The depth of Young subgroups of symmetric groups

Tim Fritzsche, Burkhard Külshammer, Carolin Reiche

Mathematisches Institut, Friedrich-Schiller-Universität, 07737 Jena, Germany

tim.fritzsche@uni-jena.de, kuelshammer@uni-jena.de, reiche@mpi-cbg.de

February 13, 2013

Abstract

Let n and k be positive integers. We consider the (ordinary) depth of several Young subgroups of the symmetric group S_n. For $k < n$, S_{n-k} has depth $2\lfloor (n-2)/k \rfloor + 1$ in S_n. Moreover, for $n \geq 2k$, we show that the depth of $S_{\{k+1,\ldots,n\}} \times S_{\{1,\ldots,k\}} \cong S_{n-k} \times S_k$ in S_n is

$$2\left\lceil \frac{2(n-1)}{k+1} \right\rceil - 1$$

in case $n \geq (k^2+2)/2$, and it is $2\lceil \log_2 k \rceil + 1$ if $n = 2k$. For $2k < n < (k^2+2)/2$, we give upper and lower bounds for the depth. Furthermore, we give the precise values for $k \leq 10$ and $n \leq 39$.

1 Introduction

The notion of depth was introduced in connection with the theory of von Neumann algebras [8, 11]. This led to the question what depth 2 means for Hopf and Frobenius algebras [5] or other algebraic structures. In particular, if H is a subgroup of a finite group G, then the inclusion of group algebras $\mathbb{C}H \subseteq \mathbb{C}G$ has depth 2 if and only if H is normal in G [12].

The depth 2 theory was then generalized to a theory of higher depths of semisimple algebras [6] and to extensions of unitary associative rings $B \subseteq A$ with $1_A = 1_B$ [3]. We only consider group algebras $A := \mathbb{C}G$, $B := \mathbb{C}H$ for a finite group G and a subgroup $H \leq G$. Then, B is said to have depth $2n$ in A if $A \otimes_B \cdots \otimes_B A$ ($n+1$ times A) is isomorphic to a direct summand of $\bigoplus_{i=1}^{a} A \otimes_B \cdots \otimes_B A$ (n times A) as A-B-bimodules (or equivalently as B-A-bimodules) for some positive integer a. Furthermore, B is said to have depth $2n+1$ in A if the same assertion holds for B-B-bimodules. Finally, B has depth 1 in A if A is isomorphic to a direct summand of $\bigoplus_{i=1}^{a} B$ as B-B-bimodules. It has been shown that the depth of B in A is finite, and B has depth $n+1$ in A if it has depth n in A. Thus, it is natural to ask for the minimal depth of B in A. For a finite group G and a subgroup H of G, the minimal (ordinary) depth of the inclusion $H \leq G$ is defined as the minimal depth of the inclusion of group algebras $\mathbb{C}H \subseteq \mathbb{C}G$. From now on, we just write depth instead of minimal depth.

As already mentioned, H has depth ≤ 2 in G if and only if H is a normal subgroup of G. Moreover, H has depth 1 in G if and only if $G = HC_G(x)$ for all $x \in H$ where $C_G(x)$ denotes the centralizer of $x \in G$ [4]. If H is not normal in G, then the determination of the depth of H in G is often done with the help of characters. Let $\mathrm{Irr}(H)$ and $\mathrm{Irr}(G)$ be the sets of irreducible characters of H and G respectively. We consider the following relation $\sim$ on $\mathrm{Irr}(H)$: $\alpha \sim \beta$ if there is some $\chi \in \mathrm{Irr}(G)$ such that the induced characters α^G and β^G have χ as a common constituent ($\alpha, \beta \in \mathrm{Irr}(H)$). The distance $d(\alpha, \beta)$ is the smallest integer m, such that there exist $\psi_1, \ldots, \psi_{m-1} \in \mathrm{Irr}(H)$ with $\alpha = \psi_0 \sim \psi_1 \sim \ldots \sim \psi_{m-1} \sim \psi_m = \beta$. Furthermore, we set $d(\alpha, \beta) = -\infty$ if there is no such chain, and $d(\alpha, \beta) = 0$ in case $\alpha = \beta$. Moreover, let $\chi \in \mathrm{Irr}(G)$ and $X \subseteq \mathrm{Irr}(H)$ be the set of all irreducible characters of H which are constituents of $\chi|_H$. Then, the distance of $\psi \in \mathrm{Irr}(H)$ and X is the minimal distance between ψ and the elements of X. We write

$$m(\chi) = \max_{\varphi \in \mathrm{Irr}(H)} \min_{\psi \in X} d(\varphi, \psi)$$

for the maximal distance between X and an irreducible character of H. Then, we obtain the depth of H in G as follows (see [6]):

Theorem 1.1. *Suppose that H is a subgroup of a finite group G.*

(i) Let $m \geq 1$. Then H has depth $\leq 2m+1$ in G if and only if the distance between any two irreducible characters of H is at most m.

(ii) Let $m \geq 2$. Then H has depth $\leq 2m$ in G if and only if $m(\chi) \leq m-1$ for all $\chi \in \mathrm{Irr}(G)$.

Apart from the description by representation theoretic methods, there is a purely group theoretic characterization of the depth of H in G (see [6] again). Let $\mathrm{Core}_G(H) := \bigcap_{g\in G} gHg^{-1}$ denote the kernel of the action of G on the set of cosets G/H.

Theorem 1.2. *Suppose that H is a subgroup of a finite group G and that $N := \mathrm{Core}_G(H)$ is the intersection of m conjugates of H. Then H has depth $\leq 2m$ in G. If N is in the center of G, then H has depth $\leq 2m-1$ in G.*

For permutation groups, we can use further group theoretic methods to find an upper bound for the depth of H in G. Suppose G acts on the finite set Ω and there is a subset $B = \{b_1, \ldots, b_r\} \subseteq \Omega$, such that the pointwise stabilizer $G_{(b_1,\ldots,b_r)} := \{g \in G : gb_i = b_i \; \forall\, i \in \{1,\ldots,r\}\}$ of B in G is trivial. Then B is called a base of G, and the base size of G is the minimum of the cardinalities of the bases of G.

Lemma 1.3. *Let H be a subgroup of a finite group G and $\widehat{G}$ be the permutation group which is induced by the action of G on G/H. Then, the base size b of $\widehat{G}$ coincides with the minimal number of conjugates of H in G such that the intersection of these conjugates is $\mathrm{Core}_G(H)$.*

Proof. Certainly, G acts transitively on G/H. Hence, a base of $\widehat{G}$ is a subset $B = \{g_1H, \ldots, g_rH\} \subseteq G/H$ whose pointwise stabilizer $G_{(g_1H,\ldots,g_rH)} := \{g \in G : gg_iH = g_iH \;\forall\, i \in \{1,\ldots,r\}\}$ satisfies $G_{(g_1H,\ldots,g_rH)} = \{g \in G : gyH = yH \;\forall\, y \in G\}$. For $y \in G$, we get $\{g \in G : gyH = yH\} = yHy^{-1}$, so

$$\{g \in G : gyH = yH \;\forall\, y \in G\} = \bigcap_{y\in G} yHy^{-1} = \mathrm{Core}_G(H)$$

and

$$G_{(g_1H,\ldots,g_rH)} = \bigcap_{i=1}^{r} g_iHg_i^{-1}.$$

This implies that B is a base of $\widehat{G}$ if and only if $\bigcap_{i=1}^{r} g_iHg_i^{-1} = \mathrm{Core}_G(H)$, and the assertion follows immediately. □

Now, Theorem 1.2 yields an upper bound for the depth of H in G depending on the base size. If, additionally, $\mathrm{Core}_G(H)$ lies in the center of G, this gives the following inequality:

Corollary 1.4. *Let H be a subgroup of a finite group G. Suppose $\mathrm{Core}_G(H)$ is contained in the center of G, and let b be the base size of the permutation group which is induced by the action of G on G/H. Then $\max\{d(\alpha,\beta) : \alpha,\beta \in \mathrm{Irr}(H)\} \leq b-1$.*

Proof. By Lemma 1.3 and Theorem 1.2, the depth of H in G is at most $2b-1$. Hence, Theorem 1.1 yields that the distance of any two irreducible characters of H is bounded by $b-1$ as claimed. □

In this paper, we want to determine the depth of Young subgroups of the form $S_{n-k} \times S_1 \times \ldots \times S_1$ respectively $S_{n-k} \times S_k$ of the symmetric group S_n for arbitrary n. That is why we need some facts from the representation theory of the symmetric groups. Overviews on this subject can be found in [10] and [14], we just collect a few assertions here.

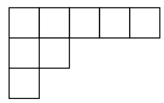
We write S_n for the symmetric group of the set $\{1,\ldots,n\}$, and S_Ω denotes the group of permutations of a finite set Ω. Moreover, we write $(\lambda_1,\ldots,\lambda_l) \vdash n$ for a partition $\lambda = (\lambda_1,\ldots,\lambda_l)$ of n, i.e. $\sum_{i=1}^{l} \lambda_i = n$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ and $\lambda_i \in \mathbb{N}$. The corresponding Young subgroup of S_n of such a partition is

$$S_\lambda = S_{\{1,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \ldots \times S_{\{n-\lambda_l+1,\ldots,n\}} \cong S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_l}.$$

The conjugacy classes as well as the irreducible characters of S_n are parametrized by the partitions of n or, equivalently, by Young diagrams. The Young diagram corresponding to the partition $\lambda = (\lambda_1,\ldots,\lambda_l) \vdash n$ is the subset

$$\{(1,1),(1,2),\ldots,(1,\lambda_1),(2,1),\ldots,(2,\lambda_2),\ldots,(l,1),\ldots,(l,\lambda_l)\}$$

of $\mathbb{N} \times \mathbb{N}$. This can be illustrated by a table of boxes, where a box at the position (i,j) is drawn if and only if $(i,j) \in \{(1,1),(1,2),\ldots,(1,\lambda_1),(2,1),\ldots,(2,\lambda_2),\ldots,(l,1),\ldots,(l,\lambda_l)\}$. For instance, the Young diagram



corresponds to the partition $(5,2,1) \vdash 8$. This way, the trivial character of S_n is assigned to the Young diagram consisting of one row with n boxes, and the alternating character of S_n corresponds to the Young diagram with n rows each containing only one box.
With the help of Young diagrams we can obtain information about the constituents of a character which is induced from $S_{n-k} \times S_k$ to S_n. Write ψ_λ for the irreducible character which is parametrized by the partition λ. An irreducible character $\psi \in \mathrm{Irr}(S_{n-k} \times S_k)$ can be written as $\psi_\mu \times \psi_\nu$ with $\psi_\mu \in \mathrm{Irr}(S_{n-k})$ and $\psi_\nu \in \mathrm{Irr}(S_k)$. The following algorithm yields $(\psi_\mu \times \psi_\nu)^{S_n}$:

Theorem 1.5 (Littlewood-Richardson rule). *Let $\mu \vdash n-k$, $\nu \vdash k$, $\psi_\mu \in \mathrm{Irr}(S_{n-k})$, $\psi_\nu \in \mathrm{Irr}(S_k)$ and $\psi = \psi_\mu \times \psi_\nu \in \mathrm{Irr}(S_{n-k} \times S_k)$ for $k \leq n/2$. Then, the Young diagrams of the irreducible constituents of ψ^{S_n} are exactly the diagrams which arise from the following construction: We attach the boxes of the Young diagram of ψ_ν to the Young diagram of ψ_μ such that any row of the resulting diagram consists of at least as many boxes as the row below. Thereby, the following conditions have to be respected:*

(i) Consider two boxes a_{ij}, a_{xj} with positions (i,j) respectively (x,j) in the Young diagram of ψ_ν. If $i < x$, then a_{ij} must occur in some row above the row containing a_{xj} in the resulting Young diagram.

(ii) Consider two boxes a_{ij}, a_{iy} with positions (i,j) respectively (i,y) in the Young diagram of ψ_ν. If $j < y$, then a_{ij} must occur in some column to the right of the column containing a_{iy} in the resulting Young diagram.

2 The depth of S_{n-k} in S_n

In this section we determine the depth of Young subgroups of the form

$$S_{n-k} \times \underbrace{S_1 \times \ldots \times S_1}_{k \text{ times}} \cong S_{n-k} \leq S_n$$

for some positive integer $k < n$. The case $k = 1$ was already done in [6] (for $n \geq 3$), but we are also able to find the depth for arbitrary $k < n$. Let $\lfloor x \rfloor$ denote the largest integer not greater than x for $x \in \mathbb{R}$.

Theorem 2.1. *The depth of S_{n-k} in S_n is $2\lfloor (n-2)/k \rfloor + 1$.*

Proof. Let $n = lk + r$ with a nonnegative integer l and $r \in \{2, \ldots, k+1\}$ and $n > k+1$. The conjugates of

$$S_{n-k} \cong S_{\{1\}} \times \cdots \times S_{\{k\}} \times S_{\{k+1,\ldots,n\}}$$

are of the form $S_{\{a_1\}} \times \ldots \times S_{\{a_k\}} \times S_{\{a_{k+1},\ldots,a_n\}}$ for pairwise distinct $a_1, \ldots, a_n \in \{1, \ldots, n\}$. Moreover, $S_{\{1\}} \times \ldots \times S_{\{n\}}$ coincides with

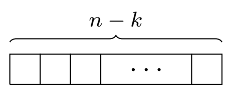
$$\begin{aligned}
& S_{\{1\}} \times \ldots \times S_{\{k\}} \times S_{\{k+1,\ldots,n\}} \\
\cap\ & S_{\{k+1\}} \times \ldots \times S_{\{2k\}} \times S_{\{1,\ldots,k,2k+1,\ldots,n\}} \\
\cap\ & \ldots \\
\cap\ & S_{\{(l-1)k+1\}} \times \ldots \times S_{\{lk\}} \times S_{\{1,\ldots,(l-1)k,lk+1,\ldots,n\}} \\
\cap\ & S_{\{lk+1\}} \times \ldots \times S_{\{lk+r-1\}} \times S_{\{1\}} \times \ldots \times S_{\{k-r+1\}} \times S_{\{k-r+2,\ldots,lk,n\}} \\
=\ & S_{\{1\}} \times \ldots \times S_{\{n\}},
\end{aligned}$$

so $\mathrm{Core}_{S_n}(S_{n-k}) = S_{\{1\}} \times \ldots \times S_{\{n\}} = 1_{S_n}$ is the intersection of at most $l+1$ conjugates. Since

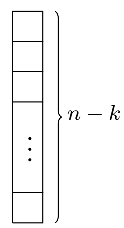
$$l = \frac{n-r}{k} \leq \frac{n-2}{k} < \frac{n+k-r}{k} = l+1,$$

we write $l = \lfloor (n-2)/k \rfloor$. Applying Theorem 1.2, we deduce that the depth of S_{n-k} in S_n is at most $2(\lfloor (n-2)/k \rfloor + 1) - 1 = 2\lfloor (n-2)/k \rfloor + 1$.

For a lower bound of the depth, we consider the distance between the trivial character ψ_{triv} and the alternating character ψ_{alt}, which is certainly positive. The corresponding Young diagrams are



for ψ_{triv} resp.



for ψ_{alt}.

By the branching rule, a special case of the Littlewood-Richardson rule, the constituents of the induced character $\psi_{\mathrm{triv}}^{S_n}$ have Young diagrams with at least $n-k$ boxes in the first row. Thus, the restricted characters (on S_{n-k}) of these constituents have constituents whose Young diagrams contain at least $n - 2k$ boxes in the first row. That means that the first row of the Young diagram of any irreducible character $\psi \in \mathrm{Irr}(S_{n-k})$ with distance $d(\psi, \psi_{\mathrm{triv}}) = 1$ has at least $n - 2k$ boxes. Similarly, the minimal number of boxes in the first row of the Young diagram of some $\psi \in \mathrm{Irr}(S_{n-k})$ with $d(\psi, \psi_{\mathrm{triv}}) = j$ is $n - (j+1)k$. Now,

$$n - \left(\left\lfloor \frac{n-2}{k} \right\rfloor - 1 + 1 \right) k \geq n - (n-2) = 2,$$

whence $d(\psi_{\mathrm{triv}}, \psi_{\mathrm{alt}}) \geq \lfloor (n-2)/k \rfloor$. Therefore, the depth of S_{n-k} in S_n is at least $2(\lfloor (n-2)/k \rfloor + 1) - 1 = 2\lfloor (n-2)/k \rfloor + 1$ by Theorem 1.1, and the assertion follows.

If $k = n-1$, then $S_{n-k} \cong S_1$ has depth 1 in S_n. This finishes the proof of the theorem. □

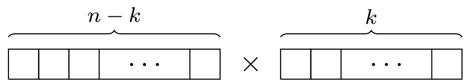
3 The depth of $S_{n-k} \times S_k$ in S_n

Let k and n be integers with $2 < 2k \leq n$. In the following, we consider Young subgroups of the form $S_{\{k+1,\ldots,n\}} \times S_{\{1,\ldots,k\}} \cong S_{n-k} \times S_k$. Whenever we write $S_{n-k} \times S_k$, we mean $S_{\{k+1,\ldots,n\}} \times S_{\{1,\ldots,k\}}$. For instance, if we determine the depth of $S_2 \times S_2$ in S_4, we do this for the subgroup $S_{\{1,2\}} \times S_{\{3,4\}}$ of $S_{\{1,2,3,4\}}$ and not for $\langle (1,2)(3,4), (1,3)(2,4) \rangle$, which is also isomorphic to $S_2 \times S_2$, but certainly has a different depth, since it is normal in S_4.

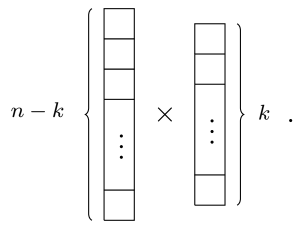
We determine various bounds for the depth of $S_{n-k} \times S_k$ in S_n, which yield the specific value of the depth for many pairs (n, k). The assertions about the base size are due to various authors. A survey on them can be found in [1].

3.1 Lower Bounds

The observations of the preceding section suggest that we should look for the distance of the trivial character ψ_{triv} of $S_{n-k} \times S_k$ with Young diagram



and the alternating character ψ_{alt} of $S_{n-k} \times S_k$, whose Young diagram is



It is clear that this distance is always positive. With the help of the Littlewood-Richardson rule we give two lower bounds for the maximum of the distances of any two irreducible characters of $S_{n-k} \times S_k$. Let $\lceil x \rceil$ denote the least integer not smaller than x for $x \in \mathbb{R}$.

Proposition 3.1. *For a subgroup $S_{n-k} \times S_k \leq S_n$, we have:*

(i) $\max\{d(\psi_1, \psi_2) : \psi_1, \psi_2 \in \text{Irr}(S_{n-k} \times S_k)\} \geq d(\psi_{triv}, \psi_{alt}) \geq \lceil \log_2 k \rceil + \lceil (n-k-1-2^{\lceil \log_2 k \rceil})/k \rceil$.

(ii) $\max\{d(\psi_1, \psi_2) : \psi_1, \psi_2 \in \text{Irr}(S_{n-k} \times S_k)\} \geq d(\psi_{triv}, \psi_{alt}) \geq \lceil 2(n-1)/(k+1) \rceil - 1$.

Proof. In both parts, we only prove the inequality on the right hand side.

(i) We choose $s \in \mathbb{N}_0$ such that $2^s < k \leq 2^{s+1}$, so $s = \lceil \log_2 k \rceil - 1$. The Young diagram of an irreducible constituent of $\psi_{\text{triv}}^{S_n}$ has at most two rows by the Littlewood-Richardson rule. Thus, the Young diagram of an irreducible constituent of $(\psi_{\text{triv}}^{S_n})_{S_{n-k} \times S_k}$ splits into two Young diagrams with $n-k$ resp. k boxes, both containing at most two rows. After a second induction-restriction step, each of the two Young diagrams has at most four rows, etc. After $s+1$ of these steps, we obtain two Young diagrams with $n-k$ resp. k boxes, where the first Young diagram has at most 2^{s+1} rows, whereas the second one has at most k rows, since $k \leq 2^{s+1}$. Now, we can construct at most k new rows in the left Young diagram in any further step. The first Young diagram of ψ_{alt} consists of $n-k-1$ rows more than the first Young diagram of ψ_{triv}. Hence, for

$$n-k-1 = 2^{s+1} + lk + r \quad \text{with } l, r \in \mathbb{N}_0,\ 1 \leq r \leq k,$$

we get

$$d(\psi_{\text{triv}}, \psi_{\text{alt}}) \geq s+1+l+1 = s+1+\left\lceil \frac{n-k-1-2^{s+1}}{k} \right\rceil = \lceil \log_2 k \rceil + \left\lceil \frac{n-k-1-2^{\lceil \log_2 k \rceil}}{k} \right\rceil.$$

(ii) Let $\psi \in \text{Irr}(S_{n-k} \times S_k)$ and $\mu_\psi \times \nu_\psi$ be the corresponding Young diagram, that means μ_ψ and ν_ψ belong to partitions of $n-k$ and k respectively. Similarly, for $\chi \in \text{Irr}(S_n)$, we write μ_χ for the corresponding Young diagram. We define functions $f : \text{Irr}(S_{n-k} \times S_k) \to \mathbb{N}_0$ and $g : \text{Irr}(S_n) \to \mathbb{N}_0$ by

$$\begin{aligned} f(\psi) &= \text{number of boxes of } \mu_\psi \text{ not contained in the first column} \\ &\quad + \text{number of boxes of } \nu_\psi \text{ not contained in the first column} \\ &\quad + \text{number of boxes of } \mu_\psi \text{ contained in the first row, but not in the first column,} \\ g(\chi) &= \text{number of boxes of } \mu_\chi \text{ not contained in the first column} \\ &\quad + \text{number of boxes of } \mu_\chi \text{ contained in the first row, but not in the first column.} \end{aligned}$$

We define the value of a box as the multiplicity, how often the box is counted by f resp. g. In particular, a box contained in the first row, but not in the first column of μ_χ has value 2, a box not contained in the first column and not in the first row has value 1 etc. For the trivial character of $S_{n-k} \times S_k$, we obtain $f(\psi_{\text{triv}}) = (k-1) + (n-k-1) + (n-k-1) = 2n-k-3 = 2(n-1) - (k+1)$, and for the alternating character, we get $f(\psi_{\text{alt}}) = 0$.

Now, we consider the absolute value of the difference $f(\psi_1) - f(\psi_2)$ for two irreducible characters $\psi_1, \psi_2 \in \operatorname{Irr}(S_{n-k} \times S_k)$ of distance 1. Let χ be an irreducible constituent of the induced character $\psi_1^{S_n}$ such that ψ_2 is a constituent of the restricted character $\chi_{S_{n-k} \times S_k}$. Then, $g(\chi) \geq f(\psi_1)$, since μ_χ arises by attaching k boxes to μ_{ψ_1} according to the Littlewood-Richardson rule, so μ_χ cannot have more rows than μ_{ψ_1} and ν_{ψ_1} together. On the other hand, a box of ν_{ψ_1}, which does not belong to the first row, cannot be attached to the first row of μ_{ψ_1}, so its value in μ_χ is at most one more than its value in ν_{ψ_1}. Certainly, the value of a box in the first row, but not in the first column of ν_{ψ_1} can also be increased by at most 1, when attaching it to μ_{ψ_1}. Finally, the value of the box in the first row and the first column of ν_{ψ_1} can be increased by at most 2, when viewed as a box of μ_χ. Since the value of each box of μ_{ψ_1} coincides with its value in μ_χ, and ν_{ψ_1} consists of exactly k boxes, we conclude $g(\chi) \leq f(\psi_1) + (k-1) + 2 = f(\psi_1) + k + 1$. Moreover, χ is also a constituent of $\psi_2^{S_n}$, so $f(\psi_2) \leq g(\chi) \leq f(\psi_2) + k + 1$. Combining these results, we get $|f(\psi_1) - f(\psi_2)| \leq k + 1$.

This implies that the absolute value of the difference $f(\psi) - f(\tilde{\psi})$ of $\psi, \tilde{\psi} \in \operatorname{Irr}(S_{n-k} \times S_k)$ with $d(\psi, \tilde{\psi}) = j$ is at most $j(k+1)$. Hence,

$$d(\psi_{\text{triv}}, \psi_{\text{alt}}) \geq \left\lceil \frac{2(n-1) - (k+1)}{k+1} \right\rceil = \left\lceil \frac{2(n-1)}{k+1} \right\rceil - 1. \qquad \square$$

Corollary 3.2. *The depth of $S_{n-k} \times S_k$ in S_n is at least*

$$\max \left\{ 2 \left(\lceil \log_2 k \rceil + \left\lceil \frac{n-k-1-2^{\lceil \log_2 k \rceil}}{k} \right\rceil \right) + 1,\ 2 \left\lceil \frac{2(n-1)}{k+1} \right\rceil - 1 \right\}.$$

Proof. This is an immediate consequence of the preceding proposition and Theorem 1.1. $\square$

3.2 Upper Bounds

We are going to establish upper bounds for the depth of $S_{\{k+1,\ldots,n\}} \times S_{\{1,\ldots,k\}} \cong S_{n-k} \times S_k$ by finding upper bounds for the base size of the permutation group G which is induced by the action of S_n on the set of k-subsets of $\{1, \ldots, n\}$. The symmetric group S_n acts transitively on the k-subsets of $\{1, \ldots, n\}$: For instance, the stabilizer of $\{1, \ldots, k\}$ is $S_{\{1,\ldots,k\}} \times S_{\{k+1,\ldots,n\}} \cong S_k \times S_{n-k}$, the stabilizer of $\{2, \ldots, k+1\}$ is $S_{\{2,\ldots,k+1\}} \times S_{\{1,k+2,\ldots,n\}}$, which is a conjugate of $S_{\{1,\ldots,k\}} \times S_{\{k+1,\ldots,n\}}$, etc. Therefore, the base size b of G coincides with the minimal number of conjugates of $S_{n-k} \times S_k$ such that the intersection of these conjugates is

$$\operatorname{Core}_{S_n}(S_{n-k} \times S_k) = S_{\{1\}} \times S_{\{2\}} \times \ldots \times S_{\{n\}}$$

by Lemma 1.3.

Obviously, a base of the action of G is a set $B = \{b_1, \ldots, b_r\}$ of k-subsets, such that for any $i, j \in \{1, \ldots, n\}$, $i \neq j$, there exists an $l \in \{1, \ldots, r\}$ with $i \in b_l$ and $j \notin b_l$ resp. $i \notin b_l$ and $j \in b_l$. Hence, for positive integers b, k, we can give a construction which yields the largest n, such that G has base size b.

Lemma 3.3. *Let G be the permutation group which is induced by the action of S_n on the k-subsets of $\{1, \ldots, n\}$. If G has base size b, then $k \leq 2^{b-1}$. Choose $t \in \mathbb{N}$ such that*

$$\sum_{i=0}^{t-1} \binom{b-1}{i} \leq k < \sum_{i=0}^{t} \binom{b-1}{i} \quad \textit{resp. } t = b \textit{ in case } k = 2^{b-1}. \tag{1}$$

Then,

$$n \leq \sum_{i=0}^{t} \binom{b}{i} + \left\lfloor \frac{b}{t+1} \left(k - \sum_{i=0}^{t-1} \binom{b-1}{i} \right) \right\rfloor . \tag{2}$$

On the other hand, if $b, k \geq 2$, $n \geq 2k$ and t are integers, such that (1) and (2) hold, then the permutation group induced by the action of S_n on the k-subsets of $\{1, \ldots, n\}$ has base size $\leq b$.

Proof. Let $k < 2^b$ at first. If $B = \{b_1, \ldots, b_r\}$ is a base of G, then, for distinct $i, j \in \{1, \ldots, n\}$, there exists a $b_l \in B$ containing exactly one of the integers i and j. Thus, there can be at most one element of $\{1, \ldots, n\}$ which does not appear in any set b_l, at most r elements of $\{1, \ldots, n\}$ which are contained in exactly one of the b_l, at most $\binom{r}{2}$ elements of $\{1, \ldots, n\}$ which are contained in exactly two sets b_l etc.
Suppose, we find a base $C = \{c_1, \ldots, c_s\}$, such that one of the integers of $\{1, \ldots, n\}$ does not lie in c_l, $l = 1, \ldots, s$, s integers of $\{1, \ldots, n\}$ lie in exactly one of the c_l, $\binom{s}{2}$ integers of $\{1, \ldots, n\}$ are contained in exactly two sets c_l, $\ldots$, $\binom{s}{u}$ integers of $\{1, \ldots, n\}$ lie in exactly u sets c_l, and $\lfloor m/(u+1) \rfloor$ integers of $\{1, \ldots, n\}$ lie in at least $u+1$ sets c_l, where $0 \leq m < (u+1)\binom{s}{u+1}$ is the number of elements in $c_1, \ldots, c_s$ which appear at least $u+1$ times counting multiplicities. Then C is a base of minimal size, that means $s = b$. Altogether, there are $b \cdot k$ elements in the sets $c_1, \ldots, c_s$ counting multiplicities. Hence, the inequality

$$\sum_{i=0}^{u} i \binom{b}{i} \leq bk < \sum_{i=0}^{u+1} i \binom{b}{i}$$

must hold for a base of minimal size. This is equivalent to

$$\sum_{i=0}^{u} \binom{b-1}{i-1} \leq k < \sum_{i=0}^{u+1} \binom{b-1}{i-1} \qquad \text{resp.} \qquad \sum_{i=0}^{u-1} \binom{b-1}{i} \leq k < \sum_{i=0}^{u} \binom{b-1}{i} .$$

Moreover, $c_1, \ldots, c_s$ contain

$$\sum_{i=0}^{u} \binom{b}{i} + \left\lfloor \frac{m}{u+1} \right\rfloor$$

distinct integers, with

$$m = bk - \sum_{i=0}^{u} i \binom{b}{i} = b \left(k - \sum_{i=0}^{u} \binom{b-1}{i-1} \right) = b \left(k - \sum_{i=0}^{u-1} \binom{b-1}{i} \right) .$$

Certainly, there is no base of size b whose elements contain more distinct integers than $c_1, \ldots, c_s$ does by construction.
Moreover, the construction above shows that it is impossible to get a base of size b containing more than

$$\sum_{i=0}^{b} \binom{b}{i} = 2^b$$

distinct integers. This yields $2^b \geq n \geq 2k$, so $k \leq 2^{b-1}$. Additionally, we obtain $n = 2^b$ in case $k = 2^{b-1}$.
It remains to show that for given integers $b, k \geq 2$, $n \geq 2k$ and t, which satisfy (1) and (2), the permutation group G induced by the action of S_n on the k-subsets of $\{1, \ldots, n\}$ has a base of size b. Suppose $k \neq 2^{b-1}$ at first. We construct a base B as follows: Clearly $B = \{B_1, \ldots, B_b\}$ shall be a set of k-subsets of $\{1, \ldots, n\}$. We label the elements of B_i by $B_{i1}, \ldots, B_{ik}$ for $i = 1, \ldots, b$. Set $B_{i1} := i$. Next, we take the integers $b+1, \ldots, b+b(b-1)/2$ and put each of them in exactly two of the B_i. The exact places are given by the 2-subsets of $\{1, \ldots, b\}$: Since there are exactly $b(b-1)/2$

of these 2-subsets, we can find a bijection between the integers $b+1,\dots,b+b(b+1)/2$ and the 2-subsets of $\{1,\dots,b\}$. If the integer c corresponds to the subset $\{r,s\}$ under this bijection, we take $i := \min\{l \in \{1,\dots,k\} : B_{rl}$ is not defined$\}$ as well as $j := \min\{l \in \{1,\dots,k\} : B_{sl}$ is not defined$\}$ and set $B_{ri} := c$ and $B_{sj} := c$. Each of the integers $1,\dots,b$ appears in exactly $b-1$ of the 2-subsets of $\{1,\dots,n\}$, so any $c \in \{b+1,\dots,b+b(b-1)/2\}$ is written on two places B_{ij} with $2 \le j \le b$. Certainly, if $c' \neq c$ is another element of $\{b+1,\dots,b+b(b-1)/2\}$, then there is a set B_i, which contains c whereas c' does not lie in B_i.
We go on with this construction writing each of the integers $\binom{b}{2}+1,\dots,\binom{b}{3}$ in exactly three different sets B_i etc. The arguments above show that after t steps, we have put $\sum_{i=1}^{t}\binom{b}{i}$ distinct integers in the first $\sum_{i=0}^{t-1}\binom{b-1}{i}$ columns of the sets $B_1,\dots,B_b$, so that for two such integers $c \neq c'$, there exists a set B_i which contains either c or c'. By our assumptions, there are

$$m := b\left(k - \sum_{i=0}^{t-1}\binom{b-1}{i}\right) < b\binom{b-1}{t}$$

places left undefined in the B_i. Therefore, we write $\lfloor m/(t+1)\rfloor$ further integers in the B_i, each one appearing in exactly $t+1$ of the B_i, so that B_{ij} is defined for $1 \le i \le b$ and $1 \le j \le k-1$. Thus, there are at most t undefined places in the last column of the B_i and any other place is defined. Assume $B_{i_1k},\dots,B_{i_uk}$ are the undefined places. If $b-u \le t$, we set $B_{i_lk} = c$ for $1 \le l \le u$, where c is the integer which appears in exactly the sets B_j for all $j \in \{1,\dots,b\} \setminus \{i_1,\dots,i_u\}$. For $t < b-u < b-1$, there is an integer c which can be found in t sets B_i, but in none of the sets $B_{i_1},\dots,B_{i_u}$. In this case we set $B_{i_lk} := c$. If $u=1$, there is an integer c which lies in $t+1$ sets B_i, but not in B_{i_1}, and we set $B_{i_1k} := c$. Then, any place of the B_i is defined and B is a base of G for

$$n' := \sum_{i=1}^{t}\binom{b}{i} + \left\lfloor \frac{b}{t+1}\left(k - \sum_{i=0}^{t-1}\binom{b-1}{i}\right)\right\rfloor.$$

Since any integer of $\{1,\dots,n'-1\}$ is contained in at least one of the B_i, B is also a base of G for $n'+1$.
Certainly, the above construction also works for $n'+1 = 2k = 2^b$. Moreover, it is clear that if $n < n'$, then the base size of the permutation group induced by the action of S_n on the k-subsets of $\{1,\dots,n\}$ is not larger than b. □

Theorem 3.4. *Let G be the permutation group which is induced by the action of S_n on the set of k-subsets of $\{1,\dots,n\}$.*

(i) If $4 \le 2k = n$, then G has base size $\lceil \log_2 k\rceil + 1$.

(ii) If $k \ge 2$ and $n \ge (k^2+2)/2$, then G has base size $\lceil 2(n-1)/(k+1)\rceil$.

Proof. (i) We have just shown $k \le 2^{b-1}$ in the proof of Lemma 3.3, whence $\log_2 k \le b-1$. Since b is an integer, we even get $b \ge \lceil \log_2 k\rceil + 1 =: c$.

On the other hand, we have $k \le 2^{c-1}$. If $k = 2^{c-1}$, then

$$n = 2k = 2^c = \sum_{i=0}^{c}\binom{c}{i}.$$

Hence, G has base size $\le c = \lceil \log 2k\rceil + 1$ by Lemma 3.3. If $k < 2^{c-1}$, then there is a positive integer $t < c$, such that

$$\sum_{i=0}^{t-1}\binom{c-1}{i} \le k < \sum_{i=0}^{t}\binom{c-1}{i}.$$

Thus,

$$\begin{aligned}
n = 2k &\le k + k + \left\lfloor \frac{c-t-1}{t+1}\left(k - \sum_{i=0}^{t-1}\binom{c-1}{i}\right)\right\rfloor - \sum_{i=0}^{t-1}\binom{c-1}{i} + \sum_{i=0}^{t-1}\binom{c-1}{i} \\
&< \sum_{i=0}^{t}\binom{c-1}{i} + \left\lfloor k + \frac{c-t-1}{t+1}\left(k - \sum_{i=0}^{t-1}\binom{c-1}{i}\right) - \sum_{i=0}^{t-1}\binom{c-1}{i}\right\rfloor + \sum_{i=0}^{t-1}\binom{c-1}{i} \\
&= \binom{c-1}{0} + \sum_{i=1}^{t}\left(\binom{c-1}{i} + \binom{c-1}{i-1}\right) + \left\lfloor \frac{c-t-1+t+1}{t+1}\left(k - \sum_{i=0}^{t-1}\binom{c-1}{i}\right)\right\rfloor \\
&= \sum_{i=0}^{t}\binom{c}{i} + \left\lfloor \frac{c}{t+1}\left(k - \sum_{i=0}^{t-1}\binom{c-1}{i}\right)\right\rfloor,
\end{aligned}$$

and we deduce that G has base size at most $c = \lceil \log 2k \rceil + 1$ by Lemma 3.3 again.

(ii) We apply Lemma 3.3 several times. Suppose the base size of G is $k-1$. Then, since $k = 1 + (k-2) + 1$,

$$n \le 1 + k - 1 + \frac{(k-1)(k-2)}{2} + \left\lfloor \frac{k-1}{3} \cdot 1 \right\rfloor \le 1 + \frac{k(k-1)}{2} + \frac{k-1}{2} = \frac{k^2+1}{2} < \frac{k^2+2}{2} \le n.$$

That is why the base size of G is at least k.

Let b be the base size of G. If $b = k$, then $k = 1 + (b-1)$, so there exists a base of size k for

$$n \le 1 + b + \frac{b(b-1)}{2} = 1 + \frac{b(b+1)}{2} = 1 + \frac{k(k+1)}{2}.$$

Thus, the base size of G is k for $(k^2+2)/2 \le n \le 1 + k(k+1)/2$. For these n, the equation $b = k = \lceil 2(n-1)/(k+1) \rceil$ certainly holds.

Now, let $b > k$, so $k < 1 + (b-1)$. Then, there exists a base of size b for

$$n \le 1 + b + \left\lfloor \frac{b}{2}(k-1) \right\rfloor = 1 + \left\lfloor \frac{b(k+1)}{2} \right\rfloor.$$

On the other hand, if G has base size $b-1$, then

$$n \le 1 + b - 1 + \left\lfloor \frac{b-1}{2}(k-1) \right\rfloor = 1 + \left\lfloor \frac{(b-1)(k+1)}{2} \right\rfloor.$$

Therefore, G has base size b if and only if

$$\left\lfloor \frac{(b-1)(k+1)}{2} \right\rfloor + 1 < n \le \left\lfloor \frac{b(k+1)}{2} \right\rfloor + 1,$$

that means, for a suitable integer $2 \le l < k+1$ and a suitable $r \in \{0, 1/2\}$, we get

$$n = \left\lfloor \frac{(b-1)(k+1)+l}{2} \right\rfloor + 1 = \frac{(b-1)(k+1)+l}{2} - r + 1.$$

Solving the equation for b, we obtain

$$b = \frac{2(n-1+r)-l}{k+1} + 1 = \frac{2(n-1)}{k+1} + \frac{2r-l+k+1}{k+1}.$$

The assumptions $0 \le r < 1$ and $2 \le l < k+1$ yield $-k-1 < 2r - l < 0$, whence the last summand lies in $(0,1)$. This implies $b = \lceil 2(n-1)/(k+1) \rceil$. $\square$

The second part of Theorem 3.4 includes a reproof of the following Theorems 3.4 and 3.5 of [7]:

Theorem 3.5 (Cáceres, Garijo, González, Márquez, Puertas)**.** *Let G be the permutation group which is induced by the action of S_n on the set of k-subsets of $\{1,\ldots,n\}$.*

(i) If b and k are positive integers with $k \leq b$ and $b > 2$ and $n = \lfloor b(k+1)/2 \rfloor + 1$, then G has base size b.

(ii) Let b, k be integers such that $2 \leq k \leq b$. Then, G has base size b for each n satisfying

$$\left\lfloor \frac{(b-1)(k+1)}{2} \right\rfloor + 1 < n < \left\lfloor \frac{b(k+1)}{2} \right\rfloor + 1.$$

Corollary 3.6. *(i) Let $4 \leq 2k = n$. Then, $S_{n-k} \times S_k$ has depth at most $2\lceil \log_2 k \rceil + 1$ in S_n.*

(ii) Let $k \geq 2$ and $n \geq (k^2+2)/2$. Then, $S_{n-k} \times S_k$ has depth at most $2\lceil 2(n-1)/(k+1) \rceil - 1$ in S_n.

Proof. This is an easy application of Theorem 3.4 as well as Lemma 1.3 and Theorem 1.2. □

Combining Corollaries 3.2 and 3.6, we obtain the following theorem.

Theorem 3.7. *Let $k \geq 2$ and $n \geq (k^2+2)/2$. Then, $S_{n-k} \times S_k$ has depth $2\lceil 2(n-1)/(k+1) \rceil - 1$ in S_n. Furthermore, for $4 \leq 2k = n$, the depth of $S_{n-k} \times S_k$ in S_n is $2\lceil \log_2 k \rceil + 1$.*
In particular, if $k \geq 2$ and b is the base size of the permutation group which is induced by the action of S_n on the set of k-subsets of $\{1,\ldots,n\}$, then

$$d(\psi_{triv}, \psi_{alt}) = \max\{d(\chi,\psi) : \chi,\psi \in \mathrm{Irr}(S_{n-k} \times S_k)\} = b - 1$$

for $n \geq (k^2+2)/2$ and $n = 2k$ respectively.

We mention that these results also hold for $k = 1$, what can be easily verified by comparing Theorem 3.7 with Theorem 2.1 and its proof.
Finding the base size of the permutation group induced by the action of S_n on the set of k-subsets of $\{1,\ldots,n\}$ for $2k < n < (k^2+2)/2$ seems to be difficult. For given n and k, it is possible to obtain the base size using Lemma 3.3, but there is no general formula known in this case. Certainly, k is an upper bound for the base size, since k is the base size if $n = (k^2+2)/2$. A further bound was established in [2]:

Theorem 3.8 (Benbenishty)**.** *Let G be the permutation group which is induced by the action of S_n on the set of k-subsets of $\{1,\ldots,n\}$. Suppose $n = lk + r$, $n < k^2$, $0 \leq r \leq n-1$ and set $c = \lfloor \log_l k \rfloor$. Then, we get $b \leq 3l(c+1)$ for the base size b of G.*

Now, the following assertion can easily be deduced:

Corollary 3.9. *Let $n = lk + r$ with $n < k^2$ and $0 \leq r \leq n-1$. Then, the depth of $S_{n-k} \times S_k$ is at most $6l(\lfloor \log_l k \rfloor + 1) - 1$.*

However, this does not yield the exact values, especially for small k. For instance, the depth d of $S_5 \times S_5$ in S_{10} is 7 by Theorem 3.7, whereas Corollary 3.9 states $d \leq 35$. On the other hand, for large k and $2k < n < (k^2+2)/2$, Corollary 3.9 gives a much better bound than $2k-1$.

3.3 Computations for small values

By Corollary 1.4, we already know

$$d(\psi_{\mathrm{triv}}, \psi_{\mathrm{alt}}) \leq \max\{d(\chi,\psi) : \chi,\psi \in \mathrm{Irr}(S_{n-k} \times S_k)\} \leq b - 1, \tag{3}$$

where b is the base size of the permutation group induced by the action of S_n on the k-subsets of $\{1,\ldots,n\}$. For $n \geq (k^2+2)/2$, we have seen in Theorem 3.7 that this inequality is, indeed, an equality. The same assertion holds for $n = 2k$ by Theorem 3.7 (for $k > 1$) and the proof of Theorem 2.1 (for $k = 1$). We want to know whether this equality also occurs in the other cases. Computations with GAP [9] (using [15]) yield the following tables which list the respective distances $d := d(\psi_{\text{triv}}, \psi_{\text{alt}})$ for $k \leq 10$ and $n \leq 39$, which we have not determined so far.

$k = 5$:

n	11	12	13
d	3	3	4

$k = 6$:

n	13	14	15	16	17	18
d	3	4	4	4	4	5

$k = 7$:

n	15	16	17	18	19	20	21	22	23	24	25
d	3	4	4	4	4	5	5	5	5	5	6

$k = 8$:

n	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
d	4	4	4	4	4	5	5	5	5	5	6	6	6	6	6	7

$k = 9$:

n	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
d	4	4	4	4	5	5	5	5	5	5	6	6	6	6	6	7	7	7	7	7	7

$k = 10$:

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
d	4	4	4	4	5	5	5	5	5	5	6	6	6	6	6	6	7	7	7

Theorem 3.10. *For $k \leq 10$ and $2k \leq n \leq 39$, the inequality (3) is an equality. In particular, the depth of $S_{n-k} \times S_k$ in S_n can be easily obtained by the above assertions for $k \leq 10$ and $2k \leq n \leq 39$.*

Proof. It suffices to show $d := d(\psi_{\text{triv}}, \psi_{\text{alt}}) \geq b - 1$ for the pairs (n, k) mentioned in the tables above, where b is the corresponding base size. To do this, we apply Lemma 3.3 to determine n for given k and b, such that the permutation group induced by S_n on the k-subsets of $\{1, \ldots, n\}$ has base size b.
For $k = 5$ and $b = 4$, we get

$$n \leq 1 + 4 + \frac{4 \cdot 3}{2} + \left\lfloor \frac{4-1}{3}(5-4) \right\rfloor = 12.$$

Moreover, the base size for $n = 13$ cannot be bigger than the base size for $n = 14 = (5^2 + 3)/2$, which is 5 by Theorem 3.4. Hence, $d \geq b - 1$ for any n in case $k = 5$, and we obtain equality.
Now, let $k = 6$. For $b = 4$, we get

$$n \leq 1 + 4 + \frac{4 \cdot 3}{2} + \left\lfloor \frac{4-1}{3}(6-4) \right\rfloor = 13,$$

and if $b = 5$, then

$$n \leq 1 + 5 + \frac{5 \cdot 4}{2} + \left\lfloor \frac{5-1}{3}(6-5) \right\rfloor = 17.$$

Moreover, we already know $n \leq 22$ if $b = 6$, so $d \geq b - 1$ also follows for any n in case $k = 6$.
The calculations for $k = 7, 8, 9, 10$ are analogous. □

We have determined the depth of $S_{n-k} \times S_k$ in S_n for a large number of pairs (n, k). For each of these pairs, it was sufficient to know the distance $d(\psi_{\text{triv}}, \psi_{\text{alt}})$ as well as the base size of the permutation group induced by the action of S_n on the k-subsets of $\{1, \ldots, n\}$. That is why we conjecture the following:

Conjecture. *Let n and k be positive integers with $n \geq 2k$. Moreover, let b be the base size of the permutation group which is induced by the action of S_n on the set of k-subsets of $\{1, \ldots, n\}$. Furthermore, let ψ_{triv} and ψ_{alt} be the trivial respectively alternating character of $S_{n-k} \times S_k$. Then, the following equation holds:*

$$d(\psi_{triv}, \psi_{alt}) = \max\{d(\chi, \psi) : \chi, \psi \in \mathrm{Irr}(S_{n-k} \times S_k)\} = b - 1.$$

Acknowledgments

The paper is based on the diploma thesis of the third author [13].

References

[1] R. F. Bailey, P. J. Cameron, *Base size, metric dimension and other invariants of groups and graphs*, Bull. LMS **43** (2011), 209–242.

[2] C. Benbenishty, *On actions of primitive groups*, Ph. D. thesis, Hebrew University of Jerusalem, unpublished.

[3] R. Boltje, S. Danz, B. Külshammer, *On the depth of subgroups and group algebra extensions*, J. Algebra **335** (2011), 258–281.

[4] R. Boltje, B. Külshammer, *Group algebra extensions of depth one*, Algebra Number Theory **5** (2011), 63–73.

[5] S. Burciu, L. Kadison, *Depth two Hopf subalgebras of a semisimple Hopf algebra*, J. Algebra **322** (1997), 162–176.

[6] S. Burciu, L. Kadison, B. Külshammer, *On subgroup depth*, Int. Electron. J. Algebra **9** (2011), 133–166.

[7] J. Cáceres, D. Garijo, A. González, A. Márquez, M. L. Puertas, *Hypergraphs for computing determining sets of Kneser graphs*, 14. Nov. 2011, arXiv:`1111.3252v1`.

[8] F. Goodman, P. de la Harpe, V.F.R. Jones, *Coxeter graphs and towers of algebras*, Springer, New York, 1989.

[9] The GAP Group, *GAP – groups, algorithms and programming*, version 4.4.12, 2008, `http://www.gap-system.org`.

[10] G. James, A. Kerber, *The representation theory of the symmetric group*, vol. 16, Cambridge University Press, 1984.

[11] V. F. R. Jones, V. S. Sunder, *Introduction to subfactors*, Springer, New York, 1989.

[12] L. Kadison, B. Külshammer, *Depth two, normality and a trace ideal condition for Frobenius extensions*, Commun. Alg. **34** (2006), 3103–3122.

[13] C. Reiche, *Die Tiefe von Young-Untergruppen symmetrischer Gruppen*, Diploma thesis, Friedrich Schiller University Jena, unpublished.

[14] B. E. Sagan, *The symmetric group*, Springer, New York, 2001.

[15] D. Traytel, *GAP 4 Package hecke*, version 1.3, 2010, `http://home.in.tum.de/~traytel/hecke/`.