\}$. Then we have

$$|D| m^{(1,B)}_{\chi\psi} = 8\widetilde{d}_{\chi\varphi_1}\widetilde{d}_{\psi\varphi_1} + \widetilde{d}_{\chi\varphi_2}\widetilde{d}_{\psi\varphi_2} \text{ for } \chi, \psi \in \mathrm{Irr}(B).$$

In particular $|D| m^{(1,B)}_{\chi\chi} \equiv 1 \pmod 4$ for a character $\chi \in \mathrm{Irr}(B)$ of height 0. For $u \in \mathcal{T} \setminus \mathrm{Z}(D)$ we have $|D| m^{(u,b_u)}_{\chi\chi} = 2$, and for $u \in \mathrm{Z}(D) \setminus \langle c \rangle$ we have $|D| m^{(u,b_u)}_{\chi\chi} = 1$. Let $u \in \langle c \rangle \setminus \{1\}$. Equation (4) and the considerations above imply $|D| m^{(u,b_u)}_{\chi\chi} \equiv 3 \pmod 4$. Now (5B) in [6] reveals the contradiction

$$|D| = \sum_{u \in \mathcal{T}} |D| m^{(u,b_u)}_{\chi\chi} \equiv |D| m^{(1,B)}_{\chi\chi} + 2^{r+1} + 2^{r-1} + 3 \cdot (2^{r-1} - 1) \equiv 2 \pmod 4. \qquad \square$$

With the proof of the last theorem we can also obtain the ordinary decomposition numbers (up to multiplication with an invertible matrix):

$$d_{\chi\varphi_1} = \begin{cases} \pm 1 & \text{if } h(\chi) = 0 \\ 0 & \text{if } h(\chi) = 1 \end{cases}, \qquad d_{\chi_i\varphi_2} = \begin{cases} \pm 1 & \text{if } 0 \le i \le 2^r \\ 0 & \text{if } 2^r < i \le k_0(B) \\ \pm 1 & \text{if } k_0(B) < i \le k(B) \end{cases}.$$

Again $\chi_1, \ldots, \chi_{k_0(B)}$ are the characters of height 0.

Since we know how $\mathcal{G}$ acts on the B-subsections, we can investigate the action of $\mathcal{G}$ on $\mathrm{Irr}(B)$.

Theorem 3.9. *The irreducible characters of height* 0 *of* B *split in* $2(r+1)$ *families of* 2*-conjugate characters. These families have sizes* $1, 1, 1, 1, 2, 2, 4, 4, \ldots, 2^{r-1}, 2^{r-1}$ *respectively. The characters of height* 1 *split in* r *families with sizes* $1, 1, 2, 4, \ldots, 2^{r-2}$ *respectively. In particular there are exactly six* 2*-rational characters in* $\mathrm{Irr}(B)$.

Proof. We start by determining the number of orbits of the action of $\mathcal{G}$ on the columns of the generalized decomposition matrix. The columns $\{d^u_{\chi\varphi_u} : \chi \in \mathrm{Irr}(B)\}$ with $u \in \mathcal{T} \setminus \mathrm{Z}(D)$ split in two orbits of length 2^{r-1}. For $i = 1, 2$ the columns $\{d^u_{\chi\varphi_i} : \chi \in \mathrm{Irr}(B)\}$ with $u \in \langle c \rangle$ split in r orbits of lengths $1, 1, 2, 4, \ldots, 2^{r-2}$ respectively. Finally, the columns $\{d^u_{\chi\varphi_u} : \chi \in \mathrm{Irr}(B)\}$ with $u \in \mathrm{Z}(D) \setminus \langle c \rangle$ consist of r orbits of lengths $1, 1, 2, 4, \ldots, 2^{r-2}$ respectively. This gives $3r + 2$ orbits altogether. By Theorem 11 in [3] there also exist exactly $3r + 2$ families of 2-conjugate characters. (Since $\mathcal{G}$ is noncyclic, one cannot conclude a priori that also the lengths of the orbits of these two actions coincide.)

By considering the column $\{d^x_{\chi\varphi_x} : \chi \in \mathrm{Irr}(B)\}$, we see that the irreducible characters of height 0 split in at most $2(r+1)$ orbits of lengths $1, 1, 1, 1, 2, 2, 4, 4, \ldots, 2^{r-1}, 2^{r-1}$ respectively. Similarly the column $\{d^c_{\chi\varphi_2} : \chi \in \mathrm{Irr}(B)\}$ shows that there are at most r orbits of lengths $1, 1, 2, 4, \ldots, 2^{r-2}$ of characters of height 1. Since $2(r+1) + r = 3r + 2$, these orbits do not merge further, and the claim is proved. $\square$

Let $M = \langle x^2, y, z\rangle$ as in Lemma 3.3. Then $D \subseteq \mathrm{T}_{\mathrm{N}_G(M)}(b_M)$. Since $e(B) = 1$, Alperin's fusion theorem implies that $\mathrm{T}_{\mathrm{N}_G(M)}(b_M)$ controls the fusion of B-subpairs. By Lemma 3.3 we also have $\mathrm{T}_{\mathrm{N}_G(M)}(b_M) \subseteq \mathrm{C}_G(c)$ for a $c \in \mathrm{Z}(D)$. This shows that B is a so called "centrally controlled block" (see [22]). In [22] it was shown that then the centers of the blocks B and b_c (regarded as blocks of FG) are isomorphic.

3.5 Dade's conjecture

In this section we will verify Dade's (ordinary) conjecture for the block B (see [12]). First, we need a lemma.

Lemma 3.10. *Let $\widetilde{B}$ be a block of RG with defect group $\widetilde{D} \cong C_{2^s} \times C_2^2$ ($s \in \mathbb{N}_0$) and inertial index 3. Then $k(\widetilde{B}) = k_0(\widetilde{B}) = |\widetilde{D}| = 2^{s+2}$ and $l(\widetilde{B}) = 3$ hold.*

Proof. Let α be an automorphism of $\widetilde{D}$ of order 3 which is induced by the inertial group. By Lemma 3.2 we have $\mathrm{C}_{\widetilde{D}}(\alpha) \cong C_{2^s}$. We choose a system of representatives $x_1, \ldots, x_k$ for the orbits of $\widetilde{D} \setminus \mathrm{C}_{\widetilde{D}}(\alpha)$ under α. Then $k = 2^s$. If $b_i \in \mathrm{Bl}(R\,\mathrm{C}_G(x_i))$ for $i = 1, \ldots, k$ and $b_u \in \mathrm{Bl}(R\,\mathrm{C}_G(u))$ for $u \in \mathrm{C}_{\widetilde{D}}(\alpha)$ are Brauer correspondents of $\widetilde{B}$, then

$$\bigcup_{i=1}^{k} \{(x_i, b_i)\} \;\cup \bigcup_{u \in \mathrm{C}_{\widetilde{D}}(\alpha)} \{(u, b_u)\}$$

is a system of representatives for the conjugacy classes of $\widetilde{B}$-subsections. Since $\alpha \notin \mathrm{C}_G(x_i)$, we have $l(b_i) = 1$ for $i = 1, \ldots, k$. In particular $k(\widetilde{B}) \le 2^{s+2}$ holds. Now we show the opposite inequality by induction on s.

For $s = 0$ the claim is well known. Let $s \ge 1$. By induction $l(b_u) = 3$ for $u \in \mathrm{C}_{\widetilde{D}}(\alpha) \setminus \{1\}$. This shows $k(\widetilde{B}) - l(\widetilde{B}) = k + (2^s - 1)3 = 2^{s+2} - 3$ and $l(\widetilde{B}) \le 3$. An inspection of the numbers $d^{x_1}_{\chi\varphi}$ implies $k(\widetilde{B}) = k_0(\widetilde{B}) = 2^{s+2} = |\widetilde{D}|$ and $l(\widetilde{B}) = 3$. (This would also follow from Theorem 1 in [46].) $\square$

Now assume $\mathrm{O}_2(G) = 1$ (this is a hypothesis of Dade's conjecture). In order to prove Dade's conjecture it suffices to consider chains

$$\sigma : P_1 < P_2 < \ldots < P_n$$

of nontrivial elementary abelian 2-subgroups of G (see [12]). (Note that also the empty chain is allowed.) In particular $P_i \trianglelefteq P_n$ and $P_n \trianglelefteq \mathrm{N}_G(\sigma)$ for $i = 1, \ldots, n$. Hence, for a block $b \in \mathrm{Bl}(R\,\mathrm{N}_G(\sigma))$ with $b^G = B$ and defect group Q we have $P_n \le Q$. Moreover, there exists a $g \in G$ such that ${}^gQ \le D$. Thus, by conjugation with g we may assume $P_n \le Q \le D$ (see also Lemma 6.9 in [12]). This shows $n \le 3$.

In the case $|P_n| = 8$ we have $P_n = \langle x^{2^{r-1}}, y, z\rangle =: E$, because this is the only elementary abelian subgroup of order 8 in D. Let $b \in \mathrm{Bl}(R\,\mathrm{N}_G(\sigma))$ with $b^G = B$. We choose a defect group Q of $\widetilde{B} := b^{\mathrm{N}_G(E)}$. Since $\Omega(Q) = P_n$, we get $\mathrm{N}_G(Q) \le \mathrm{N}_G(E)$. Then Brauer's first main theorem implies $Q = D$. Hence, $\widetilde{B}$ is the unique Brauer correspondent of B in $R\,\mathrm{N}_G(E)$. For $M := \langle x^2, y, z\rangle \le D$ we also have $\mathrm{N}_G(M) \le \mathrm{N}_G(\Omega(M)) = \mathrm{N}_G(E)$. Hence, $\widetilde{B}$ is nonnilpotent. Now consider the chain

$$\widetilde{\sigma} : \begin{cases} \varnothing & \text{if } n = 1 \\ P_1 & \text{if } n = 2 \\ P_1 < P_2 & \text{if } n = 3 \end{cases}$$

for the group $\widetilde{G} := \mathrm{N}_G(E)$. Then $\mathrm{N}_G(\sigma) = \mathrm{N}_{\widetilde{G}}(\widetilde{\sigma})$ and

$$\sum_{\substack{b \in \mathrm{Bl}(R\,\mathrm{N}_G(\sigma)),\\ b^G = B}} k^i(b) = \sum_{\substack{b \in \mathrm{Bl}(R\,\mathrm{N}_{\widetilde{G}}(\widetilde{\sigma})),\\ b^{\widetilde{G}} = \widetilde{B}}} k^i(b).$$

The chains σ and $\widetilde{\sigma}$ account for all possible chains of G. Moreover, the lengths of σ and $\widetilde{\sigma}$ have opposite parity. Thus, it seems plausible that the contributions of σ and $\widetilde{\sigma}$ in the alternating sum cancel out each other (this would imply Dade's conjecture). The question which remains is: Can we replace $(\widetilde{G}, \widetilde{B}, \widetilde{\sigma})$ by $(G, B, \widetilde{\sigma})$? We make this more precise in the following lemma.

Lemma 3.11. *Let $\mathcal{Q}$ be a system of representatives for the G-conjugacy classes of pairs (σ, b), where σ is a chain (of G) of length n with $P_n < E$ and $b \in \mathrm{Bl}(R\,\mathrm{N}_G(\sigma))$ is a Brauer correspondent of B. Similarly, let $\widetilde{\mathcal{Q}}$ be a system of representatives for the $\widetilde{G}$-conjugacy classes of pairs $(\widetilde{\sigma}, \widetilde{b})$, where $\widetilde{\sigma}$ is a chain (of $\widetilde{G}$) of length n with $P_n < E$ and $\widetilde{b} \in \mathrm{Bl}(R\,\mathrm{N}_{\widetilde{G}}(\widetilde{\sigma}))$ is a Brauer correspondent of $\widetilde{B}$. Then there exists a bijection between $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ which preserves the numbers $k^i(b)$.*

Proof. Let $b_D \in \mathrm{Bl}(R\,\mathrm{N}_G(D))$ be a Brauer correspondent of B. We consider chains of B-subpairs

$$\sigma : (P_1, b_1) < (P_2, b_2) < \ldots < (P_n, b_n) < (D, b_D),$$

where the P_i are nontrivial elementary abelian 2-subgroups such that $P_n < E$. Then σ is uniquely determined by these subgroups $P_1, \ldots, P_n$ (see Theorem 1.7 in [36]). Moreover, the empty chain is also allowed. Let $\mathcal{U}$ be a system of representatives for G-conjugacy classes of such chains. For every chain $\sigma \in \mathcal{U}$ we define

$$\widetilde{\sigma} : (P_1, \widetilde{b_1}) < (P_2, \widetilde{b_2}) < \ldots < (P_n, \widetilde{b_n}) < (D, b_D)$$

with $\widetilde{b_i} \in \mathrm{Bl}(R\,\mathrm{C}_{\widetilde{G}}(P_i))$ for $i = 1, \ldots, n$. Finally we set $\widetilde{\mathcal{U}} := \{\widetilde{\sigma} : \sigma \in \mathcal{U}\}$. By Alperin's fusion theorem $\widetilde{\mathcal{U}}$ is a system of representatives for the $\widetilde{G}$-conjugacy classes of corresponding chains for the group $\widetilde{B}$. Hence, it suffices to show the existence of bijections f (resp. $\widetilde{f}$) between $\mathcal{U}$ (resp. $\widetilde{\mathcal{U}}$) and $\mathcal{Q}$ (resp. $\widetilde{\mathcal{Q}}$) such that the following property is satisfied: If $f(\sigma) = (\tau, b)$ and $\widetilde{f}(\widetilde{\sigma}) = (\widetilde{\tau}, \widetilde{b})$, then $k^i(b) = k^i(\widetilde{b})$ for all $i \in \mathbb{N}_0$.

Let $\sigma \in \mathcal{U}$. Then we define the chain τ by only considering the subgroups of σ, i. e. $\tau : P_1 < \ldots < P_n$. This gives $\mathrm{C}_G(P_n) \subseteq \mathrm{N}_G(\tau)$, and we can define

$$f : \mathcal{U} \to \mathcal{Q},\ \sigma \mapsto \big(\tau, b_n^{\mathrm{N}_G(\tau)}\big).$$

Now let $(\sigma, b) \in \mathcal{Q}$ arbitrary. We write $\sigma : P_1 < \ldots < P_n$. By Theorem 5.5.15 in [29] there exists a Brauer correspondent $\beta_n \in \mathrm{Bl}(R\,\mathrm{C}_G(P_n))$ of b. Since (P_n, β_n) is a B-subpair, we may assume $(P_n, \beta_n) < (D, b_D)$ after a suitable conjugation. Then there are uniquely determined blocks $\beta_i \in \mathrm{Bl}(R\,\mathrm{C}_G(P_i))$ for $i = 1, \ldots, n-1$ such that

$$(P_1, \beta_1) < (P_2, \beta_2) < \ldots < (P_n, \beta_n) < (D, b_D).$$

This shows that f is surjective.

Now let $\sigma_1, \sigma_2 \in \mathcal{U}$ be given. We write

$$\sigma_i : (P_1^i, \beta_1^i) < \ldots < (P_n^i, \beta_n^i)$$

for $i = 1, 2$. Let us assume that $f(\sigma_1) = (\tau_1, b_1)$ and $f(\sigma_2) = (\tau_2, b_2)$ are conjugate in G, i. e. there is a $g \in G$ such that

$$\big(\tau_2, ({}^g\beta_n^1)^{\mathrm{N}_G(\tau_2)}\big) = {}^g(\tau_1, b_1) = (\tau_2, b_2) = \big(\tau_2, (\beta_n^2)^{\mathrm{N}_G(\tau_2)}\big).$$

Since ${}^g\beta_n^1 \in \mathrm{Bl}(R\,\mathrm{C}_G(P_n^2))$ and β_n^2 are covered by b_2, there is $h \in \mathrm{N}_G(\tau_2)$ with ${}^{hg}\beta_n^1 = \beta_n^2$. Then

$${}^{hg}(P_n^1, \beta_n^1) = (P_n^2, \beta_n^2).$$

Since the blocks β_j^i for $i = 1, 2$ and $j = 1, \ldots, n-1$ are uniquely determined by P_j^i, we also have ${}^{gh}\sigma_1 = \sigma_2 = \sigma_1$. This proves the injectivity of f. Analogously, we define the map $\widetilde{f}$.

It remains to show that f and $\widetilde{f}$ satisfy the property given above. For this let $\sigma \in \mathcal{U}$ with $\sigma : (P_1, b_1) < \ldots < (P_n, b_n)$, $\widetilde{\sigma} : (P_1, \widetilde{b_1}) < \ldots < (P_n, \widetilde{b_n})$, $f(\sigma) = \big(\tau, b_n^{\mathrm{N}_G(\tau)}\big)$ and $\widetilde{f}(\widetilde{\sigma}) = \big(\tau, \widetilde{b_n}^{\mathrm{N}_{\widetilde{G}}(\tau)}\big)$. We have to prove $k^i\big(b_n^{\mathrm{N}_G(\tau)}\big) = k^i\big(\widetilde{b_n}^{\mathrm{N}_{\widetilde{G}}(\tau)}\big)$ for $i \in \mathbb{N}_0$.

Let Q be a defect group of $b_n^{\mathrm{N}_G(\tau)}$. Then $Q\,\mathrm{C}_G(Q) \subseteq \mathrm{N}_G(\tau)$, and there is a Brauer correspondent $\beta_n \in \mathrm{Bl}(RQ\,\mathrm{C}_G(Q))$ of $b_n^{\mathrm{N}_G(\tau)}$. In particular (Q, β_n) is a B-Brauer subpair. As in Lemma 3.1 we may assume $Q \in \{D, M, \langle x, z\rangle, \langle xy, z\rangle\}$. The same considerations also work for the defect group $\widetilde{Q}$ of $\widetilde{b_n}^{\mathrm{N}_{\widetilde{G}}(\tau)}$. Since $b_n^{D\,\mathrm{C}_G(P_n)} = b_D^{D\,\mathrm{C}_G(P_n)} = \widetilde{b_n}^{D\,\mathrm{C}_G(P_n)}$, we get:

$$Q = D \Longleftrightarrow D \subseteq \mathrm{N}_G(\tau) \Longleftrightarrow D \subseteq \mathrm{N}_{\widetilde{G}}(\tau) \Longleftrightarrow \widetilde{Q} = D.$$

Let us consider the case $Q = D$ $(= \widetilde{Q})$. Let $b_M \in \mathrm{Bl}(R\,\mathrm{C}_G(M))$ such that $(M, b_M) \le (D, b_D)$ and $\alpha \in \mathrm{T}_{\mathrm{N}_G(M)}(b_M) \setminus D\,\mathrm{C}_G(M) \subseteq \mathrm{N}_G(M) \subseteq \widetilde{G}$. Then:

$$b_n^{\mathrm{N}_G(\tau)} \text{ is nilpotent} \iff \alpha \notin \mathrm{N}_G(\tau) \iff \alpha \notin \mathrm{N}_{\widetilde{G}}(\tau) \iff \widetilde{b_n}^{\mathrm{N}_{\widetilde{G}}(\tau)} \text{ is nilpotent.}$$

Thus, the claim holds in this case. Now let $Q < D$ (and $\widetilde{Q} < D$). Then we have $Q\,\mathrm{C}_G(Q) = \mathrm{C}_G(Q) \subseteq \mathrm{C}_G(P_n)$. Since $\beta_n^{\mathrm{C}_G(P_n)}$ is also a Brauer correspondent of $b_n^{\mathrm{N}_G(\tau)}$, the blocks $\beta_n^{\mathrm{C}_G(P_n)}$ and b_n are conjugate. In particular b_n (and $\widetilde{b_n}$) has defect group Q. Hence, we obtain $Q = \widetilde{Q}$. If $Q \in \{\langle x, z\rangle, \langle xy, z\rangle\}$, then $b_n^{\mathrm{N}_G(\tau)}$ and $\widetilde{b_n}^{\mathrm{N}_{\widetilde{G}}(\tau)}$ are nilpotent, and the claim holds. Thus, we may assume $Q = M$. Then as before:

$$b_n^{\mathrm{N}_G(\tau)} \text{ is nilpotent} \iff \alpha \notin \mathrm{N}_G(\tau) \iff \alpha \notin \mathrm{N}_{\widetilde{G}}(\tau) \iff \widetilde{b_n}^{\mathrm{N}_{\widetilde{G}}(\tau)} \text{ is nilpotent.}$$

We may assume that the nonnilpotent case occurs. Then $t\big(b_n^{\mathrm{N}_G(\tau)}\big) = t\big(\widetilde{b_n}^{\mathrm{N}_{\widetilde{G}}(\tau)}\big) = 3$, and the claim follows from Lemma 3.10. $\square$

As explained in the beginning of the section, the Dade conjecture follows.

Theorem 3.12. *The Dade conjecture holds for B.*

3.6 Alperin's weight conjecture

In this section we prove Alperin's weight conjecture for B. Let (P, β) be a weight for B, i. e. P is a 2-subgroup of G and β is a block of $R[\mathrm{N}_G(P)/P]$ with defect 0. Moreover, β is dominated by a Brauer correspondent $b \in \mathrm{Bl}(R\,\mathrm{N}_G(P))$ of B. As usual, one can assume $P \le D$. If $\mathrm{Aut}(P)$ is a 2-group, then $\mathrm{N}_G(P)/\,\mathrm{C}_G(P)$ is also a 2-group. Then P is a defect group of b, since β has defect 0. Moreover, β is uniquely determined by b. By Brauer's first main theorem we have $P = D$. Thus, in this case there is exactly one weight for B up to conjugation.

Now let us assume that $\mathrm{Aut}(P)$ is not a 2-group (in particular $P < D$). As usual, β covers a block $\beta_1 \in \mathrm{Bl}(R[\mathrm{C}_G(P)/P])$. By the Fong-Reynolds theorem (see [29] for example) also β_1 has defect 0. Hence, β_1 is dominated by exactly one block $b_1 \in \mathrm{Bl}(R\,\mathrm{C}_G(P))$ with defect group P. Since $\beta\beta_1 \ne 0$, we also have $bb_1 \ne 0$, i. e. b covers b_1. Thus, the situation is as follows:

$$\begin{array}{ccc}
\beta \in \mathrm{Bl}(R[\mathrm{N}_G(P)/P]) & \longleftrightarrow & b \in \mathrm{Bl}(R\,\mathrm{N}_G(P)) \\
\updownarrow & & \updownarrow \\
\beta_1 \in \mathrm{Bl}(R[\mathrm{C}_G(P)/P]) & \longleftrightarrow & b_1 \in \mathrm{Bl}(R\,\mathrm{C}_G(P))
\end{array}$$

By Theorem 5.5.15 in [29] we have $b_1^{\mathrm{N}_G(P)} = b$ and $b_1^G = B$. This shows that (P, b_1) is a B-Brauer subpair. Then $P = M$ $(=\langle x^2, y, z\rangle)$ follows. By Brauer's first main theorem b is uniquely determined (independent of β). Now we prove that also β is uniquely determined by b.

In order to do so it suffices to show that β is the only block with defect 0 which covers β_1. By the Fong-Reynolds theorem it suffices to show that β_1 is covered by only one block of $R\,\mathrm{T}_{\mathrm{N}_G(M)/M}(\beta_1) = R[\mathrm{T}_{\mathrm{N}_G(M)}(b_1)/M]$ with defect 0. For convenience we write $\overline{\mathrm{C}_G(M)} := \mathrm{C}_G(M)/M$, $\overline{\mathrm{N}_G(M)} := \mathrm{N}_G(M)/M$ and $\overline{\mathrm{T}} := \mathrm{T}_{\mathrm{N}_G(M)}(b_1)/M$. Let $\chi \in \mathrm{Irr}(\beta_1)$. The irreducible constituents of $\mathrm{Ind}_{\overline{\mathrm{C}_G(M)}}^{\overline{\mathrm{T}}}(\chi)$ belong to blocks which covers β_1 (where Ind denote induction). Conversely, every block of $R\overline{\mathrm{T}}$ which covers β_1 arises in this way (see Lemma 5.5.7 in [29]). Let

$$\mathrm{Ind}_{\overline{\mathrm{C}_G(M)}}^{\overline{\mathrm{T}}}(\chi) = \sum_{i=1}^{t} e_i\psi_i$$

with $\psi_i \in \mathrm{Irr}(\overline{\mathrm{T}})$ and $e_i \in \mathbb{N}$ for $i = 1, \ldots, t$. Then

$$\sum_{i=1}^{t} e_i^2 = |\overline{\mathrm{T}} : \overline{\mathrm{C}_G(M)}| = |\,\mathrm{T}_{\mathrm{N}_G(M)}(b_1) : \mathrm{C}_G(M)| = 6$$

(see page 84 in [17]). Thus, there is some $i \in \{1, \ldots, t\}$ with $e_i = 1$, i.e. χ is extendible to $\overline{\mathrm{T}}$. We may assume $e_1 = 1$. By Corollary 6.17 in [17] it follows that $t = |\operatorname{Irr}(\overline{\mathrm{T}}/\overline{\mathrm{C}_G(M)})| = |\operatorname{Irr}(S_3)| = 3$ and

$$\{\psi_1, \psi_2, \psi_3\} = \{\psi_1\tau : \tau \in \operatorname{Irr}(\overline{\mathrm{T}}/\overline{\mathrm{C}_G(M)})\},$$

where the characters in $\operatorname{Irr}(\overline{\mathrm{T}}/\overline{\mathrm{C}_G(M)})$ were identified with their inflations in $\operatorname{Irr}(\overline{\mathrm{T}})$. Thus, we may assume $e_2 = 1$ and $e_3 = 2$. Then it is easy to see that ψ_1 and ψ_2 belong to blocks with defect at least 1. Hence, only the block with contains ψ_3 is allowed. This shows uniqueness.

Finally we show that there is in fact a weight of the form (M, β). For this we choose b, b_1, β_1, χ and ψ_i as above. Then χ vanishes on all nontrivial 2-elements. Moreover, ψ_1 is an extension of χ. Let $\tau \in \operatorname{Irr}(\overline{\mathrm{T}}/\overline{\mathrm{C}_G(M)})$ be the character of degree 2. Then τ vanishes on all nontrivial 2-elements of $\overline{\mathrm{T}}/\overline{\mathrm{C}_G(M)}$. Hence, $\psi_3 = \psi_1\tau$ vanishes on all nontrivial 2-elements of $\overline{\mathrm{T}}$. This shows that ψ_3 belongs in fact to a block $\widetilde{\beta} \in \operatorname{Bl}(R\overline{\mathrm{T}})$ with defect 0. Then $\big(M, \widetilde{\beta}^{\overline{\mathrm{N}_G(M)}}\big)$ is the desired weight for B.

Hence, we have shown that there are exactly two weights for B up to conjugation. Since $l(B) = 2$, Alperin's weight conjecture is satisfied.

Theorem 3.13. *Alperin's weight conjecture holds for B.*

3.7 The gluing problem

Finally we show that the gluing problem (see Conjecture 4.2 in [26]) for the block B has a unique solution. We will not recall the very technical statement of the gluing problem. Instead we refer to [37] for most of the notations. Observe that the field F is denoted by k in [37].

Theorem 3.14. *The gluing problem for B has a unique solution.*

Proof. As in [37] we denote the fusion system induced by B with $\mathcal{F}$. Then the $\mathcal{F}$-centric subgroups of D are given by $M_1 := \langle x^2, y, z\rangle$, $M_2 := \langle x, z\rangle$, $M_3 := \langle xy, z\rangle$ and D. We have seen so far that $\operatorname{Aut}_{\mathcal{F}}(M_1) \cong \operatorname{Out}_{\mathcal{F}}(M_1) \cong S_3$, $\operatorname{Aut}_{\mathcal{F}}(M_i) \cong D/M_i \cong C_2$ for $i = 2, 3$ and $\operatorname{Aut}_{\mathcal{F}}(D) \cong D/\mathrm{Z}(D) \cong C_2^2$ (see proof of Lemma 3.3). Using this, we get $\mathrm{H}^i(\operatorname{Aut}_{\mathcal{F}}(\sigma), F^\times) = 0$ for $i = 1, 2$ and every chain σ of $\mathcal{F}$-centric subgroups (see proof of Corollary 2.2 in [37]). Hence, $\mathrm{H}^0([S(\mathcal{F}^c)], \mathcal{A}^2_{\mathcal{F}}) = \mathrm{H}^1([S(\mathcal{F}^c)], \mathcal{A}^1_{\mathcal{F}}) = 0$. Now the claim follows from Theorem 1.1 in [37]. $\square$

4 The case $r = s > 1$

In the section we assume that B is a nonnilpotent block of RG with defect group

$$D := \langle x, y \mid x^{2^r} = y^{2^r} = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$$

for $r \geq 2$. As before we define $z := [x, y]$. Since $|D/\Phi(D)| = 4$, 2 and 3 are the only prime divisors of $|\operatorname{Aut}(D)|$. In particular $t(B) \in \{1, 3\}$. If $t(B) = 1$, then B would be nilpotent by Theorem 2.4. Thus, we have $t(B) = 3$.

4.1 The B-subsections

We investigate the automorphism group of D.

Lemma 4.1. *Let $\alpha \in \operatorname{Aut}(D)$ be an automorphism of order 3. Then z is the only nontrivial fixed-point of $\mathrm{Z}(D)$ under α.*

Proof. Since $D' = \langle z\rangle$, z remains fixed under all automorphisms of D. Moreover, $\alpha(x) \in y\,\mathrm{Z}(D) \cup xy\,\mathrm{Z}(D)$, because α acts nontrivially on $D/\mathrm{Z}(D)$. In both cases we have $\alpha(x^2) \neq x^2$. This shows that $\alpha|_{\mathrm{Z}(D)} \in \operatorname{Aut}(\mathrm{Z}(D))$ is also an automorphism of order 3. Obviously α induces an automorphism of order 3 on $\mathrm{Z}(D)/\langle z\rangle \cong C^2_{2^{r-1}}$. But this automorphism is fixed-point-free (see Lemma 1 in [27]). The claim follows. $\square$

Using this, we can find a system of representatives for the conjugacy classes of B-subsections.

Lemma 4.2. *Let $b \in \mathrm{Bl}(RD\,\mathrm{C}_G(D))$ be a Brauer correspondent of B, and for $Q \le D$ let b_Q be the unique block of $RQ\,\mathrm{C}_G(Q)$ with $(Q, b_Q) \le (D, b)$. We choose a system $\mathcal{S} \subseteq \mathrm{Z}(D)$ of representatives for the orbits of $\mathrm{Z}(D)$ under the action of $\mathrm{T}_{\mathrm{N}_G(D)}(b)$. We set $\mathcal{T} := \mathcal{S} \cup \{y^i x^{2j} : i, j \in \mathbb{Z},\ i \text{ odd}\}$. Then*

$$\bigcup_{a \in \mathcal{T}} \left\{ \left(a, b_{\mathrm{C}_D(a)}^{\mathrm{C}_G(a)}\right) \right\}$$

is a system of representatives for the conjugacy classes of B-subsections. Moreover,

$$|\mathcal{T}| = \frac{5 \cdot 2^{2(r-1)} + 4}{3}.$$

Proof. Proposition 2.12.(ii) in [34] states the desired system wrongly. More precisely the claim $I_D = \mathrm{Z}(D)$ in the proof is false. Indeed Lemma 4.1 shows $I_D = \mathcal{S}$. Now the claim follows easily. $\square$

From now on we write $b_a := b_{\mathrm{C}_D(a)}^{\mathrm{C}_G(a)}$ for $a \in \mathcal{T}$. We are able to determine the difference $k(B) - l(B)$.

Proposition 4.3. *We have*

$$k(B) - l(B) = \frac{5 \cdot 2^{2(r-1)} + 7}{3}.$$

Proof. Consider $l(b_a)$ for $1 \ne a \in \mathcal{T}$.

Case 1: $a \in \mathrm{Z}(D)$.
Then b_a is a block with defect group D. Moreover, b_a and B have a common Brauer correspondent in $\mathrm{Bl}(RD\,\mathrm{C}_{\mathrm{C}_G(a)}(D)) = \mathrm{Bl}(RD\,\mathrm{C}_G(D))$. In case $a \ne z$ we have $t(b_a) = 1$ by Lemma 4.1. Hence, b_a is nilpotent and $l(b_a) = 1$. Now let $a = z$. Then there exists a block $\overline{b_z}$ of $\mathrm{C}_G(z)/\langle z \rangle$ with defect group $D/\langle z \rangle \cong C_{2^r}^2$ and $l(\overline{b_z}) = l(b_z)$. By Theorem 1.5(iv) in [33], $t(\overline{b_z}) = t(b_z) = 3$ holds. Thus, Theorem 2 in [43] implies $l(b_z) = l(\overline{b_z}) = 3$.

Case 2: $a \notin \mathrm{Z}(D)$.
Then $b_{\mathrm{C}_P(a)} = b_M$ is a block with defect group $M := \langle x^2, y, z \rangle$. Since $b_M^{D\,\mathrm{C}_G(M)} = b_D^{D\,\mathrm{C}_G(M)}$, also $b_M^{\mathrm{C}_G(a)} = b_a$ has defect group M. For every automorphism $\alpha \in \mathrm{Aut}(D)$ of order 3 we have $\alpha(M) \ne M$. Since D controls the fusion of B-subpairs, we get $t(b_a) = l(b_a) = 1$.

Now the conclusion follows from $k(B) = \sum_{a \in \mathcal{T}} l(b_a)$. $\square$

The next result concerns the Cartan matrix of B.

Lemma 4.4. *The elementary divisors of the Cartan matrix of B are contained in $\{1, 2, |D|\}$. The elementary divisor 2 occurs twice and $|D|$ occurs once (as usual). In particular $l(B) \ge 3$.*

Proof. Let C be the Cartan matrix of B. As in Lemma 3.7 we use the notion of lower defect groups. For this let $P < D$ such that $|P| \ge 4$, and let $b \in \mathrm{Bl}(R\,\mathrm{N}_G(P))$ be a Brauer correspondent of B with defect group $Q \le D$. Brauer's first main theorem implies $P < Q$. By Proposition 1.3 in [33] there exists a block $\beta \in \mathrm{Bl}(R\,\mathrm{C}_G(P))$ with $\beta^{\mathrm{N}_G(P)} = b$ such that at most $l(\beta)$ lower defect groups of b contain a conjugate of P. Let $S \le Q$ be a defect group of β. First, we consider the case $S = D$. Then $P \subseteq \mathrm{Z}(D)$. By Lemma 4.1 we have $l(\beta) = 1$, since $|P| \ge 4$. It follows that $m_b^1(P) = m_b(P) = 0$, because P is contained in the (lower) defect group Q of b.

Now assume $S < D$. In particular S is abelian. If S is even metacyclic, then $l(\beta) = 1$ and $m_b^1(P) = 0$, since $P \subseteq \mathrm{Z}(\mathrm{C}_G(P))$. Thus, let us assume that S is nonmetacyclic. By (3C) in [5], $x^2 \in \mathrm{Z}(D)$ is conjugate to an element of $\mathrm{Z}(S)$. This shows $S \cong C_{2^k} \times C_{2^l} \times C_2$ with $k \in \{r, r-1\}$ and $1 \le l \le r$. If $1, k, l$ are pairwise distinct, then $l(\beta) = 1$ and $m_b^1(P) = 0$ follow from Lemma 2.3. Let $k = l$. Then every automorphism of S of order 3 has only one nontrivial fixed-point. Since $|P| \ge 4$, it follows again that $l(\beta) = 1$ and $m_b^1(P) = 0$.

Now let $S \cong C_{2^k} \times C_2^2$ with $2 \le k \in \{r-1, r\}$. Assume first that P is noncyclic. Then S/P is metacyclic. If S/P is not a product of two isomorphic cyclic groups, then $l(\beta) = 1$ and $m_b^1(P) = 0$. Hence, we may assume

$S/P \cong C_2^2$. It is easy to see that there exists a subgroup $P_1 \le P$ with $S/P_1 \cong C_4 \times C_2$. We get $l(\beta) = 1$ and $m_b^1(P) = 0$ also in this case.

Finally, let $P = \langle u \rangle$ be cyclic. Then (u, β) is a B-subsection. Since $|P| \ge 4$, u is not conjugate to z. As in the proof of Proposition 4.3 we have $l(\beta) = 1$ and $m_b^1(P) = 0$. This shows $m_B^1(P) = 0$. Since P was arbitrary, the multiplicity of $|P|$ as an elementary divisor of C is 0.

It remains to consider the case $|P| = 2$. We write $P = \langle u \rangle \le D$. As before let $b \in \operatorname{Bl}(R\operatorname{N}_G(P))$ be a Brauer correspondent of B. Then (u, b) is a B-subsection. If (u, b) is not conjugate to (z, b_z), then $l(b) = 1$ and $m_b^1(P) = 0$ as in the proof of Proposition 4.3. Since we can replace P by a conjugate, we may assume $P = \langle z \rangle$ and $(u, b) = (z, b_z)$. Then $l(b) = 3$ and D is a defect group of b. Now let $\overline{b} \in \operatorname{Bl}(R[\operatorname{N}_G(P)/P])$ be the block which is dominated by b. By Corollary 1 in [16] the elementary divisors of the Cartan matrix of $\overline{b}$ are $1, 1, |D|/2$. Hence, the elementary divisors of the Cartan matrix of b are $2, 2, |D|$. This shows

$$2 = \sum_{\substack{Q \in \mathcal{P}(\operatorname{N}_G(P)),\\ |Q|=2}} m_b^1(Q),$$

where $\mathcal{P}(\operatorname{N}_G(P))$ is a system of representatives for the conjugacy classes of p-subgroups of $\operatorname{N}_G(P)$. The same arguments applied to b instead of B imply $m_b^1(Q) = 0$ for $P \ne Q \le \operatorname{N}_G(P)$ with $|Q| = 2$. Hence, $2 = m_b^1(P) = m_B^1(P)$, and 2 occurs as elementary divisors of C twice. $\square$

As in Section 3 we write $\operatorname{IBr}(b_u) = \{\varphi_u\}$ for $u \in \mathcal{T} \setminus \langle z \rangle$. In a similar manner we define the integers a_i^u. If $u \in \mathcal{T} \setminus \langle z \rangle$ with $|\langle u \rangle| = 2^k > 2$, then the 2^{k-1} distinct subsections of the form ${}^\gamma(u, b_u)$ for $\gamma \in \mathcal{G}$ are pairwise nonconjugate (same argument as in the case $r > s = 2$). Hence, Lemma 3.4 carries over in a corresponding form. Apart from that we can also carry over Lemma (6.B) in [20]:

Lemma 4.5. *Let $\chi \in \operatorname{Irr}(B)$ and $u \in \mathcal{T} \setminus \operatorname{Z}(D)$. Then χ has height 0 if and only if the sum*

$$\sum_{i=0}^{2^{r-1}-1} a_i^u(\chi)$$

is odd.

Proof. If χ has height 0, the sum is odd by Proposition 1 in [9]. The other implication follows easily from (5G) in [6]. $\square$

The next lemma is the analogon to Lemma 3.5.

Lemma 4.6. *Let $u \in \operatorname{Z}(D) \setminus \langle z \rangle$ of order 2^k. Then for all $\chi \in \operatorname{Irr}(B)$ we have:*

(i) $2^{h(\chi)} \mid a_i^u(\chi)$ *for* $i = 0, \ldots, 2^{k-1} - 1$,

(ii) $\sum_{i=0}^{2^{k-1}-1} a_i^u(\chi) \equiv 2^{h(\chi)} \pmod{2^{h(\chi)+1}}$.

As in the case $r > s = 1$, Lemma 1.1 in [39] implies

$$k(B) \le \sum_{i=0}^{\infty} 2^{2i} k_i(B) \le |D|. \tag{6}$$

In particular Brauer's $k(B)$-conjecture holds. Moreover, Theorem 3.1 in [39] gives $k_0(B) \le |D|/2 = |D : D'|$, i. e. Olsson's conjecture is satisfied. Using this, we can improve the inequality (6) to

$$|D| \ge k_0(B) + 4(k(B) - k_0(B)) = 4k(B) - 3k_0(B) \ge 4k(B) - \frac{3|D|}{2}$$

and

$$\frac{5 \cdot 2^{2(r-1)} + 16}{3} \le k(B) - l(B) + l(B) = k(B) \le \frac{5|D|}{8} = 5 \cdot 2^{2(r-1)}.$$

We will improve this further. Let $\overline{b_z}$ be the block of $\mathrm{Bl}(R\,\mathrm{C}_G(z)/\langle z\rangle)$ which is dominated by b_z. Then $\overline{b_z}$ has defect group $D/\langle z\rangle \cong C_{2^r}^2$. Using the existence of a perfect isometry (see [44, 45, 38]), one can show that the Cartan matrix of $\overline{b_z}$ is equivalent to

$$\overline{C} := \frac{1}{3}\begin{pmatrix} 2^{2r}+2 & 2^{2r}-1 & 2^{2r}-1 \\ 2^{2r}-1 & 2^{2r}+2 & 2^{2r}-1 \\ 2^{2r}-1 & 2^{2r}-1 & 2^{2r}+2 \end{pmatrix}.$$

Hence, the Cartan matrix of b_z is equivalent to $2\overline{C}$. Now inequality $(**)$ in [24] yields

$$k(B) \le 2\frac{2^{2r}+8}{3} = \frac{|D|+16}{3}.$$

(Notice that the proof of Theorem A in [24] also works for b_z instead of B, since the generalized decomposition numbers corresponding to (z, b_z) are integral. See also Lemma 3 in [42].)

In addition we have

$$k_i(B) = 0 \text{ for } i \ge 4$$

by Corollary (6D) in [7]. This means that the heights of the characters in $\mathrm{Irr}(B)$ are bounded independently of r. We remark also that Alperin's weight conjecture is equivalent to

$$l(B) = l(b)$$

for the Brauer correspondent $b \in \mathrm{Bl}(R\,\mathrm{N}_G(D))$ of B (see Consequence 5 in [1]). Since $z \in \mathrm{Z}(\mathrm{N}_G(D))$, $l(B) = l(b) = 3$ and $k(B) = (5 \cdot 2^{2(r-1)} + 16)/3$ would follow in this case (see proof of Proposition 4.3).

4.2 The gluing problem

As in section 3.7 we use the notations of [37].

Theorem 4.7. *The gluing problem for B has a unique solution.*

Proof. Let $\mathcal{F}$ be the fusion system induced by B. Then the $\mathcal{F}$-centric subgroups of D are given by $M := \langle x^2, y, z\rangle$ and D (up to conjugation in $\mathcal{F}$). We have $\mathrm{Aut}_{\mathcal{F}}(M) \cong D/M \cong C_2$ and $\mathrm{Aut}_{\mathcal{F}}(D) \cong A_4$. This shows $\mathrm{H}^2(\mathrm{Aut}_{\mathcal{F}}(\sigma), F^\times) = 0$ for every chain σ of $\mathcal{F}$-centric subgroups. Consequently, $\mathrm{H}^0([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^2) = 0$. On the other hand, we have $\mathrm{H}^1(\mathrm{Aut}_{\mathcal{F}}(D), F^\times) \cong \mathrm{H}^1(C_3, F^\times) \cong C_3$ and $\mathrm{H}^1(\mathrm{Aut}_{\mathcal{F}}(\sigma), F^\times) = 0$ for all chains $\sigma \ne D$. Hence, the situation is as in Case 3 of the proof of Theorem 1.2 in [37]. However, the proof in [37] is pretty short. For the convenience of the reader, we give a more complete argument.

Since $[S(\mathcal{F}^c)]$ is partially ordered by taking subchains, one can view $[S(\mathcal{F}^c)]$ as a category, where the morphisms are given by the pairs of ordered chains. In particular $[S(\mathcal{F}^c)]$ has exactly five morphisms. With the notations of [47] the functor $\mathcal{A}_{\mathcal{F}}^1$ is a *representation* of $[S(\mathcal{F}^c)]$ over $\mathbb{Z}$. Hence, we can view $\mathcal{A}_{\mathcal{F}}^1$ as a module $\mathcal{M}$ over the incidence algebra of $[S(\mathcal{F}^c)]$. More precisely, we have

$$\mathcal{M} := \bigoplus_{a \in \mathrm{Ob}[S(\mathcal{F}^c)]} \mathcal{A}_{\mathcal{F}}^1(a) = \mathcal{A}_{\mathcal{F}}^1(D) \cong C_3.$$

Now we can determine $\mathrm{H}^1([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1)$ using Lemma 6.2(2) in [47]. For this let $d : \mathrm{Hom}[S(\mathcal{F}^c)] \to \mathcal{M}$ a derivation. Then we have $d(\alpha) = 0$ for all $\alpha \in \mathrm{Hom}[S(\mathcal{F}^c)]$ with $\alpha \ne (D, D) =: \alpha_1$. However,

$$d(\alpha_1) = d(\alpha_1\alpha_1) = (\mathcal{A}_{\mathcal{F}}^1(\alpha_1))(d(\alpha_1)) + d(\alpha_1) = 2d(\alpha_1) = 0.$$

Hence, $\mathrm{H}^1([S(\mathcal{F}^c)], \mathcal{A}_{\mathcal{F}}^1) = 0$. $\square$

4.3 Special cases

Since the general methods do not suffice to compute the invariants of B, we restrict ourself to certain special situations.

Proposition 4.8. *If* $\mathrm{O}_2(G) \neq 1$, *then*

$$k(B) = \frac{5 \cdot 2^{2(r-1)} + 16}{3}, \qquad k_0(B) \geq \frac{2^{2r} + 8}{3}, \qquad l(B) = 3.$$

Proof. Let $1 \neq Q := \mathrm{O}_2(G)$. Then $Q \subseteq D$. In the case $Q = D'$ we have $\mathrm{C}_G(z) = \mathrm{N}_G(Q) = G$ and $B = b_z$. Then the assertions on $k(B)$ and $l(B)$ are clear. Moreover, b_z dominates a block $\overline{b_z} \in \mathrm{Bl}(R\,\mathrm{C}_G(z)/\langle z\rangle)$ with defect group $C_{2^r}^2$. By Theorem 2 in [43] we have

$$k_0(B) \geq k_0(\overline{b_z}) = k(\overline{b_z}) = \frac{2^{2r} + 8}{3}.$$

Hence, we may assume $Q \neq D'$. With the same argument we may also assume $Q < D$. In particular Q is abelian. We consider a B-subpair (Q, b_Q). Then D or M is a defect group of b_Q (see proof of Lemma 4.2). If D is a defect group of b_Q, then $D \subseteq \mathrm{C}_G(Q)$ and $Q \subseteq \mathrm{Z}(D)$. By Lemma 4.1 it follows that b_Q is nilpotent.

Now let us assume that M is a defect group of b_Q. Since D controls the fusions of B-subpairs, we have $t(b_Q) = 1$ (see Case 2 in the proof of Proposition 4.3). Hence, again b_Q is nilpotent. Thus, in both cases B is an extension of a nilpotent block of $\mathrm{Bl}(R\,\mathrm{C}_G(Q))$. In this situation the Külshammer-Puig theorem applies. In particular we can replace B by a block with normal defect group (see [23]). Hence, $B = b_z$, and the claim follows as before. □

Since $\mathrm{N}_G(D) \subseteq \mathrm{C}_G(z)$, B is a "centrally controlled block" (see [22]). In [22] it was shown that then an epimorphism $\mathrm{Z}(B) \to \mathrm{Z}(b_z)$ exists, where one has to regard B (resp. b_z) as blocks of FG (resp. $F\,\mathrm{C}_G(z)$). Moreover, we conjecture that the blocks B and b_z are Morita-equivalent. For the similar defect group Q_8 this holds in fact (see [18]). In this context the work [11] is also interesting. There is was shown that there is a perfect isometry between any two blocks with the same quaternion group as defect group and the same fusion of subpairs. Thus, it would be also possible that there is a perfect isometry between B and b_z.

Proposition 4.9. *In order to determine $k(B)$ (and thus also $l(B)$), we may assume that $\mathrm{O}_2(G)$ is trivial and $\mathrm{O}_{2'}(G) = \mathrm{Z}(G) = \mathrm{F}(G)$ is cyclic. Moreover, we can assume that G is an extension of a solvable group by a quasisimple group. In particular G has only one nonabelian composition factor.*

Proof. By Proposition 4.8 we may assume $\mathrm{O}_2(G) = 1$. Now we consider $\mathrm{O}(G) := \mathrm{O}_{2'}(G)$. Using Clifford theory we may assume that $\mathrm{O}(G)$ is central and cyclic (see e.g. Theorem X.1.2 in [15]). Since $\mathrm{O}_2(G) = 1$, we get $\mathrm{O}(G) = \mathrm{Z}(G)$. Let $\mathrm{E}(G)$ be the normal subgroup of G generated by the components. As usual, B covers a block b of $\mathrm{E}(G)$. By Fong-Reynolds we can assume that b is stable in G. Then $d := D \cap \mathrm{E}(G)$ is a defect group of b. By the Külshammer-Puig result we may assume that b is nonnilpotent. In particular d has rank at least 2. Let $C_1, \ldots, C_n$ be the components of G. Then $\mathrm{E}(G)$ is the central product of $C_1, \ldots, C_n$. Since $[C_i, C_j] = 1$ for $i \neq j$, b covers exactly one block β_i of RC_i for $i = 1, \ldots, n$. Then b is dominated by the block $\beta_1 \otimes \ldots \otimes \beta_n$ of $R[C_1 \times \ldots \times C_n]$. Since $\mathrm{Z}(C_1)$ is abelian and subnormal in G, it must have odd order. Hence, we may identify b with $\beta_1 \otimes \ldots \otimes \beta_n$ (see Proposition 1.5 in [13]). In particular $d = \delta_1 \times \ldots \times \delta_n$, where $\delta_i := d \cap C_i$ is a defect group of β_i for $i = 1, \ldots, n$. Assume that δ_1 is cyclic. Then β_1 is nilpotent and isomorphic to $(R\delta_1)^{m \times m}$ for some $m \in \mathbb{N}$ by Puig. Let $\{C_1, \ldots, C_k\}$ be the orbit of C_1 under the conjugation action of G ($k \leq n$). Then $\beta_1 \otimes \ldots \otimes \beta_k \cong (R\delta_1)^{m_1 \times m_1}$ (for some $m_1 \in \mathbb{N}$) is a block of $R[C_1 \ldots C_k]$ with $l(\beta_1 \otimes \ldots \otimes \beta_k) = 1$. Lemma 2.1(v) implies $k \leq 2$ or $k = 3$ and $|\delta_1| = 2$. In the first case Theorem 2 in [43] shows that $\beta_1 \otimes \ldots \otimes \beta_k$ is nilpotent. This also holds in the second case by [25]. Since $C_1 \ldots C_k \trianglelefteq G$, B is an extension of a nilpotent block. This shows that we can assume that the groups δ_i are noncyclic for $i = 1, \ldots, n$. By Lemma 2.1(v), d has rank at most 3. Hence, $n = 1$ and $\mathrm{E}(G) = C_1$.

That means in order to determine the invariants of the block B we may assume that G contains only one component. Let $\mathrm{F}(G)$ (resp. $\mathrm{F}^*(G)$) be the Fitting subgroup (resp. generalized Fitting subgroup) of G. Since $\mathrm{F}(G) = \mathrm{Z}(G)$, we have $\mathrm{C}_G(\mathrm{E}(G)) = \mathrm{C}_G(\mathrm{F}^*(G)) \leq \mathrm{F}(G)$. Hence, $\mathrm{C}_G(\mathrm{E}(G))$ is nilpotent. On the other hand, the quotient $G/\mathrm{C}_G(\mathrm{E}(G))$ is isomorphic to a subgroup of the automorphism group of the quasisimple group $\mathrm{E}(G)$.

Consider the canonical map $f : \operatorname{Aut}(\mathrm{E}(G)) \to \operatorname{Aut}(\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G)))$. Let $\alpha \in \ker f$. Then $\alpha(g)g^{-1} \in \mathrm{Z}(\mathrm{E}(G))$ for all $g \in \mathrm{E}(G)$. Hence, we get a map $\beta : \mathrm{E}(G) \to \mathrm{Z}(\mathrm{E}(G))$, $g \mapsto \alpha(g)g^{-1}$. Moreover, it is easy to see that β is a homomorphism. Since $\mathrm{E}(G)$ is perfect, we get $\beta = 1$ and thus $\alpha = 1$. This shows $\operatorname{Aut}(\mathrm{E}(G)) \le \operatorname{Aut}(\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G)))$. By Schreier's conjecture (which can be proven using the classification) $\operatorname{Aut}(\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G)))$ is an extension of the solvable group $\operatorname{Out}(\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G)))$ by the simple group $\operatorname{Inn}(\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G))) \cong \mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G))$. Taking these facts together, we see that G has only one nonabelian composition factor. In particular G is an extension of a solvable group by a quasisimple group. $\square$

Now we consider blocks of maximal defect, i. e. D is a Sylow 2-subgroup of G. These include principal blocks.

Proposition 4.10. *If B has maximal defect, then G is solvable. In particular Alperin's weight conjecture is satisfied, and we have*

$$\begin{aligned} k(B) &= \frac{5 \cdot 2^{2(r-1)} + 16}{3}, \\ k_0(B) &= \frac{2^{2r} + 8}{3}, \\ k_1(B) &= \frac{2^{2(r-1)} + 8}{3}, \\ l(B) &= 3. \end{aligned}$$

Proof. By Feit-Thompson we may assume $\mathrm{O}_{2'}(G) = 1$ in order to show that G is solvable. We apply the Z^*-theorem. For this let $g \in G$ such that ${}^g z \in D$. Since all involutions of D are central (in D), we get ${}^g z \in \mathrm{Z}(D)$. By Burnside's fusion theorem there exists $h \in \mathrm{N}_G(D)$ such that ${}^h z = {}^g z$. (For principal blocks this would also follow from the fact that D controls fusion.) Since $D' = \langle z \rangle$, we have ${}^g z = z$. Now the Z^*-theorem implies $z \in \mathrm{Z}(G)$. Then $D/\langle z \rangle \cong C_{2^r}^2$ is a Sylow 2-subgroup of $G/\langle z \rangle$. By Theorem 1 in [4], $G/\langle z \rangle$ is solvable. Hence, also G is solvable. Since Alperin's weight conjecture holds for solvable groups, we obtain the numbers $k(B)$ and $l(B)$.

It is also known that the Alperin-McKay-conjecture holds for solvable groups (see [32]). Thus, in order to determine $k_0(B)$ we may assume $D \trianglelefteq G$. Then we can apply the results of [21]. For this let $L := D \rtimes C_3$. Then $B \cong (RL)^{n \times n}$ for some $n \in \mathbb{N}$. Hence, $k_0(B)$ is just the number of irreducible characters of L with odd degree. By Clifford, every irreducible character of L is an extension or an induction of a character of D. Thus, it suffices to count the characters of L which arise from linear characters of D. These linear characters of D are just the inflations of $\operatorname{Irr}(D/D')$. They spilt into the trivial character and orbits of length 3 under the action of L by Brauer's permutation lemma. The three inflations of $\operatorname{Irr}(L/D)$ are the extensions of the trivial character of D. The other linear characters of D remain irreducible after induction. Characters in the same orbit amount to the same character of L. This shows

$$k_0(B) = 3 + \frac{|D/D'| - 1}{3} = \frac{2^{2r} + 8}{3}.$$

By Theorem 1.4 in [28] we have $k_i(B) = 0$ for $i \ge 2$. We conclude

$$k_1(B) = k(B) - k_0(B) = \frac{5 \cdot 2^{2(r-1)} + 16}{3} - \frac{2^{2r} + 8}{3} = \frac{2^{2(r-1)} + 8}{3}. \qquad \square$$

The last result implies that Brauer's height zero conjecture is also satisfied for blocks of maximal defect. Moreover, the Dade-conjecture holds for solvable groups (see [40]).

Finally we consider the case $r = 2$ (i. e. $|D| = 32$) for arbitrary groups G.

Proposition 4.11. *If $r = 2$, we have*

$$k(B) = 12, \qquad k_0(B) = 8, \qquad k_1(B) = 4, \qquad l(B) = 3.$$

There are two pairs of 2-conjugate characters of height 0. The remaining characters are 2-rational. Moreover, the Cartan matrix of B is equivalent to

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 12 \end{pmatrix}.$$

Proof. The proof is somewhat lengthy and consists entirely of technical calculations. For this reason we will only outline the argumenation. Since $k_0(B)$ is divisible by 4, inequality (6) implies $k_0(B) \geq 8$. Since there are exactly two pairs of 2-conjugate B-subsections, Brauer's permutation lemma implies that we also have two pairs of 2-conjugate characters. Hence, the column a_1^y contains at most four nonvanishing entries. Since $(a_1^y, a_1^y) = 8$, there are just two nonvanishing entries, both are ± 2. Now Lemma 4.5 implies $k_0(B) = 8$. This shows $(k(B), k_1(B), l(B)) \in \{(12, 4, 3), (14, 6, 5)\}$.

By way of contradiction, we assume $k(B) = 14$. Then one can determine the numbers $d^u_{\chi\varphi}$ for $u \neq 1$ with the help of the contributions. However, there are many possibilities. The ordinary decomposition matrix Q can be computed as the orthogonal space of the other columns of the generalized decomposition matrix. Finally we obtain the Cartan matrix of B as $C = Q^{\mathrm{T}} Q$. In all cases is turns out that C has the wrong determinant (see Lemma 4.4). This shows $k(B) = 12$, $k_1(B) = 4$ and $l(B) = 3$.

Again we can determine the numbers $d^u_{\chi\varphi}$ for $u \neq 1$. This yields the heights of the 2-conjugate characters. We also obtain some informations about the Cartan invariants in this way. We regard the Cartan matrix C as a quadratic form. Using the tables [31, 30] we conclude that C has the form given in the statement of the proposition. $\square$

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References

[1] J. L. Alperin, *Weights for finite groups*, in The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 369–379, Amer. Math. Soc., Providence, RI, 1987.

[2] H. Bender, *Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt*, J. Algebra **17** (1971), 527–554.

[3] R. Brauer, *On the connection between the ordinary and the modular characters of groups of finite order*, Ann. of Math. (2) **42** (1941), 926–935.

[4] R. Brauer, *Some applications of the theory of blocks of characters of finite groups. II*, J. Algebra **1** (1964), 307–334.

[5] R. Brauer, *On blocks and sections in finite groups. I*, Amer. J. Math. **89** (1967), 1115–1136.

[6] R. Brauer, *On blocks and sections in finite groups. II*, Amer. J. Math. **90** (1968), 895–925.

[7] R. Brauer, *Some applications of the theory of blocks of characters of finite groups. IV*, J. Algebra **17** (1971), 489–521.

[8] R. Brauer, *On 2-blocks with dihedral defect groups*, in Symposia Mathematica, Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), 367–393, Academic Press, London, 1974.

[9] M. Broué, *On characters of height zero*, in The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), 393–396, Amer. Math. Soc., Providence, R.I., 1980.

[10] D. A. Buell, *Binary quadratic forms*, Springer-Verlag, New York, 1989.

[11] M. Cabanes and C. Picaronny, *Types of blocks with dihedral or quaternion defect groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **39** (1992), 141–161.

[12] E. C. Dade, *Counting characters in blocks. I*, Invent. Math. **109** (1992), 187–210.

[13] O. Düvel, *On Donovan's conjecture*, J. Algebra **272** (2004), 1–26.

[14] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1990.

[15] W. Feit, *The representation theory of finite groups*, North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam, 1982.

[16] M. Fujii, *On determinants of Cartan matrices of p-blocks*, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), 401–403.

[17] I. M. Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006.

[18] R. Kessar and M. Linckelmann, *On perfect isometries for tame blocks*, Bull. London Math. Soc. **34** (2002), 46–54.

[19] H. Kurzweil and B. Stellmacher, *The theory of finite groups*, Universitext, Springer-Verlag, New York, 2004.

[20] B. Külshammer, *On 2-blocks with wreathed defect groups*, J. Algebra **64** (1980), 529–555.

[21] B. Külshammer, *Crossed products and blocks with normal defect groups*, Comm. Algebra **13** (1985), 147–168.

[22] B. Külshammer and T. Okuyama, *On centrally controlled blocks of finite groups*, unpublished.

[23] B. Külshammer and L. Puig, *Extensions of nilpotent blocks*, Invent. Math. **102** (1990), 17–71.

[24] B. Külshammer and T. Wada, *Some inequalities between invariants of blocks*, Arch. Math. (Basel) **79** (2002), 81–86.

[25] P. Landrock, *On the number of irreducible characters in a* 2*-block*, J. Algebra **68** (1981), 426–442.

[26] M. Linckelmann, *Fusion category algebras*, J. Algebra **277** (2004), 222–235.

[27] V. D. Mazurov, *Finite groups with metacyclic Sylow 2-subgroups*, Sibirsk. Mat. Ž. **8** (1967), 966–982.

[28] A. Moretó and G. Navarro, *Heights of characters in blocks of p-solvable groups*, Bull. London Math. Soc. **37** (2005), 373–380.

[29] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press Inc., Boston, MA, 1989.

[30] G. Nebe and N. Sloane, *The Brandt-Intrau-Schiemann table of even ternary quadratic forms*, `http://www2.research.att.com/~njas/lattices/Brandt_2.html`.

[31] G. Nebe and N. Sloane, *The Brandt-Intrau-Schiemann table of odd ternary quadratic forms*, `http://www2.research.att.com/~njas/lattices/Brandt_1.html`.

[32] T. Okuyama and M. Wajima, *Irreducible characters of p-solvable groups*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), 309–312.

[33] J. B. Olsson, *On* 2*-blocks with quaternion and quasidihedral defect groups*, J. Algebra **36** (1975), 212–241.

[34] J. B. Olsson, *On the subsections for certain* 2*-blocks*, J. Algebra **46** (1977), 497–510.

[35] J. B. Olsson, *Lower defect groups*, Comm. Algebra **8** (1980), 261–288.

[36] J. B. Olsson, *On subpairs and modular representation theory*, J. Algebra **76** (1982), 261–279.

[37] S. Park, *The gluing problem for some block fusion systems*, J. Algebra **323** (2010), 1690–1697.

[38] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and cyclic inertial quotients of order* 4, J. Algebra **172** (1995), 205–213.

[39] G. R. Robinson, *On Brauer's $k(B)$ problem*, J. Algebra **147** (1992), 450–455.

[40] G. R. Robinson, *Dade's projective conjecture for p-solvable groups*, J. Algebra **229** (2000), 234–248.

[41] L. Rédei, *Das „schiefe Produkt“ in der Gruppentheorie*, Comment. Math. Helv. **20** (1947), 225–264.

[42] B. Sambale, *Cartan matrices and Brauer's $k(B)$-conjecture*, Journal of Algebra (to appear), `http://www.sciencedirect.com/science/article/B6WH2-51H1NP7-1/2/591ade320b5d95adf75b60f4fadb8ada`.

[43] B. Sambale, *Fusion systems on metacyclic* 2*-groups*, `http://arxiv.org/abs/0908.0783`.

[44] Y. Usami, *On p-blocks with abelian defect groups and inertial index* 2 *or* 3. *I*, J. Algebra **119** (1988), 123–146.

[45] Y. Usami, *On p-blocks with abelian defect groups and inertial index* 2 *or* 3. *II*, J. Algebra **122** (1989), 98–105.

[46] A. Watanabe, *Notes on p-blocks of characters of finite groups*, J. Algebra **136** (1991), 109–116.

[47] P. Webb, *An introduction to the representations and cohomology of categories*, in Group representation theory, 149–173, EPFL Press, Lausanne, 2007.