CROSSED PRODUCTS AND BLOCKS WITH NORMAL DEFECT GROUPS

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Blocks having a normal defect group D are fairly well understood. R. Brauer [3] reduced the analysis of their block idempotents to blocks of defect zero in the quotient of the centralizer of D by the center of D. A reduction for the ordinary and modular characters in such blocks to the twisted group algebra of a naturally defined group having D as normal Sylow p-subgroup was given by Reynolds [10]. Later Dade [4] extended Brauer's work to study the behavior of arbitrary block idempotents with respect to normal subgroups. Here we use Dade's methods to generalize Reynolds' theorem. Since our reduction produces a crossed product it seems natural to start with crossed products, too. So we work in this slightly more general context. As a corollary we get the following version of Reynolds' result:

A. THEOREM. *Let G be a finite group and R a complete discrete valuation ring with residue class field F of prime characteristic p. Let $B \longleftrightarrow E$ be a block of the group algebra RG having a normal defect group D, and choose a block $b \longleftrightarrow e$ of $R\mathrm{C}_G(D)$ with $Ee \neq 0$. Denote by $G(b)$ the set of all elements $g \in G$ such that $e^g = e$, and assume that F is a splitting field for $\mathrm{Z}b$.*

Then $D\mathrm{C}_G(D)/\mathrm{C}_G(D)$ is a normal Sylow p-subgroup of $G(b)/\mathrm{C}_G(D)$, and B is isomorphic as an R-algebra to $S \otimes_R R_\gamma DH$ for some central separable R-algebra S and some twisted group algebra $R_\gamma DH$ of DH over R where H denotes a complement of $D\mathrm{C}_G(D)/\mathrm{C}_G(D)$ in $G(b)/D\mathrm{C}_G(D)$ and DH the semidirect product of D with H.

Let us fix some notation. Throughout the following, R denotes a complete discrete valuation ring with residue class field F. (We also allow the case that R is a field.) All R-algebras are assumed to have an identity and to be free R-modules of finite rank. For subsets X, Y of an R-algebra A we denote by

$$\mathrm{C}_X(Y) := \{x \in X \mid xy = yx \text{ for } y \in Y\}$$

the centralizer of Y in X. In particular, $\mathrm{C}_A(A) = \mathrm{Z}A$ is the center of A. We denote by $\mathrm{J}A$ the Jacobson radical and by $\mathrm{U}A$ the group of units of A. The group $\operatorname{Aut} A$ of all R-algebra automorphisms of A has as normal subgroup the group $\operatorname{Inn} A$ consisting of all inner automorphisms $A \longrightarrow A,\ a \mapsto u^{-1}au = a^u$, for $u \in \mathrm{U}A$. For a natural number n, $\mathrm{Mat}(n, A)$ will be the R-algebra of all $n \times n$-matrices with coefficients in A. We denote blocks of A by $B \longleftrightarrow E$ where E is the block idempotent and $B = EA$ the block ideal. If G is a group acting on the center of an R-algebra A we denote by $G(B) := \{g \in G \mid E^g = E\}$ the stabilizer of a block $B \longleftrightarrow E$ of A.

1. CROSSED PRODUCTS

We us the approach given by Dade in his elegant paper [5]. Proofs of the elementary facts on crossed products are left to the reader. Some of them may be found in [4,5].

Let G be a finite group, and let A be an R-algebra with fixed decomposition $A = \bigoplus_{g\in G} A_g$ where A_g is an R-submodule of A containing a unit of A such that $A_gA_h \subseteq A_{gh}$ for all $g, h \in G$. Then A is called a **crossed product of G over** A_1, and for $g \in G$ the R-submodule A_g of A is called the g-**component** of A.

It is easy to see that A_1 is an R-subalgebra of A containing the identity of A. For any unit u_g of A in A_g we have $A_g = A_1u_g = u_gA_1$. Therefore $A_gA_h = A_{gh}$ for $g, h \in G$. The structure of A is usually described by an automorphism system and a factor system.

A map $\alpha : G \longrightarrow \mathrm{Aut}\, A_1$ is called an **automorphism system for** A if, for any $g \in G$, there is a unit u_g of A in A_g such that $x^{\alpha(g)} = u_g^{-1}xu_g$ for $x \in A_1$. If α is an automorphism system for A then all automorphism systems are given by the maps $\iota\alpha : G \longrightarrow \mathrm{Aut}\, A_1,\ g \mapsto \iota(g)\alpha(g)$, where $\iota : G \longrightarrow \mathrm{Inn}\, A_1$ is an arbitrary map. Therefore the induced map $G \longrightarrow \mathrm{Aut}\, A_1/\mathrm{Inn}\, A_1,\ g \mapsto \alpha(g)\mathrm{Inn}\, A_1$ is independent of the choice of the automorphism system α and a homomorphism of groups. The usual problem is that α itself need not be a homomorphism. It is possible, however, to define an action of G on $\mathrm{C}_A(A_1)$ by setting $c^g := u_g^{-1}cu_g$ for elements $g \in G$, $c \in \mathrm{C}_A(A_1)$ and any unit u_g of A in A_g. Since for different units u_g, v_g of A in A_g the element $v_g^{-1}u_g$ lies in $\mathrm{U}A_1$ this action is in fact well-defined. By restriction we get an action of G on $\mathrm{Z}A_1$. It can be shown that the map $\alpha : G \longrightarrow \mathrm{Aut}\, A_1$ determines the isomorphism type of A (as a crossed product of G over A_1) up to an element in the second cohomology group $\mathrm{H}^2(G, \mathrm{UZ}A_1)$ but we shall not go into this here in detail.

For any subgroup H of G we set $A_H := \sum_{h\in H} A_h$. Then A_H is a crossed product of H over A_1. Furthermore, for any normal subgroup N of G, A is a crossed product of G/N over A_N with gN-component $\sum_{n\in N} A_{gn}$ for $g \in G$. Moreover, if I is an ideal of A containing $(\mathrm{J}R)A$ such that $I = \bigoplus_{g\in G} I \cap A_g$ then A/I is a crossed product of G over the F-algebra $A_1 + I/I \cong A_1/A_1 \cap I$ with g-component $(A/I)_g = A_g + I/I$ for $g \in G$.

Now let H and K be subgroups of G with $K \leq H$, and choose a transversal T for K in H. Then, for any element $c \in \mathrm{C}_A(A_K)$, the element $\sum_{t\in T} c^t$ is easily seen to be in $\mathrm{C}_A(A_H)$ and independent of the choice of T. We denote this element by $\tau_K^H(c)$. Then we get a relative trace map $\tau_K^H : \mathrm{C}_A(A_K) \longrightarrow \mathrm{C}_A(A_H)$ which commutes with multiplication by elements in $\mathrm{C}_A(A_H)$. It is obvious that τ_K^H will have the usual properties of a relative trace map (cf. [7]). The following elementary result is one of the main tools in our analysis.

B. LEMMA. *Let A be a crossed product of G over A_1 with automorphism system α. Let S be an R-subalgebra of A_1 such that for any element $g \in G$ there is a unit $u(g)$ in A_1 satisfying $s^{\alpha(g)} = u(g)su(g)^{-1}$ for any $s \in S$. Then the following holds:*

(i) $\mathrm{C}_A(S)$ *is a crossed product of G over* $\mathrm{C}_{A_1}(S)$.

(ii) *The map $\alpha' : G \longrightarrow \mathrm{Aut}\, A_1$ with $x^{\alpha'(g)} = x^{\alpha(g)u(g)}$ for $x \in A_1$ is an automorphism system for A such that its restriction $G \longrightarrow \mathrm{Aut}\, \mathrm{C}_{A_1}(S)$ is an automorphism system for* $\mathrm{C}_A(S)$.

(iii) $A = A_1\mathrm{C}_A(S) = \mathrm{C}_A(S)A_1$.

Proof. We choose units u_g of A in A_g such that $x^{\alpha(g)} = u_g^{-1} x u_g$ for $x \in A_1$ and $g \in G$. Then the elements $u_g u(g)$ are units of A in A_g centralizing S. Since S is contained in A_1 we get

$$\mathrm{C}_A(S) = \bigoplus_{g\in G} \mathrm{C}_A(S) \cap A_g = \bigoplus_{g\in G} \mathrm{C}_{A_1}(S) u_g u(g),$$

and

$$A = \bigoplus_{g\in G} A_1 u_g u(g) = \bigoplus_{g\in G} u_g u(g) A_1.$$

Therefore $A = A_1 \mathrm{C}_A(S) = \mathrm{C}_A(S) A_1$, and the result follows.

It is easily seen that the hypothesis of lemma B does depend on the choice of the automorphism system. A first application is the following:

C. THEOREM. *Let A be a crossed product of G over A_1 with automorphism system α, and set $N := \{g \in G \mid \alpha(g) \in \mathrm{Inn}\, A_1\}$. Then the following holds:*

(i) *N is a normal subgroup of G.*

(ii) *The For any transversal T for N in G the map $\alpha' : G/N \longrightarrow \mathrm{Aut}\, A_N$ satisfying $(xc)^{\alpha'(gN)} = x^{\alpha(g)} c^g$ for elements $x \in A_1$, $c \in \mathrm{C}_{A_N}(A_1)$, $g \in T$ is an automorphism system for the crossed product A of G/N over A_N.*

Proof. N is the kernel of the homomorphism $G \longrightarrow \mathrm{Aut}\, A_1/\mathrm{Inn}\, A_1$, $g \mapsto \alpha(g)\mathrm{Inn}\, A_1$, so it is a normal subgroup of G. Then A may be considered as a crossed product of G/N over A_N, and A_N is a crossed product of N over A_1. We may apply lemma B with N in place of G and A_1 in place of S. This gives $A_N = A_1 \mathrm{C}_{A_N}(A_1) = \mathrm{C}_{A_N}(A_1) A_1$, and $\mathrm{C}_{A_N}(A_1)$ is a crossed product of N over $\mathrm{Z}A_1$. To prove (ii) it is enough to show that $A_1 \otimes_{\mathrm{Z}A_1} \mathrm{C}_{A_N}(A_1)$ and A_N are free R-modules of equal rank. But $\mathrm{C}_{A_N}(A_1)$ is free of rank $|N|$ as a module over $\mathrm{Z}A_1$, so $A_1 \otimes_{\mathrm{Z}A_1} \mathrm{C}_{A_N}(A_1)$ is free of rank $|N|$ over A_1. Since the same is true for A_N, (ii) follows.

(iv) is an immediate consequence of lemma B.

Now choose a unit u_g of A in A_g for $g \in T$ such that $x^{u_g} = x^{\alpha(g)}$ for $x \in A_1$. Then $A = \bigoplus_{g\in T} A_N u_g$, and $(xc)^{u_g} = x^{\alpha(g)} c^g$ for $x \in A_1$, $c \in \mathrm{C}_{A_N}(A_1)$, $g \in T$. This implies (v).

In applications of theorem C it may be difficult to determine the structure of $\mathrm{C}_{A_N}(A_1)$. We introduce further restrictions in order to simplify the situation. Azumaya [1] has shown that any R-algebra A such that $A/\mathrm{J}A$ is a separable F-algebra contains an R-separable subalgebra S such that $A = S + \mathrm{J}A$. Furthermore, S is uniquely determined up to inner automorphisms of A. Such a subalgebra S is called an **inertial subalgebra of** A. For elementary facts on separable algebras we refer to [2,6].

The following lemma will later be applied to a block of $\mathrm{C}_{A_N}(A_1)$.

D. LEMMA. *Let A be an R-algebra and Z a local subalgebra of $\mathrm{Z}A$ such that A is free over Z and $A/(\mathrm{J}Z)A$ is a separable F-algebra. Then the following holds:*

(i) $\mathrm{J}A = (\mathrm{J}Z)A$, *so A contains an inertial subalgebra .*

(ii) $Z/\mathrm{J}Z$ *is a separable F-algebra, so Z contains one and only one inertial subalgebra T.*

(iii) T *is a complete discrete valuation ring satisfying* $\mathrm{J}T = (\mathrm{J}R)T$.

(iv) $T = S \cap Z$ *for any inertial subalgebra S of A, and the map* $Z \otimes_T S \longrightarrow A,\ z \otimes s \mapsto zs$, *is an isomorphism of R-algebras.*

Proof. (i) follows immediately from our hypothesis.

(ii) By [2, III.2.19] $Z/\mathrm{J}Z$ is a separable F-algebra, and $A/(\mathrm{J}Z)A$ is a separable $Z/\mathrm{J}Z$-algebra. Then, by Azumaya's theorem, Z contains a unique inertial subalgebra T.

(iii) Since T is separable over R we get $\mathrm{J}T = (\mathrm{J}R)T$. T is a local R-algebra since T is contained in Z. Since T is torsion-free as an R-module it is an integral domain. Therefore T is a discrete valuation ring and complete since the rank of T over R is finite.

(iv) It is easy to see that the hypothesis of lemma D is also satisfied with T in place of R. Let S be a T-separable inertial subalgebra of A. Then T is also an R-separable inertial subalgebra of A, and $T \subseteq S \cap Z$.

We choose a basis $s_1, \dots, s_n$ for S over T. Then $s_1 + \mathrm{J}S, \dots, s_n + \mathrm{J}S$ is a basis of $S/\mathrm{J}S = S/(\mathrm{J}T)S$ over $T/\mathrm{J}T$, and $s_1 + \mathrm{J}A, \dots, s_n + \mathrm{J}A$ is a basis of $S + \mathrm{J}A/\mathrm{J}A = A/\mathrm{J}A$ over $T/\mathrm{J}T \cong Z/\mathrm{J}Z$. This implies that $s_1, \dots, s_n$ is a basis for A over Z. In particular the map $Z \otimes_T S \longrightarrow A,\ z \otimes s \mapsto zs$, is an isomorphism of R-algebras. We get $T = S \cap Z$ since $s_1, \dots, s_n$ are linearly independent over T and over $S \cap Z$. Since property (iv) is invariant under conjugation with units the result follows.

The following is a well-known property of central separable R-algebras.

E. LEMMA. *Let A be an arbitrary R-algebra and S a central separable subalgebra of A. Then the map* $\mathrm{C}_A(S) \otimes_R S \longrightarrow A,\ c \otimes s \mapsto cs$, *is an isomorphism of R-algebras.*

Proof. See [2, III.4.3]

We want to study the hypothesis of lemma B when S is a separable R-algebra. Therefore the following well-known lemma is useful.

F. LEMMA. *Let S be a separable R-algebra, A an arbitrary R-algebra and* $\sigma, \sigma' : S \longrightarrow A$ *homomorphisms of R-algebras such that* $s^\sigma + \mathrm{J}A = s^{\sigma'} + \mathrm{J}A$ *for* $s \in S$. *Then there is an element* $x \in \mathrm{J}A$ *such that* $(1-x)^{-1} s^\sigma (1-x) = s^{\sigma'}$ *for* $s \in S$.

Proof. The maps σ and σ' turn $\mathrm{J}A$ into an S-bimodule by $sxt := s^\sigma x t^{\sigma'}$ for elements $x \in \mathrm{J}A$, $s, t \in S$. Then the map $S \longrightarrow \mathrm{J}A,\ s \mapsto s^\sigma - s^{\sigma'}$, is easily seen to be a derivation. By the cohomological characterization of separable algebras any derivation of S is inner ([6, p. 76]). So there is an element $x \in \mathrm{J}A$ such that $s^\sigma - s^{\sigma'} = s^\sigma x = x s^{\sigma'}$ for $s \in S$. Then $s^\sigma(1-x) = (1-x)s^{\sigma'}$, and the result follows since $1-x$ is a unit in A.

More can be said when S is even central separable.

G. LEMMA. *Let A be a primary and S a central separable R-algebra. Then for any two homomorphisms $\sigma, \sigma' : S \longrightarrow A$ of R-algebras there is a unit u in A such that $u^{-1}s^{\sigma}u = s^{\sigma'}$ for $s \in S$.*

Here we call A primary if $A/\mathrm{J}A$ is a simple F-algebra.

Proof. σ and σ' induce homomorphisms $S/\mathrm{J}S \longrightarrow A/\mathrm{J}A$; here $A/\mathrm{J}A$ is a simple F-algebra and $S/\mathrm{J}S$ is a central simple F-algebra. If the induced homomorphisms are conjugate by a unit of $A/\mathrm{J}A$ then lemma F implies that σ and σ' are conjugate by a unit in A. Therefore we may assume that $R = F$ and A is simple. In this case the lemma can be proves as in [9, (7.21)].

We not try to put things together.

H. THEOREM. *Let A be a crossed product of G over A_1 with automorphism system α, and set $N := \{g \in G \mid \alpha(g) \in \mathrm{Inn}\, A_1\}$. Let $b \longleftrightarrow e$ be a block of A_N such that $\mathrm{C}_b(A_1)/(\mathrm{JZ}A_1)\mathrm{C}_b(A_1)$ is a separable F-algebra and $e \in \mathrm{Z}A$. Denote by S an inertial subalgebra of $\mathrm{C}_b(A_1)$ and by Q a maximal central separable R-subalgebra of S. Then the following holds:*

(i) *eA is a crossed product of G/N over b.*

(ii) *The map $\mathrm{C}_{eA}(Q) \otimes_R Q \longrightarrow eA,\ c \otimes q \mapsto cq$, is an isomorphism of R-algebras.*

(iii) *$\mathrm{C}_{eA}(Q)$ is a crossed product of G/N over $\mathrm{C}_b(Q)$.*

(iv) *The map $eA_1 \otimes_{e\mathrm{Z}A_1 \cap S} \mathrm{C}_S(Q) \longrightarrow \mathrm{C}_b(Q),\ x \otimes c \mapsto xc$, is an isomorphism of R-algebras.*

(v) *$\mathrm{C}_S(Q)$ is a separable local R-algebra.*

(vi) *For any transversal T for N in G there are units $u(g)$ in $\mathrm{C}_b(A_1)$ such that the map $\alpha' : G/N \longrightarrow \mathrm{Aut}\, \mathrm{C}_b(Q)$ with $(xc)^{\alpha'(gN)} = x^{\alpha(g)}c^{gu(g)}$ for $x \in eA_1$, $c \in \mathrm{C}_S(Q)$, $g \in T$ is an automorphism system for $\mathrm{C}_{eA}(Q)$.*

Proof. Obviously A is a crossed product of G/N over A_N. Since e is central in A, (i) holds.

(ii) is an immediate consequence of lemma E.

By theorem C, the map $A_1 \otimes_{\mathrm{Z}A_1} \mathrm{C}_{A_N}(A_1) \longrightarrow A_N,\ x \otimes c \mapsto xc$, is an isomorphism of R-algebras. Therefore the map $eA_1 \otimes_{e\mathrm{Z}A_1} \mathrm{C}_b(A_1) \longrightarrow b,\ x \otimes c \mapsto xc$, is an isomorphism, too. By theorem C, $\mathrm{C}_{A_N}(A_1)$ is free as a module over $\mathrm{Z}A_1$, so $\mathrm{C}_b(A_1)$ is projective as a module over $e\mathrm{Z}A_1$. But $e\mathrm{Z}A_1$ is local since b is a block ideal, so $\mathrm{C}_b(A_1)$ is free over $e\mathrm{Z}A_1$, and we may apply lemma D to $\mathrm{C}_b(A_1)$ and $e\mathrm{Z}A_1$ to get an isomorphism $e\mathrm{Z}A_1 \otimes_{e\mathrm{Z}A_1 \cap S} S \longrightarrow \mathrm{C}_b(A_1),\ z \otimes_s \mapsto zs$. This implies that the map $eA_1 \otimes_{e\mathrm{Z}A_1 \cap S} S \longrightarrow b,\ x \otimes s \mapsto xs$, is an isomorphism. In particular, S has exactly one block. Since S is separable this implies that S is primary.

The group G acts on $\mathrm{C}_A(A_1)$, so it also acts on $\mathrm{C}_{A_N}(A_1)$ and on $\mathrm{C}_b(A_1)$. For any element $g \in G$, S^g is another inertial subalgebra of $\mathrm{C}_b(A_1)$. By Azumaya's theorem [1] there exists a unit $v(g)$ of $\mathrm{C}_b(A_1)$ such that $S^{gv(g)} = S$. Then by lemma G we can find a unit $w(g)$ of S such that $q^{gv(g)w(g)} = q$ for $q \in Q$. This shows that we may apply lemma B to get that $\mathrm{C}_{eA}(Q)$ is a crossed product of G/N over $\mathrm{C}_b(Q)$, and we have proved (iii).

By lemma E the map $\mathrm{C}_S(Q) \otimes_R Q \longrightarrow S,\ c \otimes q \mapsto cq$, is an isomorphism. Therefore $b \cong eA_1 \otimes_{e\mathrm{Z}A_1 \cap S} \mathrm{C}_S(Q) \otimes_R Q$. On the other hand, $b \cong \mathrm{C}_b(Q) \otimes_R Q$, again by lemma E. This implies (iv).

Since S is primary and $\mathrm{C}_S(Q) \otimes_R Q \cong S$, $\mathrm{C}_S(Q)$ is primary, too. Since Q is a maximal central separable R-subalgebra of S, $\mathrm{C}_S(Q)$ contains only trivial idempotents, so $\mathrm{C}_S(Q)$ is local. The separability of S implies that of $\mathrm{C}_S(Q)$, so (v) is proved.

Now choose a transversal T for N in G and units u_g of A in A_g such that $x^{\alpha(g)} = u_g^{-1} x u_g$ for $x \in A_1$ and $g \in T$. Then the elements $eu_g v(g) w(g)$ are units in $\mathrm{C}_{eA}(Q)$ contained in eA_{gN} satisfying $(xc)^{eu_g v(g) w(g)} = x^{\alpha(g)} c^{gv(g)w(g)}$ for $x \in eA_1$, $c \in \mathrm{C}_S(Q)$, $g \in T$. This shows that (vi) is true.

It follows that apart from the two separable algebras Q and $\mathrm{C}_S(Q)$ the structure of eA is determined by a crossed product of G/N over eA_1. So we have been able to remove the normal subgroup N. It is well-known that the hypothesis $e \in \mathrm{Z}A$ can be removed by first passing to the stabilizer of $b \longleftrightarrow e$.

I. THEOREM. *Let A be a crossed product of G over A_1, and let $b \longleftrightarrow e$ be a block of A_1. Choose a transversal $g_1, \ldots, g_t$ for $G(b)$ in G and units $u_1, \ldots, u_t$ of A in $A_{g_1}, \ldots, A_{g_t}$, respectively. Then the following holds:*

(i) *$eAe = eA_{G(b)}$ is a crossed product of $G(b)$ over b with g-component $eA_g = A_g e$ for $g \in G(b)$.*

(ii) *If α is an automorphism system for A then its restriction is an automorphism system for eAe.*

(iii) *$\tau^G_{G(b)}(e)$ is an idempotent in $\mathrm{Z}A$ with $AeA = \tau^G_{G(b)}(e)A$.*

(iv) *The maps*

$$\mathrm{Mat}(t, eAe) \longrightarrow AeA,\ [a_{ij}]_{i,j=1}^t \mapsto \sum_{i,j=1}^t u_i^{-1} a_{ij} u_j$$

and

$$AeA \longrightarrow \mathrm{Mat}(t, eAe),\ x \mapsto [eu_i x u_j^{-1} e]_{i,j=1}^t,$$

are isomorphisms of R-algebras and inverse to each other.

(v) *The maps*

$$\mathrm{Z}(eAe) \longrightarrow \mathrm{Z}(AeA),\ z \mapsto \tau^G_{G(b)}(z),$$

and

$$\mathrm{Z}(AeA) \longrightarrow \mathrm{Z}(eAe),\ y \mapsto ey,$$

are isomorphisms of R-algebras and inverse to each other.

<u>Proof.</u> If g and h are elements in G lying in different cosets of $G(b)$ then e^g and e^h are different idempotents in $\mathrm{Z}A_1$, so $e^g e^h = 0$. This implies

$$eA_g e = eA_1 u_g e = eA_1 u_g e^g e = 0$$

for any element $g \in G \backslash G(b)$ and any unit u_g of A in A_g, and

$$eA_g = eA_1 u_g = A_1 e u_g = A_1 u_g e^g = A_g e = eA_g e$$

for any element $g \in G(b)$ and any unit u_g of A in A_g. Furthermore, $eu_g^{-1} e u_g = ee^g = e$. Therefore eu_g is a unit of eAe in eA_g, and (i) and (ii) are proved.

From $e \in \mathrm{C}_A(A_{G(b)})$ we conclude $\tau^G_{G(b)}(e) \in \mathrm{C}_A(A_G) = \mathrm{Z}A$ and $\tau^G_{G(b)}(e) = \sum_{i=1}^{t} u_i^{-1} e u_i \in AeA$. On the other hand, $e = e\tau^G_{G(b)}(e)$, so $AeA = A\tau^G_{G(b)}(e)A = \tau^G_{G(b)}(e)A$. Since $\tau^G_{G(b)}(e)$ is a sum of pairwise orthogonal idempotents it is an idempotent, too. So (iii) follows.

For elements $a_{ij}, b_{kl} \in eAe$ and $x \in AeA$ we have

$$\begin{aligned}\sum_{i,j,k,l} u_i^{-1} a_{ij} u_j u_k^{-1} b_{kl} u_l &= \sum_{i,j,k,l} u_i^{-1} a_{ij} u_j e^{g_j} e^{g_k} u_k^{-1} b_{kl} u_l \\ &= \sum_{i,j,l} u_i^{-1} a_{ij} e b_{jl} u_l \\ &= \sum_{i,l} u_i^{-1} \left(\sum_j a_{ij} b_{jl} \right) u_l, \\ e u_k \left(\sum_{i,j} u_i^{-1} a_{ij} u_j \right) u_l^{-1} e &= \sum_{i,j} u_k e^{g_k} e^{g_i} u_i^{-1} a_{ij} u_j e^{g_j} e^{g_l} u_l^{-1} \\ &= e a_{kl} e = a_{kl}\end{aligned}$$

and

$$\sum_{i,j} u_i^{-1} e u_i x u_j^{-1} e u_j = \tau^G_{G(b)}(e) x \tau^G_{G(b)}(e) = x.$$

This implies (iv).

In particular, we get an isomorphism $\mathrm{ZMat}(t, eAe) \longrightarrow \mathrm{Z}(AeA)$. When we multiply this map with the natural isomorphism $\mathrm{Z}(eAe) \longrightarrow \mathrm{ZMat}(t, eAe)$ we get the first map in (v). Its inverse is given by the second map since

$$e\tau^G_{G(b)}(z) = e\tau^G_{G(b)}(ez) = ez = z$$

for $z \in \mathrm{Z}(eAe)$. This completes the proof of theorem I.

At this point it seems natural to introduce some splitting condition in order to avoid all the extensions of R.

J. THEOREM. *Let A be a crossed product of G over A_1 with automorphism system α, and set $N := \{g \in G \mid \alpha(g) \in \mathrm{Inn}\, A_1\}$. Let $b \longleftrightarrow e$ be a block of A_N such that $\mathrm{C}_b(A_1)/(\mathrm{JZ}A_1)\mathrm{C}_b(A_1)$ is a central simple F-algebra, and denote by S an inertial subalgebra of $\mathrm{C}_b(A_1)$.*

Then the R-algebras $\tau^G_{G(b)}(e)A$ and $\mathrm{Mat}([G : G(b)], C \otimes_R S)$ are isomorphic for a crossed product C of $G(b)/N$ over eA_1 with automorphism system $\alpha' : G(b)/N \longrightarrow \mathrm{Aut}\,(eA_1)$ satisfying $x^{\alpha'(gN)} = x^{\alpha(g)}$ for $x \in eA_1$, $g \in T$ where T is an arbitrary transversal for N in $G(b)$.

<u>Proof.</u> An application of theorem I to the crossed product A of G/N over A_N shows that the R-algebras $\tau^G_{G(b)}(e)A$ and $\mathrm{Mat}([G : G(b)], eA_{G(b)})$ are isomorphic. Furthermore, $A_{G(b)}$ is a crossed product of $G(b)$ over A_1 satisfying the conditions of theorem H. $\mathrm{Z}(\mathrm{C}_b(A_1)/(\mathrm{JZ}A_1)\mathrm{C}_b(A_1)) = Fe + (\mathrm{JZ}A_1)\mathrm{C}_b(A_1)$ implies $\mathrm{Z}S = Re$, so S itself is a central separable R-algebra. Our result follows

from theorem H.

2. GROUP ALGEBRAS

In order to apply theorem J to blocks with normal defect groups several other things have to proved. The following statement also occurs in [4]. Since our general hypothesis is different from that in [4] we cannot use the proof given there.

K. LEMMA. *For any natural number n not divisible by the characteristic of F, any commutative R-algebra A and any element $x \in 1 + \mathrm{J}A$ there is exactly one element $y \in 1 + \mathrm{J}A$ such that $x = y^n$.*

Proof. Since A is complete and $1 + \mathrm{J}A$ is closed in the topology defined by $\mathrm{J}A$ it is enough to show that for any $k \in \mathbb{N}$ there is a unique element $y_k + (\mathrm{J}A)^k \in A/(\mathrm{J}A)^k$ satisfying

$$y_k^n + (\mathrm{J}A)^k = x + (\mathrm{J}A)^k$$

and

$$y_k + \mathrm{J}A = 1 + \mathrm{J}A.$$

This is straight forward by induction on k.

We need lemma K to prove the following.

L. PROPOSITION. *Let G be a finite group such that $|G|$ is not divisible by the characteristic of F, and let A be a crossed product of G over A_1. Then $\mathrm{H}^n(G, 1 + \mathrm{JZ}A_1) = 0$ for $n \in \mathbb{N}$.*

Proof. Lemma K and [8, I.16.20].

In our setup a twisted group algebra of G over R is nothing but a crossed product of G over R. Thus we obtain:

M. LEMMA. *Let D be a finite p-group and G a p'-subgroup of* $\mathrm{Aut}\, D$*. Let A be a crossed product of G over the group algebra RD such that the automorphism system for A is induced by the action of G on D. Then A is a twisted group algebra of the semidirect product DG of D with G over R.*

Proof. We choose units u_g of A in A_g such that $u_g^{-1} d u_g = d^g$ for $d \in D$, $g \in G$. Then

$$u_h^{-1} u_g^{-1} u_{gh} d u_{gh}^{-1} u_g u_h = d^{h^{-1} g^{-1} gh} = d$$

for $d \in D$, $g, h \in G$, so $u_{gh}^{-1} u_g u_h \in A_1 \cap \mathrm{C}_A(RD) = \mathrm{Z}RD$. The map $(g, h) \mapsto u_{gh}^{-1} u_g u_h$ defines a 2-cocycle in $\mathrm{Z}^2(G, \mathrm{UZ}RD)$. Since

$$\mathrm{H}^2(G, \mathrm{UZ}RD) \cong \mathrm{H}^2(G, \mathrm{U}R) \times \mathrm{H}^2(G, 1 + \mathrm{JZ}RD) \cong \mathrm{H}^2(G, \mathrm{U}R)$$

by proposition L this 2-cocycle is equivalent to a 2-cocycle with values in $\mathrm{U}R$. Therefore we may assume $u_{gh}^{-1} u_g u_h \in \mathrm{U}R$ for $g, h \in G$. But then the elements

du_g ($d \in D$, $g \in G$) are a basis for A such that

$$(cu_g)(du_h) = cd^{g^{-1}} u_g u_h \in (\mathrm{U}R)cd^{g^{-1}} u_{gh}$$

for elements $c, d \in D$, $g, h \in G$. The result follows.

We also need a result on the behavior of defect groups with respect to normal subgroups. A proof of these results can be obtained from [4]. Note however, that we do not have any splitting hypothesis on R.

N. PROPOSITION. *Let $B \longleftrightarrow E$ be a block of the group algebra RG, and let $b \longleftrightarrow e$ be a block of RN with $Ee \neq 0$ where N is a normal subgroup of G. Then Ee is a block idempotent in $RG(b)$, any defect group D of the corresponding block is a defect group of $B \longleftrightarrow E$, and $D \cap N$ is a defect group of $b \longleftrightarrow e$.*

We are now in a position to prove theorem A.

Proof of theorem A. By proposition N, Ee is a block idempotent in $RG(b)$, and the corresponding block has defect group D. Since D is normal in G we have

$$\mathrm{Z}RG = (\mathrm{Z}RG \cap R\mathrm{C}_G(D)) + \mathrm{JZ}RG,$$

so E is contained in $R\mathrm{C}_G(D)$. This implies $Ee = e$, and $E = \tau^G_{G(b)}(e)$ by theorem I. We choose an element $c \in \mathrm{C}_{RG(b)}(D)$ with $e = \tau_D^{G(b)}(c)$. Since $e \in R\mathrm{C}_G(D)$ we may assume $c \in R\mathrm{C}_G(D)$. After replacing c by ece we may also assume $c \in b$. Then

$$\tau_D^{D\mathrm{C}_G(D)}(c) = \tau_{\mathrm{Z}D}^{\mathrm{C}_G(D)}(c) \in \mathrm{Z}b$$

so $\tau_D^{D\mathrm{C}_G(D)}(c) \equiv \alpha e \pmod{\mathrm{JZ}R\mathrm{C}_G(D)}$ for some $\alpha \in R$. This implies

$$e = \tau_D^{G(b)}(c) \equiv \tau^{G(b)}_{D\mathrm{C}_G(D)}(\alpha e) \equiv [G(b) : D\mathrm{C}_G(D)]\alpha e \pmod{\mathrm{JZ}RG(b)},$$

so $[G(b) : D\mathrm{C}_G(D)]$ is not divisible by p. Thus $D\mathrm{C}_G(D)/\mathrm{C}_G(D)$ is a Sylow p-subgroup of $G(b)/\mathrm{C}_G(D)$, so possesses a complement H by the Schur-Zassenhaus theorem.

We regard RG as a crossed product of G/D over RD. The elements of H induce p'-automorphisms on FD where as the elements in $\mathrm{U}RD$ induce p-automorphisms on FD by conjugation. Therefore the normal subgroup N of G/D inducing inner automorphisms of RD must be $D\mathrm{C}_G(D)/D$. $\mathrm{C}_{RD\mathrm{C}_G(D)}(RD)$ is a crossed product of $D\mathrm{C}_G(D)/D$ over $\mathrm{Z}RD$ by theorem C. Thus $\mathrm{C}_{RD\mathrm{C}_G(D)}(RD)/(\mathrm{JZ}RD)\mathrm{C}_{RD\mathrm{C}_G(D)}(RD)$ is a crossed product of $D\mathrm{C}_G(D)/D$ over F. Since $\mathrm{C}_G(D) \subseteq \mathrm{C}_{RD\mathrm{C}_G(D)}(RD)$ we even get

$$\mathrm{C}_{RD\mathrm{C}_G(D)}(RD)/(\mathrm{JZ}RD)\mathrm{C}_{RD\mathrm{C}_G(D)}(RD) \cong F[D\mathrm{C}_G(D)/D].$$

Obviously the image of e in $F[D\mathrm{C}_G(D)/D]$ is a block idempotent of defect zero, so $\mathrm{C}_b(RD)/(\mathrm{JZ}RD)\mathrm{C}_b(RD)$ is a central simple F-algebra. An application of theorem J shows that B is isomorphic to $C \otimes_R S$ for some central separable R-algebra S and some crossed product C of H over eRD. Since $eRD \cong RD$ our result follows from lemma M.

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