Blocks with defect group $D_{2^n} \times C_{2^m}$

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Abstract

We determine the numerical invariants of blocks with defect group $D_{2^n} \times C_{2^m}$, where D_{2^n} denotes a dihedral group of order 2^n and C_{2^m} denotes a cyclic group of order 2^m. This generalizes Brauer's results [2] for $m = 0$. As a consequence, we prove Brauer's $k(B)$-conjecture, Olsson's conjecture (and more generally Eaton's conjecture), Brauer's height zero conjecture, the Alperin-McKay conjecture, Alperin's weight conjecture and Robinson's ordinary weight conjecture for these blocks. Moreover, we show that the gluing problem has a unique solution in this case.

Keywords: 2-blocks, dihedral defect groups, Alperin's weight conjecture, ordinary weight conjecture
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1 Introduction

Let R be a discrete complete valuation ring with quotient field K of characteristic 0. Moreover, let (π) be the maximal ideal of R and $F := R/(\pi)$. We assume that F is algebraically closed of characteristic 2. We fix a finite group G, and assume that K contains all $|G|$-th roots of unity. Let B be a 2-block of RG with defect group D. We denote the number of irreducible ordinary characters of B by $k(B)$. These characters split in $k_i(B)$ characters of height $i \in \mathbb{N}_0$. Here the *height* of a character χ in B is the largest integer $h(\chi) \geq 0$ such that $2^{h(\chi)}|G : D|_2 \mid \chi(1)$, where $|G : D|_2$ denotes the highest 2-power dividing $|G : D|$. Finally, let $l(B)$ be the number of irreducible Brauer characters of B.

If D is a dihedral group, then all invariants of B are known (see [2]). Thus, it seems natural to consider the case, where D is a direct product of a dihedral group and a cyclic group. We write

$$D := \langle x, y, z \mid x^{2^{n-1}} = y^2 = z^{2^m} = [x, z] = [y, z] = 1,\ yxy^{-1} = x^{-1} \rangle = \langle x, y \rangle \times \langle z \rangle \cong D_{2^n} \times C_{2^m},$$

where $n \geq 2$ and $m \geq 0$. In the case $n = 2$ and $m = 0$ we get a four-group. Then the invariants of B have been known for a long time. If $n = 2$ and $m = 1$, D is elementary abelian of order 8, and the block invariants are also known (see [9]). Finally, in the case $n = 2 \leq m$ there exists a perfect isometry between B and its Brauer correspondent (see [18]). Thus, also in this case the block invariants are known, and the major conjectures are satisfied. Hence, we assume $n \geq 3$ for the rest of the paper. We allow $m = 0$, since the results are completely consistent in this case.

In contrast to Brauer's work we use a more modern language and give shorter proofs. In addition we apply the theory of lower defect groups and the theory of centrally controlled blocks (see [10]). The main reason that these blocks are accessible lies in the fact that certain inequalities for $k(B)$ and $k_i(B)$ are sharp.

2 Subsections

Lemma 2.1. *The automorphism group* $\mathrm{Aut}(D)$ *is a* 2*-group.*

Proof. This is known for $m = 0$. For $m \geq 1$ the subgroups $\Phi(D) < \Phi(D)\,\mathrm{Z}(D) < \langle x, z\rangle < D$ are characteristic in D. By Theorem 5.3.2 in [6] every automorphism of $\mathrm{Aut}(D)$ of odd order acts trivially on $D/\Phi(D)$. The claim follows from Theorem 5.1.4 in [6]. □

It follows that the inertial index $e(B)$ of B equals 1. Now we investigate the fusion system $\mathcal{F}$ of the B-subpairs. For this we use the notation of [16, 12], and we assume that the reader is familiar with these articles. Let b_D be a Brauer correspondent of B in $RD\,\mathrm{C}_G(D)$. Then for every subgroup $Q \leq D$ there is a unique block b_Q of $RQ\,\mathrm{C}_G(Q)$ such that $(Q, b_Q) \leq (D, b_D)$. We denote the inertial group of b_Q in $\mathrm{N}_G(Q)$ by $\mathrm{N}_G(Q, b_Q)$.

Lemma 2.2. *Let* $Q_1 := \langle x^{2^{n-2}}, y, z\rangle \cong C_2^2 \times C_{2^m}$ *and* $Q_2 := \langle x^{2^{n-2}}, xy, z\rangle \cong C_2^2 \times C_{2^m}$*. Then* Q_1 *and* Q_2 *are the only candidates for proper* $\mathcal{F}$*-centric,* $\mathcal{F}$*-radical subgroups up to conjugation. In particular the fusion of subpairs is controlled by* $\mathrm{N}_G(Q_1, b_{Q_1}) \cup \mathrm{N}_G(Q_2, b_{Q_2}) \cup D$*. Moreover, one of the following cases occurs:*

(aa) $\mathrm{N}_G(Q_1, b_{Q_1})/\,\mathrm{C}_G(Q_1) \cong S_3$ *and* $\mathrm{N}_G(Q_2, b_{Q_2})/\,\mathrm{C}_G(Q_2) \cong S_3$*.*

(ab) $\mathrm{N}_G(Q_1, b_{Q_1}) = \mathrm{N}_D(Q_1)\,\mathrm{C}_G(Q_1)$ *and* $\mathrm{N}_G(Q_2, b_{Q_2})/\,\mathrm{C}_G(Q_2) \cong S_3$*.*

(ba) $\mathrm{N}_G(Q_1, b_{Q_1})/\,\mathrm{C}_G(Q_1) \cong S_3$ *and* $\mathrm{N}_G(Q_2, b_{Q_2}) = \mathrm{N}_D(Q_2)\,\mathrm{C}_G(Q_2)$*.*

(bb) $\mathrm{N}_G(Q_1, b_{Q_1}) = \mathrm{N}_D(Q_1)\,\mathrm{C}_G(Q_1)$ *and* $\mathrm{N}_G(Q_2, b_{Q_2}) = \mathrm{N}_D(Q_2)\,\mathrm{C}_G(Q_2)$*.*

In case (bb) the block B *is nilpotent.*

Proof. Let $Q < D$ be $\mathcal{F}$-centric and $\mathcal{F}$-radical. Then $z \in \mathrm{Z}(D) \subseteq \mathrm{C}_D(Q) \subseteq Q$ and $Q = (Q \cap \langle x, y\rangle) \times \langle z\rangle$. Since $\mathrm{Aut}(Q)$ is not a 2-group, $Q \cap \langle x, y\rangle$ and thus Q must be abelian (see Lemma 2.1). Let us consider the case $Q = \langle x, z\rangle$. Then $m = n - 1$ (this is not important here). The group $D \subseteq \mathrm{N}_G(Q, b_Q)$ acts trivially on $\Omega(Q) \subseteq \mathrm{Z}(D)$, while a nontrivial automorphism of $\mathrm{Aut}(Q)$ of odd order acts nontrivially on $\Omega(Q)$ (see Theorem 5.2.4 in [6]). This contradicts $\mathrm{O}_2(\mathrm{Aut}_{\mathcal{F}}(Q)) = 1$. Hence, Q is isomorphic to $C_2^2 \times C_{2^m}$, and contains an element of the form $x^i y$. After conjugation with a suitable power of x we may assume $Q \in \{Q_1, Q_2\}$. This shows the first claim. The second claim follows from Alperin's fusion theorem.

Let $S \leq D$ be an arbitrary subgroup isomorphic to $C_2^2 \times C_{2^m}$. If $z \notin S$, the group $\langle S, z\rangle = (\langle S, z\rangle \cap \langle x, y\rangle) \times \langle z\rangle$ is abelian and of order at least 2^{m+3}. Hence, $\langle S, z\rangle \cap \langle x, y\rangle$ would be cyclic. This contradiction shows $z \in S$. Thus, S is conjugate to $Q \in \{Q_1, Q_2\}$. Since $|\mathrm{N}_D(Q)| = 2^{m+3}$, we derive that Q is fully $\mathcal{F}$-normalized (see Definition 2.2 in [12]). In particular $\mathrm{N}_D(Q)\,\mathrm{C}_G(Q)/\,\mathrm{C}_G(Q) \cong \mathrm{N}_D(Q)/Q \cong C_2$ is a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{N}_G(Q, b_Q)/\,\mathrm{C}_G(Q)$ by Proposition 2.5 in [12]. In particular $\mathrm{O}_{2'}(\mathrm{Aut}_{\mathcal{F}}(Q))$ has index 2 in $\mathrm{Aut}_{\mathcal{F}}(Q)$. Assume $\mathrm{N}_D(Q)\,\mathrm{C}_G(Q) < \mathrm{N}_G(Q, b_Q)$. Lemma 5.4 in [12] shows $\mathrm{O}_2(\mathrm{Aut}_{\mathcal{F}}(Q)) = 1$. If $m \neq 1$, we have $|\mathrm{Aut}(Q)| = 2^k \cdot 3$ for some $k \in \mathbb{N}$, since $\Phi(Q) < \Omega(Q)\Phi(Q) \leq Q$ are characteristic subgroups. Then $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{N}_G(Q, b_Q)/\,\mathrm{C}_G(Q) \cong S_3$. Hence, we may assume $m = 1$. Then $\mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(Q) \cong \mathrm{GL}(3, 2)$. Since the normalizer of a Sylow 7-subgroup of $\mathrm{GL}(3, 2)$ has order 21, it follows that $|\mathrm{O}_{2'}(\mathrm{Aut}_{\mathcal{F}}(Q))| \neq 7$. Since this normalizer is selfnormalizing in $\mathrm{GL}(3, 2)$, we also have $|\mathrm{O}_{2'}(\mathrm{Aut}_{\mathcal{F}}(Q))| \neq 21$. This shows $|\mathrm{O}_{2'}(\mathrm{Aut}_{\mathcal{F}}(Q))| = 3$ and $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{N}_G(Q, b_Q)/\,\mathrm{C}_G(Q) \cong S_3$, because $|\mathrm{GL}(3, 2)| = 2^3 \cdot 3 \cdot 7$.

The last claim follows from Alperin's fusion theorem and $e(B) = 1$. □

The naming of these cases is adopted from [2]. Since the cases (ab) and (ba) are symmetric, we ignore case (ba) for the rest of the paper. It is easy to see that Q_1 and Q_2 are not conjugate in D. Hence, by Alperin's fusion theorem the subpairs (Q_1, b_{Q_1}) and (Q_2, b_{Q_2}) are not conjugate in G. It is also easy to see that Q_1 and Q_2 are always $\mathcal{F}$-centric.

Lemma 2.3. *Let* $Q \in \{Q_1, Q_2\}$ *such that* $\mathrm{N}_G(Q, b_Q)/\,\mathrm{C}_G(Q) \cong S_3$*. Then*

$$\mathrm{C}_Q(\mathrm{N}_G(Q, b_Q)) \in \{\langle z\rangle,\ \langle x^{2^{n-2}} z\rangle\}.$$

In particular $z^{2j} \in \mathrm{C}_Q(\mathrm{N}_G(Q, b_Q))$ *and* $x^{2^{n-2}} z^{2j} \notin \mathrm{C}_Q(\mathrm{N}_G(Q, b_Q))$ *for* $j \in \mathbb{Z}$*.*

Proof. We consider only the case $Q = Q_1$ (the other case is similar). It is easy to see that the elements in $Q \setminus \mathrm{Z}(D)$ are not fixed under $\mathrm{N}_D(Q) \subseteq \mathrm{N}_D(Q, b_Q)$. Since D acts trivially on $\mathrm{Z}(D)$, it suffices to determine the fixed points of an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ of order 3 in $\mathrm{Z}(D)$. By Lemma 3.2 in [21] $\mathrm{C}_Q(\alpha) = \langle a \rangle$ has order 2^m. First we show that $a \in \mathrm{Z}(D)$. Suppose the contrary. Let $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q)$ be the automorphism induced by $x^{2^{n-3}} \in \mathrm{N}_D(Q) \subseteq \mathrm{N}_G(Q, b_Q)$. Then we have $\beta(a) \neq a$. Since $\beta\alpha\beta^{-1} = \alpha^{-1}$, we have $\alpha(\beta(a)) = \beta(\alpha^{-1}(a)) = \beta(a)$. Thus, $\beta(a) \in \mathrm{C}_Q(\alpha) = \langle a \rangle$. This gives the contradiction $\beta(a)a^{-1} \in D' \cap \langle a \rangle = \langle x^2 \rangle \cap \langle a \rangle = 1$. Now in case $m \neq 1$ the claim is clear. Thus, assume $m = 1$ and $a = x^{2^{n-2}}$. Then β acts trivially on $Q/\langle a \rangle$ and α acts nontrivially on $Q/\langle a \rangle$. This contradicts $\beta\alpha\beta^{-1}\alpha = 1$. □

It is not possible to decide whether $\mathrm{C}_Q(\mathrm{N}_G(Q, b_Q))$ is $\langle z \rangle$ or $\langle x^{2^{n-2}} z \rangle$ in Lemma 2.3, since we can replace z by $x^{2^{n-2}} z$. For a subgroup $Q \leq D$ and an element $u \in \mathrm{Z}(Q)$ we write $b_u := b_{\langle u \rangle} = b_Q^{\mathrm{C}_G(u)}$, where $b_Q^{\mathrm{C}_G(u)}$ denotes the Brauer correspondent of b_Q in $R\,\mathrm{C}_G(u)$.

Lemma 2.4.

(i) In case (aa) the subsections $(x^i z^j, b_{x^i z^j})$ ($i = 0, 1, \ldots, 2^{n-2}$, $j = 0, 1, \ldots, 2^m - 1$) form a set of representatives for the conjugacy classes of B-subsections.

(ii) In case (ab) the subsections $(x^i z^j, b_{x^i z^j})$ and (yz^j, b_{yz^j}) ($i = 0, 1, \ldots, 2^{n-2}$, $j = 0, 1, \ldots, 2^m - 1$) form a set of representatives for the conjugacy classes of B-subsections.

Proof. We investigate the set $\mathrm{A}_0(D, b_D)$ (see [16]) and apply (6C) in [3]. Since $D \in \mathrm{A}_0(D, b_D)$ and $e(B) = 1$ there are 2^{m+1} major subsections (z^j, b_{z^j}) and $(x^{2^{n-2}} z^j, b_{x^{2^{n-2}} z^j})$ ($j = 0, 1, \ldots, 2^m - 1$) which are pairwise nonconjugate. Now let $Q \in \mathrm{A}_0(D, b_D)$. As in the proof of Lemma 2.2, we have $Q = (Q \cap \langle x, y \rangle) \times \langle z \rangle$ (see Lemma (3.1) in [16]). If $Q \cap \langle x, y \rangle$ is a nonabelian dihedral group, then $\mathrm{Z}(Q) = \mathrm{Z}(D)$, and there are no subsections corresponding to (Q, b_Q). On the other hand we have $Q := \langle x, z \rangle \in \mathrm{A}_0(D, b_D)$ by Lemma 1.7 in [14]. Suppose that $\mathrm{Aut}_{\mathcal{F}}(Q)$ is not a 2-group. Then $m = n - 1$ and $D\,\mathrm{C}_G(Q)/\,\mathrm{C}_G(Q)$ is a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Since $\mathrm{Aut}(D)$ is a 2-group, Lemma 5.4 in [12] shows $\mathrm{O}_2(\mathrm{Aut}_{\mathcal{F}}(Q)) = 1$. However, this contradicts Lemma 2.2, since Q is $\mathcal{F}$-centric. This shows $\mathrm{N}_G(Q, b_Q) = D\,\mathrm{C}_G(Q)$. For a subsection (u, b) with $u \in Q$ we must check whether $|\mathrm{N}_G(Q, b_Q) \cap \mathrm{C}_G(u) : Q\,\mathrm{C}_G(Q)|$ is odd. It is easy to see that this holds if and only if $u \notin \mathrm{Z}(D)$. The action of D on $Q \setminus \mathrm{Z}(D)$ gives the following subsections: $(x^i z^j, b_{x^i z^j})$ ($i = 1, \ldots, 2^{n-2} - 1$, $j = 0, 1, \ldots, 2^m - 1$).

Now suppose $Q = Q_2$ and $u \in Q \setminus \mathrm{Z}(D)$. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ be an automorphism of order 3. As in the proof of Lemma 2.3 we have $\mathrm{C}_Q(\alpha) \subseteq \mathrm{Z}(D)$. Thus, $u\alpha(u)\alpha^{-1}(u) \in \mathrm{C}_Q(\alpha) \subseteq \mathrm{Z}(D)$. It follows that $\alpha(u) \in \mathrm{Z}(D)$ or $\alpha^{-1}(u) \in \mathrm{Z}(D)$, since $\mathrm{Z}(D)$ has index 2 in Q. Let $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q)$ be the automorphism induced by $x^{2^{n-3}} \in \mathrm{N}_D(Q) \subseteq \mathrm{N}_G(Q, b_Q)$. Then one of the 2-elements $\alpha\beta\alpha^{-1}$ or $\alpha^{-1}\beta\alpha$ fixes u. This shows $2 \mid |\mathrm{N}_G(Q, b_Q) \cap \mathrm{C}_G(u) : \mathrm{C}_G(Q)|$ for every $u \in Q$. Hence, there are no subsections corresponding to (Q_2, b_{Q_2}). In case (aa) the same holds for (Q_1, b_{Q_1}). This proves part (i). Let us consider $Q = Q_1$ in case (ab). By way of contradiction, suppose $Q \notin \mathrm{A}_0(D, b_D)$. Then we get the same set of representatives for the conjugacy classes of subsections as in case (aa). In particular the subpair $(\langle y \rangle, b_y)$ is conjugate to a subpair $(\langle u \rangle, b_u)$ with $u \in \mathrm{Z}(D)$. However, this contradicts Alperin's fusion theorem. Hence, $Q \in \mathrm{A}_0(D, b_D)$. Then we have $|\mathrm{N}_G(Q, b_Q) \cap \mathrm{C}_G(u) : Q\,\mathrm{C}_G(Q)| = |\mathrm{N}_D(Q)\,\mathrm{C}_G(Q) \cap \mathrm{C}_G(u) : \mathrm{C}_G(Q)| = |\mathrm{C}_G(Q)(\mathrm{N}_D(Q) \cap \mathrm{C}_G(u)) : \mathrm{C}_G(Q)| = |\mathrm{N}_D(Q) \cap \mathrm{C}_G(u) : Q|$ for $u \in Q$. Thus, we have to take the subsections (u, b) with $u \in Q \setminus \mathrm{Z}(D)$ up to $\mathrm{N}_D(Q)$-conjugation. This shows part (ii). □

3 The numbers $k(B)$, $k_i(B)$ and $l(B)$

Now we study the generalized decomposition numbers of B. If $l(b_u) = 1$, then we denote the unique irreducible modular character of b_u by φ_u. In this case the generalized decomposition numbers $d^u_{\chi\varphi_u}$ for $\chi \in \mathrm{Irr}(B)$ form a column $d(u)$. Let 2^k be the order of u, and let $\zeta := \zeta_{2^k}$ be a primitive 2^k-th root of unity. Then the entries of $d(u)$ lie in the ring of integers $\mathbb{Z}[\zeta]$. Hence, there exist integers $a_i^u := (a_i^u(\chi))_{\chi \in \mathrm{Irr}(B)} \in \mathbb{Z}^{k(B)}$ such that

$$d^u_{\chi\varphi_u} = \sum_{i=0}^{2^{k-1}-1} a_i^u(\chi)\zeta^i.$$

We extend this by

$$a^u_{i+2^{k-1}} := -a^u_i$$

for all $i \in \mathbb{Z}$.

Let $|G| = 2^a r$ where $2 \nmid r$. We may assume $\mathbb{Q}(\zeta_{|G|}) \subseteq K$. Then $\mathbb{Q}(\zeta_{|G|}) \mid \mathbb{Q}(\zeta_r)$ is a Galois extension, and we denote the corresponding Galois group by

$$\mathcal{G} := \operatorname{Gal}\big(\mathbb{Q}(\zeta_{|G|}) \mid \mathbb{Q}(\zeta_r)\big).$$

Restriction gives an isomorphism

$$\mathcal{G} \cong \operatorname{Gal}\big(\mathbb{Q}(\zeta_{2^a}) \mid \mathbb{Q}\big).$$

In particular $|\mathcal{G}| = 2^{a-1}$. For every $\gamma \in \mathcal{G}$ there is a number $\widetilde{\gamma} \in \mathbb{N}$ such that $\gcd(\widetilde{\gamma}, |G|) = 1$, $\widetilde{\gamma} \equiv 1 \pmod r$, and $\gamma(\zeta_{|G|}) = \zeta_{|G|}^{\widetilde{\gamma}}$ hold. Then $\mathcal{G}$ acts on the set of subsections by

$$^{\gamma}(u, b) := (u^{\widetilde{\gamma}}, b).$$

For every $\gamma \in \mathcal{G}$ we get

$$d(u^{\widetilde{\gamma}}) = \sum_{s \in \mathcal{S}} a^u_s \zeta_{2^k}^{s\widetilde{\gamma}} \tag{1}$$

for every system $\mathcal{S}$ of representatives of the cosets of $2^{k-1}\mathbb{Z}$ in $\mathbb{Z}$. It follows that

$$a^u_s = 2^{1-a} \sum_{\gamma \in \mathcal{G}} d\big(u^{\widetilde{\gamma}}\big)\zeta_{2^k}^{-\widetilde{\gamma}s} \tag{2}$$

for $s \in \mathcal{S}$.

Next, we introduce a general result which does not depend on D.

Lemma 3.1. *Let (u, b_u) be a B-subsection with $|\langle u\rangle| = 2^k$ and $l(b_u) = 1$.*

(i) If $\chi \in \operatorname{Irr}(B)$ has height 0, then the sum

$$\sum_{i=0}^{2^{k-1}-1} a^u_i(\chi) \tag{3}$$

is odd.

(ii) If (u, b_u) is major and $k \le 1$, then $2^{h(\chi)} \mid d^u_{\chi\varphi_u} = a^u_0(\chi)$ and $2^{h(\chi)+1} \nmid d^u_{\chi\varphi_u}$ for all $\chi \in \operatorname{Irr}(B)$.

Proof. Let $Q \le D$ be a defect group of b_u. Since $l(b_u) = 1$, we have $|Q|m^{(u,b_u)}_{\chi\chi} = d^u_{\chi\varphi_u}\overline{d^u_{\chi\varphi_u}}$ for the contribution $m^{(u,b_u)}_{\chi\chi}$ (see Eq. (5.2) in [1]). Assume that χ has height 0. By Corollary 2 in [4] it follows that

$$|Q|m^{(u,b_u)}_{\chi\chi} = |Q|\big(\chi^{(u,b_u)}, \chi\big)_G \not\equiv 0 \pmod{(\pi)}$$

and $d^u_{\chi\varphi_u} \not\equiv 0 \pmod{(\pi)}$. Since $\zeta_{2^k} \equiv 1 \pmod{(\pi)}$, the sum (3) is odd.

Now assume that (u, b_u) is major and $k \le 1$. Then $d^u_{\chi\varphi_u} = a^u_0(\chi) \in \mathbb{Z}$ for all $\chi \in \operatorname{Irr}(B)$. If $\psi \in \operatorname{Irr}(B)$ has height 0 (ψ always exists), part (i) shows that $d^u_{\psi\varphi_u}$ is odd. By (5H) in [1] we have $2^{h(\chi)} \mid |D|m^{(u,b_u)}_{\chi\psi} = d^u_{\chi\varphi_u}d^u_{\psi\varphi_u}$ and $2^{h(\chi)+1} \nmid |D|m^{(u,b_u)}_{\chi\psi}$. This proves part (ii). □

Lemma 3.2. *Olsson's conjecture $k_0(B) \le 2^{m+2} = |D : D'|$ is satisfied in all cases.*

Proof. Let $\gamma \in \mathcal{G}$ such that the restriction of γ to $\mathbb{Q}(\zeta_{2^a})$ is the complex conjugation. Then $x^{\widetilde{\gamma}} = x^{-1}$. The block b_x has defect group $\langle x, z\rangle$ (see the proof of (6F) in [3]). Since we have shown that $\operatorname{Aut}_{\mathcal{F}}(\langle x, z\rangle)$ is a 2-group, b_x is nilpotent. In particular $l(b_x) = 1$. Since the subsections (x, b_x) and $(x^{-1}, b_{x^{-1}}) = (x^{-1}, b_x) = {}^{\gamma}(x, b_x)$ are conjugate by y, we have $d(x) = d(x^{\widetilde{\gamma}})$ and

$$a^x_j(\chi) = a^x_{-j}(\chi) = -a^x_{2^{n-2}-j}(\chi) \tag{4}$$

for all $\chi \in \operatorname{Irr}(B)$ by Eq. (1). In particular $a^x_{2^{n-3}}(\chi) = 0$ (cf. (4.16) in [2]). By the orthogonality relations we have $(d(x), d(x)) = |\langle x, z\rangle| = 2^{n-1+m}$. On the other hand the subsections (x, b_x) and $(x^i, b_{x^i}) = (x^i, b_x)$ are not conjugate for odd $i \in \{3, 5, \ldots, 2^{n-2} - 1\}$. Eq. (2) implies

$$(a^x_0, a^x_0) = 2^{2(1-a)} \sum_{\gamma,\delta\in\mathcal{G}} \big(d(x^{\widetilde{\gamma}}), d(x^{\widetilde{\delta}})\big) = 2^{2(1-a)}2^{2a-n+1}(d(x), d(x)) = 2^{m+2}$$

(cf. Proposition (4C) in [2]). Combining Eq. (4) with Lemma 3.1(i) we see that $a^x_0(\chi) \neq 0$ is odd for characters $\chi \in \operatorname{Irr}(B)$ of height 0. This proves the lemma. □

We remark that Olsson's conjecture in case (bb) also follows from Lemma 2.2. Moreover, in case (ab) Olsson's conjecture follows easily from Theorem 3.1 in [19].

Theorem 3.3. *In all cases we have*

$$k(B) = 2^m(2^{n-2} + 3), \qquad k_0(B) = 2^{m+2}, \qquad k_1(B) = 2^m(2^{n-2} - 1).$$

Moreover,

$$l(B) = \begin{cases} 1 & \textit{in case (bb)} \\ 2 & \textit{in case (ab)} \\ 3 & \textit{in case (aa)} \end{cases}.$$

In particular Brauer's $k(B)$-conjecture, Brauer's height zero conjecture and the Alperin-McKay conjecture hold.

Proof. Assume first that case (bb) occurs. Then B is nilpotent and $k_i(B)$ is just the number $k_i(D)$ of irreducible characters of D of degree 2^i ($i \geq 0$) and $l(B) = 1$. Since C_{2^m} is abelian, we get $k_i(B) = 2^m k_i(D_{2^n})$. The claim follows in this case. Thus, we assume that case (aa) or case (ab) occurs. We determine the numbers $l(b)$ for the subsections in Lemma 2.4 and apply (6D) in [3]. Let us begin with the nonmajor subsections. Since $\operatorname{Aut}_{\mathcal{F}}(\langle x, z\rangle)$ is a 2-group, the block $b_{\langle x,z\rangle}$ with defect group $\langle x, z\rangle$ is nilpotent. Hence, we have $l(b_{x^i z^j}) = 1$ for all $i = 1, \ldots, 2^{n-2} - 1$ and $j = 0, 1, \ldots, 2^m - 1$. The blocks b_{yz^j} ($j = 0, 1, \ldots, 2^m - 1$) have Q_1 as defect group. Since $\operatorname{N}_G(Q_1, b_{Q_1}) = \operatorname{N}_D(Q_1) \operatorname{C}_G(Q_1)$, they are also nilpotent, and it follows that $l(b_{yz^j}) = 1$.

We divide the (nontrivial) major subsections into three sets:

$$\begin{aligned} U &:= \{x^{2^{n-2}} z^{2j} : j = 0, 1, \ldots, 2^{m-1} - 1\}, \\ V &:= \{z^j : j = 1, \ldots, 2^m - 1\}, \\ W &:= \{x^{2^{n-2}} z^{2j+1} : j = 0, 1, \ldots, 2^{m-1} - 1\}. \end{aligned}$$

By Lemma 2.3 case (bb) occurs for b_u, and we get $l(b_u) = 1$ for $u \in U$. The blocks b_v with $v \in V$ dominate unique blocks $\overline{b_v}$ of $R\operatorname{C}_G(v)/\langle v\rangle$ with defect group $D/\langle v\rangle \cong D_{2^n} \times C_{2^m/|\langle v\rangle|}$ such that $l(b_v) = l(\overline{b_v})$ (see Theorem 5.8.11 in [13] for example). The same argument for $w \in W$ gives blocks $\overline{b_w}$ with defect group $D/\langle w\rangle \cong D_{2^n}$. This allows us to apply induction on m (for the blocks b_v and b_w). The beginning of this induction ($m = 0$) is satisfied by Brauer's result (see [2]). Thus, we may assume $m \geq 1$. By Theorem 1.5 in [14] the cases for b_v (resp. b_w) and $\overline{b_v}$ (resp. $\overline{b_w}$) coincide.

Suppose that case (ab) occurs. By Lemma 2.3 case (ab) occurs for exactly $2^m - 1$ blocks in $\{b_v : v \in V\} \cup \{b_w : w \in W\}$ and case (bb) occurs for the other 2^{m-1} blocks. Induction gives

$$\sum_{v\in V} l(b_v) + \sum_{w\in W} l(b_w) = \sum_{v\in V} l(\overline{b_v}) + \sum_{w\in W} l(\overline{b_w}) = 2(2^m - 1) + 2^{m-1}.$$

Taking all subsections together, we derive

$$k(B) - l(B) = 2^m(2^{n-2} + 3) - 2.$$

In particular $k(B) \geq 2^m(2^{n-2} + 3) - 1$. Let $u := x^{2^{n-2}} \in \operatorname{Z}(D)$. Lemma 3.1(ii) implies $2^{h(\chi)} \mid d^u_{\chi\varphi_u}$ and $2^{h(\chi)+1} \nmid d^u_{\chi\varphi_u}$ for $\chi \in \operatorname{Irr}(B)$. In particular $d^u_{\chi\varphi_u} \neq 0$. Lemma 3.2 gives

$$2^{n+m} - 4 \leq k_0(B) + 4(k(B) - k_0(B)) \leq \sum_{\chi\in\operatorname{Irr}(B)} \big(d^u_{\chi\varphi_u}\big)^2 = (d(u), d(u)) = |D| = 2^{n+m}. \tag{5}$$

Hence, we have

$$d^u_{\chi\varphi_u} = \begin{cases} \pm 1 & \text{if } h(\chi) = 0 \\ \pm 2 & \text{otherwise} \end{cases},$$

and the claim follows in case (ab).

Now suppose that case (aa) occurs. Then by the same argument as in case (ab) we have

$$\sum_{v\in V} l(b_v) + \sum_{w\in W} l(b_w) = \sum_{v\in V} l(\overline{b_v}) + \sum_{w\in W} l(\overline{b_w}) = 3(2^m - 1) + 2^{m-1}.$$

Observe that this sum does not depend on which case actually occurs for b_z (for example). In fact all three cases for b_z are possible. Taking all subsections together, we derive

$$k(B) - l(B) = 2^m(2^{n-2} + 3) - 3.$$

Here it is not clear a priori whether $l(B) > 1$. Brauer delayed the discussion of the possibility $l(B) = 1$ until section 7 of [2]. Here we argue differently via lower defect groups and centrally controlled blocks. First we consider the case $m \ge 2$. By Lemma 2.3 we have $\langle D, \mathrm{N}_G(Q_1, b_{Q_1}), \mathrm{N}_G(Q_2, b_{Q_2})\rangle \subseteq \mathrm{C}_G(z^2)$, i. e. B is centrally controlled (see [10]). By Theorem 1.1 in [10] we get $l(B) \ge l(b_{z^2}) = 3$. Hence, the claim follows with Ineq. (5).

Now consider the case $m = 1$. By Lemma 2.3 there is a (unique) nontrivial fixed point $u \in \mathrm{Z}(D)$ of $\mathrm{N}_G(Q_1, b_{Q_1})$. Then $l(b_u) > 1$. By Proposition (4G) in [2] the Cartan matrix of b_u has 2 as an elementary divisor. With the notation of [15] we have $m^{(1)}_{b_u}(Q) \ge 1$ for some $Q \le \mathrm{C}_G(u) = \mathrm{N}_G(\langle u\rangle)$ with $|Q| = 2$ (see the remark on page 285 in [15]). In particular Q is a lower defect group of b_u (see Theorem (5.4) in [15]). Since $\langle u\rangle \le \mathrm{Z}(\mathrm{C}_G(u))$, Corollary (3.7) in [15] implies $Q = \langle u\rangle$. By Theorem (7.2) in [15] we have $m^{(1)}_B(\langle u\rangle) \ge 1$. In particular 2 occurs as elementary divisor of the Cartan matrix of B. This shows $l(B) \ge 2$. Now the claim follows again with Ineq. (5). □

We add some remarks. For trivial reasons also Eaton's conjecture is satisfied which provides a generalization of Brauer's $k(B)$-conjecture and Olsson's conjecture (see [5]). Brauer's $k(B)$-conjecture already follows from Theorem 2 in [22]. The principal blocks of D, $S_4 \times C_{2^m}$ and $\mathrm{GL}(3, 2) \times C_{2^m}$ give examples for the cases (bb), (ab) and (aa) respectively (at least for $n = 3$). Moreover, the principal block of S_6 shows that also $\mathrm{C}_{Q_1}(\mathrm{N}_G(Q_1, b_{Q_1})) \ne \mathrm{C}_{Q_2}(\mathrm{N}_G(Q_2, b_{Q_2}))$ is possible in case (aa). This gives an example, where B is not centrally controlled (and $m = 1$). However, B cannot be a block of maximal defect of a simple group for $m \ge 1$ by the main theorem in [7].

4 Alperin's weight conjecture

Alperin's weight conjecture asserts that $l(B)$ is the number of conjugacy classes of weights for B. Here a weight is a pair (Q, β), where Q is a 2-subgroup of G and β is a block of $R[\mathrm{N}_G(Q)/Q]$ with defect 0. Moreover, β is dominated by a Brauer correspondent b of B in $R\,\mathrm{N}_G(Q)$.

Theorem 4.1. *Alperin's weight conjecture holds for B.*

Proof. We use Proposition 5.4 in [8]. For this, let $Q \le D$ be $\mathcal{F}$-centric and $\mathcal{F}$-radical. By Lemma 2.2 we have $\mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$ or $\mathrm{Out}_{\mathcal{F}}(Q) = 1$ (if $Q = D$). In particular $\mathrm{Out}_{\mathcal{F}}(Q)$ has trivial Schur multiplier. Moreover, $F\,\mathrm{Out}_{\mathcal{F}}(Q)$ has precisely one block of defect 0. Now the claim follows from Theorem 3.3 and Proposition 5.4 in [8]. □

5 Ordinary weight conjecture

In this section we prove Robinson's ordinary weight conjecture (OWC) for B (see [20]). If OWC holds for all groups and all blocks, then also Alperin's weight conjecture holds. However, for our particular block B this implication is not known. In the same sense OWC is equivalent to Dade's projective conjecture (see [5]). Uno has proved Dade's invariant conjecture in the case $m = 0$ (see [23]). For $\chi \in \mathrm{Irr}(B)$ let $d(\chi) := n + m - h(\chi)$ be the *defect* of χ. We set $k^i(B) = |\{\chi \in \mathrm{Irr}(B) : d(\chi) = i\}|$ for $i \in \mathbb{N}$.

Theorem 5.1. *The ordinary weight conjecture holds for B.*

Proof. We prove the version in Conjecture 6.5 in [8]. For this, let $Q \le D$ be $\mathcal{F}$-centric and $\mathcal{F}$-radical. In the case $Q = D$ we have $\mathrm{Out}_{\mathcal{F}}(D) = 1$ and $\mathcal{N}_D$ consists only of the trivial chain (with the notations of [8]). Then it follows easily that $\mathbf{w}(D, d) = k^d(D) = k^d(B)$ for all $d \in \mathbb{N}$. Now let $Q \in \{Q_1, Q_2\}$ such that $\mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q) \cong S_3$. It suffices to show that $\mathbf{w}(Q, d) = 0$ for all $d \in \mathbb{N}$. Since Q is abelian, we have $\mathbf{w}(Q, d) = 0$ unless $d = m + 2$. Thus, let $d = m + 2$. Up to conjugation $\mathcal{N}_Q$ consists of the trivial chain $\sigma : 1$ and that chain $\tau : 1 < C$, where $C \le \mathrm{Out}_{\mathcal{F}}(Q)$ has order 2.

We consider the chain σ first. Here $I(\sigma) = \mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$ acts faithfully on $\Omega(Q) \cong C_2^3$ and thus fixes a four-group. Hence, the characters in $\mathrm{Irr}(Q)$ split in 2^m orbits of length 3 and 2^m orbits of length 1 under $I(\sigma)$ (see also Lemma 2.3). For a character $\chi \in \mathrm{Irr}(D)$ lying in an orbit of length 3 we have $I(\sigma, \chi) \cong C_2$ and thus $w(Q, \sigma, \chi) = 0$. For the 2^m stable characters $\chi \in \mathrm{Irr}(D)$ we get $w(Q, \sigma, \chi) = 1$, since $I(\sigma, \chi) = \mathrm{Out}_{\mathcal{F}}(Q)$ has precisely one block of defect 0.

Now consider the chain τ. Here $I(\tau) = C$ and the characters in $\mathrm{Irr}(Q)$ split in 2^m orbits of length 2 and 2^{m+1} orbits of length 1 under $I(\tau)$. For a character $\chi \in \mathrm{Irr}(D)$ in an orbit of length 2 we have $I(\tau, \chi) = 1$ and thus $w(Q, \tau, \chi) = 1$. For the 2^{m+1} stable characters $\chi \in \mathrm{Irr}(D)$ we get $I(\tau, \chi) = I(\tau) = C$ and $w(Q, \tau, \chi) = 0$.

Taking both chains together, we derive

$$\mathbf{w}(Q, d) = (-1)^{|\sigma|+1} 2^m + (-1)^{|\tau|+1} 2^m = 2^m - 2^m = 0.$$

This proves OWC. □

5.1 The gluing problem

Finally we show that the gluing problem (see Conjecture 4.2 in [11]) for the block B has a unique solution. This was done for $m = 0$ in [17]. We will not recall the very technical statement of the gluing problem. Instead we refer to [17] for most of the notations. Observe that the field F is denoted by k in [17].

Theorem 5.2. *The gluing problem for B has a unique solution.*

Proof. We will show that $\mathrm{H}^i(\mathrm{Aut}_{\mathcal{F}}(\sigma), F^\times) = 0$ for $i = 1, 2$ and every chain σ of $\mathcal{F}$-centric subgroups of D. Then it follows that $\mathcal{A}^i_{\mathcal{F}} = 0$ and $\mathrm{H}^0([S(\mathcal{F}^c)], \mathcal{A}^2_{\mathcal{F}}) = \mathrm{H}^1([S(\mathcal{F}^c)], \mathcal{A}^1_{\mathcal{F}}) = 0$. Hence, by Theorem 1.1 in [17] the gluing problem has only the trivial solution.

Let $Q \le D$ be the largest ($\mathcal{F}$-centric) subgroup occurring in σ. Then as in the proof of Lemma 2.2 we have $Q = (Q \cap \langle x, y\rangle) \times \langle z\rangle$. If $Q \cap \langle x, y\rangle$ is nonabelian, $\mathrm{Aut}(Q)$ is a 2-group by Lemma 2.1. In this case we get $\mathrm{H}^i(\mathrm{Aut}_{\mathcal{F}}(\sigma), F^\times) = 0$ for $i = 1, 2$ (see proof of Corollary 2.2 in [17]). Hence, we may assume that $Q \in \{Q_1, Q_2\}$ and $\mathrm{Aut}_{\mathcal{F}}(Q) \cong S_3$ (see proof of Lemma 2.4 for the case $Q = \langle x, z\rangle$). Then σ only consists of Q and $\mathrm{Aut}_{\mathcal{F}}(\sigma) = \mathrm{Aut}_{\mathcal{F}}(Q)$. Hence, also in this case we get $\mathrm{H}^i(\mathrm{Aut}_{\mathcal{F}}(\sigma), F^\times) = 0$ for $i = 1, 2$. □

It seems likely that one can prove similar results about blocks with defect group $Q_{2^n} \times C_{2^m}$ or $SD_{2^n} \times C_{2^m}$, where Q_{2^n} denotes the quaternion group and SD_{2^n} denotes the semidihedral group of order 2^n. This would generalize Olsson's results for $m = 0$ (see [14]).

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