A NOTE ON BLOCKS WITH DIHEDRAL DEFECT GROUPS

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Abstract. We prove that the 2-regular subspace of the center of a block with a dihedral defect group is multiplicatively closed.

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In this paper we show that, for a 2-block B with a dihedral defect group D, the 2-regular subspace of the center of B is multiplicatively closed. In [8], Yun Fan and the second author had proved a similar result for blocks with abelian defect groups. Our result here is the block version of a result by H. Meyer [15] who showed that the 2-regular subspace of the center of a group algebra FG, for a finite group G with a dihedral Sylow 2-subgroup and an algebraically closed field F of characteristic 2, is multiplicatively closed.

In the following, we fix a finite group G and an algebraically closed field F of characteristic $p > 0$. We denote the set of conjugacy classes of G by $\mathrm{Cl}(G)$, and the set of p-regular conjugacy classes of G by $\mathrm{Cl}_{p'}(G)$. (Recall that a conjugacy class of G is called p-regular if the common order of its elements is not divisible by p.) For $C \in \mathrm{Cl}(G)$, we denote its class sum in the group algebra FG by

$$C^+ := \sum_{x \in C} x.$$

The class sums C^+ ($C \in \mathrm{Cl}(G)$) form an F-basis for the center ZFG of FG, and we denote by $Z_{p'}FG$ the F-subspace of ZFG spanned by all C^+ ($C \in \mathrm{Cl}_{p'}(G)$). In general, $Z_{p'}FG$ is not multiplicatively closed. Suppose that FG has the block decomposition

$$FG = B_1 \oplus \cdots \oplus B_t.$$

Then there is a similar decomposition of the center ZFG:

$$ZFG = ZB_1 \oplus \cdots \oplus ZB_t.$$

For $i = 1, \ldots, t$, we set $Z_{p'}B_i := B_i \cap Z_{p'}FG = Z_{p'}FG \cdot 1_{B_i}$ where the last equality comes from a result by Iizuka (cf. [11]). Thus we have

$$Z_{p'}FG = Z_{p'}B_1 \oplus \cdots \oplus Z_{p'}B_t.$$

The interest in $Z_{p'}B_i$ has to do with the fact that its dimension equals the number of isomorphism classes of simple B_i-modules, usually denoted by $\ell(B_i)$. In general, $Z_{p'}B_i$ is not multiplicatively closed. We start with the following observation:

Proposition 1. *Let B be a block of the group algebra FG, and suppose that $\ell(b_u) = 1$ for every element u of order p in G and every block b_u of $FC_G(u)$ such that $(b_u)^G = B$. Then $Z_{p'}B$ is multiplicatively closed.*

Proof. Assume that there are elements $y, z \in Z_{p'}B$ such that $yz \notin Z_{p'}B$, i.e. $yz \notin Z_{p'}FG$. We write $yz = \sum_{g \in G} \alpha_g g$ with $\alpha_g \in F$ for $g \in G$. Then there is a p-singular element $h \in G$ such that $\alpha_h \neq 0$. (Recall that an element in G is called p-singular if its order is divisible by p.) Choose $m \in \mathbb{N}$ such that $u := h^m$ has order p, and let $\mathrm{Br}_u : ZFG \longrightarrow ZFC_G(u)$ denote the corresponding Brauer homomorphism. Then

$$\mathrm{Br}_u(y)\mathrm{Br}_u(z) = \mathrm{Br}_u(yz) = \sum_{g \in C_G(u)} \alpha_g g \notin Z_{p'}FC_G(u),$$

since $h \in C_G(u)$ and $\alpha_h \neq 0$. Thus there exists a block b of $FC_G(u)$ such that

$$\mathrm{Br}_u(y)1_b \cdot \mathrm{Br}_u(z)1_b = \mathrm{Br}_u(yz)1_b \notin Z_{p'}b;$$

in particular, we have $0 \neq \mathrm{Br}_u(y)1_b = \mathrm{Br}_u(y1_B)1_b = \mathrm{Br}_u(y)\mathrm{Br}_u(1_B)1_b$ and hence $0 \neq \mathrm{Br}_u(1_B)1_b$. This implies that $b^G = B$. Thus our hypothesis forces $\ell(b) = 1$, so that $Z_{p'}b = F1_b$, by a result of Osima (cf. [11]). But then $\mathrm{Br}_u(y)1_b, \mathrm{Br}_u(z)1_b \in Z_{p'}FC_G(u)1_b = Z_{p'}b = F1_b$, and therefore $\mathrm{Br}_u(y)1_b \cdot \mathrm{Br}_u(z)1_b \in F1_b = Z_{p'}b$ as well, a contradiction. □

In the situation of Proposition 1, we can make use of the theory of lower defect groups in order to show that the radical $JZ_{p'}B$ of $Z_{p'}B$ is contained in the socle $\mathrm{Soc}(B)$ of B. The theory of lower defect groups originates from Brauer [2]. Later accounts include Olsson [16], Broué [4] and Broué-Olsson [5]. For any p-subgroup P of G, we denote by $I_P(FG)$ the ideal of ZFG spanned by the class sums C^+ where C ranges over all conjugacy classes of G having a defect group contained in P. (Recall that the defect groups of $C \in \mathrm{Cl}(G)$ are the Sylow p-subgroups of the centralizers $C_G(g)$ of the elements $g \in C$.) Then $I_P(FG) = I_Q(FG)$ whenever P and Q are conjugate p-subgroups of G, and we set

$$I_{<P}(FG) := \sum_{R<P} I_R(FG).$$

For a block B of FG and a p-subgroup P of G,

$$m_B(P) := \dim(ZB \cap I_P(FG)) - \dim(ZB \cap I_{<P}(FG))$$

is called the multiplicity of P as a lower defect group of B. We will make use of the related number

$$m'_B(P) := \dim(Z_{p'}B \cap I_P(FG)) - \dim(Z_{p'}B \cap I_{<P}(FG))$$

which is the p-regular multiplicity of P as a lower defect group of B. It is known that

$$\sum_P m_B(P) = \dim ZB \quad \text{and} \quad \sum_P m'_B(P) = \dim Z_{p'}B$$

where P ranges over a transversal $\mathcal{P}$ for the conjugacy classes of p-subgroups of G. Also, we have

$$m_B(P) = m'_B(P) = 0$$

unless P is conjugate to a subgroup of a defect group D of B. Moreover, we are going to use the fact that

$$m'_B(D) = 1,$$

so that $\dim Z_{p'}B/(Z_{p'}B \cap I_{<D}(FG)) = 1$. (When $Z_{p'}B$ is multiplicatively closed this means that $JZ_{p'}B = Z_{p'}B \cap I_{<D}(FG)$.)

Proposition 2. *In the situation of Proposition 1, suppose that B has positive defect. Then we have $JZ_{p'}B = ZB \cap I_1(FG) \subseteq \mathrm{Soc}(B)$; in particular, $(JZ_{p'}B)^2 = 0$.*

Proof. We assume that $JZ_{p'}B \neq ZB \cap I_1(FG)$, choose an element $z \in JZ_{p'}B \setminus I_1(FG)$, and write $z = \sum_{g\in G} \alpha_g g$ with $\alpha_g \in F$ for $g \in G$. Then there exists an element $h \in G$ such that $\alpha_h \neq 0$ and $|C_G(h)| \equiv 0 \pmod p$. Let u be a p-element in $C_G(h)$, and let $\mathrm{Br}_u : ZFG \longrightarrow ZFC_G(u)$ denote the corresponding Brauer homomorphism. Then

$$\mathrm{Br}_u(z) = \sum_{g\in C_G(u)} \alpha_g g \neq 0$$

since $h \in C_G(u)$ and $\alpha_h \neq 0$. Thus there exists a block b of $FC_G(u)$ such that $0 \neq \mathrm{Br}_u(z)1_b = \mathrm{Br}_u(z1_B)1_b = \mathrm{Br}_u(z)\mathrm{Br}_u(1_B)1_b$. In particular, $\mathrm{Br}_u(1_B)1_b \neq 0$, so that $b^G = B$. Hence our hypothesis implies that $1 = \ell(b) = \dim Z_{p'}b$, so that $Z_{p'}b = F1_b$, and $\mathrm{Br}_u(z)1_b \in Z_{p'}b = F1_b$. On the other hand, $\mathrm{Br}_u(z)1_b$ is nilpotent, so we obtain the contradiction $\mathrm{Br}_u(z)1_b = 0$.

This contradiction shows that $JZ_{p'}B = ZB \cap I_1(FG)$, and it is well-known that $I_1(FG)$ is contained in the Reynolds ideal of ZFG which in turn is contained in $\mathrm{Soc}(FG)$ (cf. [11]). Thus $JZ_{p'}B \subseteq \mathrm{Soc}(B)$, and therefore $(JZ_{p'}B)^2 = 0$. □

The theory of lower defect groups implies that, in the situation above, every elementary divisor of the Cartan matrix of B is either equal to 1 or to the order $|D|$ of a defect group D of B (and, as usual, $|D|$ has multiplicity 1 as an elementary divisor of this Cartan matrix).

Next we show that blocks with dihedral defect groups satisfy the hypothesis of Proposition 1. (This is certainly known to the experts; but we include a proof for the convenience of the reader.)

Proposition 3. *Let B be a block of the group algebra FG with a dihedral defect group D. Moreover, let u be an element of order 2 in G, and let b_u be a block of $FC_G(u)$ such that $(b_u)^G = B$. Then b_u is nilpotent; in particular, we have $\ell(b_u) = 1$.*

We recall that a block B of FG is called nilpotent if $N_G(Q, b_Q)/C_G(Q)$ is a p-group, for every p-subgroup Q of G and every block b_Q of $FQC_G(Q)$ such that $(b_Q)^G = B$; here

$$N_G(Q, b_Q) := \{g \in N_G(Q) : gb_Qg^{-1} = b_Q\}$$

denotes the stabilizer of b_Q in $N_G(Q)$.

Proof. We set $H := C_G(u)$ and choose a defect group P of b_u. Then P is conjugate to a subgroup of D, so we may assume that $P \leq D$. Let Q be a 2-subgroup of H, and let b_Q be a block of $FQC_H(Q)$ such that $(b_Q)^H = b_u$. Then Q is conjugate to a subgroup of P, and we may assume that $Q \leq P \leq D$. Since D is dihedral, Q is cyclic, a Klein four group or dihedral. If Q is not a Klein four group then $\mathrm{Aut}(Q)$ and therefore $N_H(Q, b_Q)/C_H(Q)$ are 2-groups. Thus we may assume that Q is a Klein four group. Then $\mathrm{Aut}(Q)$ is isomorphic to the symmetric group S_3. Assume that $N_H(Q, b_Q)/C_H(Q)$ is not a 2-group. Then $N_H(Q, b_Q)/C_H(Q)$ contains an element $\overline{g} = gC_H(Q)$ of order 3 which permutes the involutions in Q transitively.

On the other hand, we have $u \in Z(H)$, which forces $u \in P$. But then we must have $u \in Q$; for otherwise $\langle u, Q\rangle$ would be an elementary abelian subgroup of order 8 in the dihedral group D, which is impossible. Since u is certainly fixed under conjugation with g, we have arrived at a contradiction.

This contradiction shows that b_u is nilpotent. Then Puig's theorem (cf. [16], or [12]) implies that $\ell(b_u) = 1$. □

A combination of the results above now implies the main result of this paper:

Theorem 4. *Let B be a block of the group algebra FG with a dihedral defect group D. Then $Z_{2'}B$ is multiplicatively closed, and $JZ_{2'}B \subseteq \mathrm{Soc}(B)$; in particular, $(JZ_{2'}B)^2 = 0$.*

We note that Theorem 4 generalizes Meyer's result [15] on group algebras of finite groups with a dihedral Sylow 2-subgroup. Its proof does not make use of the detailed structure theory of blocks with dihedral defect groups, as developed by Brauer [3] and Erdmann [7]. (A more recent account can be found in [6]; see also [13].) Brauer's results imply that $\ell(B) \in \{1, 2, 3\}$. So, in combination with Theorem 4, we get:

Corollary 5. *In the situation of Theorem 4, the following holds:*
(i) If $\ell(B) = 1$ then $Z_{2'}B \cong F$;
(ii) if $\ell(B) = 2$ then $Z_{2'}B \cong F[X]/(X^2)$;
(iii) if $\ell(B) = 3$ then $Z_{2'}B \cong F[X, Y]/(X^2, XY, Y^2)$.

We also note that, for a block B with quaternion or semidihedral defect group, the subspace $Z_{2'}B$ is not multiplicatively closed, in general; counterexamples are provided by SL(2,3) and GL(2,3), as already observed by Meyer [14]. However, if B is a block of FG with a defect group

$$D = \langle x, y : x^{2^{n-1}} = y^2 = 1,\ yxy^{-1} = x^{2^{n-2}+1} \rangle$$

for some $n \geq 4$ then $Z_{2'}B$ is multiplicatively closed. This follows immediately from the fact that every such block is nilpotent. For the convenience of the reader, we provide a proof of this fact. Before we do so, we need to list some elementary properties of D. Their proofs are left to the reader (cf. 5.4 in [9]). The group D is sometimes called a modular 2-group.

Lemma 6. *Let D be a modular 2-group as above, with $n \geq 4$. Then the following holds:*
(i) For $i = 1, \dots, n-2$, the subgroup $\Omega_i(D) := \langle g \in D : g^{2^i} = 1 \rangle = \langle x^{2^{n-1-i}}, y \rangle$ is abelian of type $(2^i, 2)$.
(ii) If Q is a proper noncyclic subgroup of D then $Q = \Omega_i(D)$ for some $i \in \{1, \dots, n-2\}$.
(iii) $\mathrm{Aut}(D)$ *is a 2-group.*

Now we can prove the following result which is the block version of Satz IV.3.5 in [10].

Theorem 7. *Let B be a block of FG with a defect group D which is a modular 2-group. Then B is nilpotent; in particular, $Z_{2'}B$ is multiplicatively closed.*

Proof. Let (D, b_D) be a maximal B-subpair, let Q be a subgroup of D, and let b_Q be the unique block of $FQC_G(Q)$ such that $(Q, b_Q) \leq (D, b_D)$ (cf. [1]). We need to show that $N_G(Q, b_Q)/C_G(Q)$ is a 2-group. If $Q = D$ then this is a consequence of Lemma 6 (iii). So we may assume that $Q < D$. If Q is cyclic or abelian of type $(2^i, 2)$ for some $i > 1$ then $\mathrm{Aut}(Q)$ is again a 2-group. Thus, by Lemma 6 (ii), it remains to consider the case $Q = \Omega_1(D)$. Since $Q \trianglelefteq D$, we must have $(Q, b_Q) \trianglelefteq (D, b_D)$ (cf. Proposition 3.4 in [1]). The block $(b_Q)^{DC_G(Q)} = (b_D)^{DC_G(Q)}$ of $FDC_G(Q)$ has defect group D and covers the block b_Q of $FQC_G(Q)$. Thus b_Q has defect group $D \cap QC_G(Q) = C_D(Q) = \Omega_{m-2}(D) =: R$. Since $N_G(Q, b_Q)$ permutes the defect groups of b_Q, the Frattini argument implies that

$$N_G(Q, b_Q) = [N_G(Q, b_Q) \cap N_G(R)]QC_G(Q) = [N_G(Q, b_Q) \cap N_G(R)]C_G(Q).$$

Thus

$$|N_G(Q, b_Q) : C_G(Q)| = |[N_G(Q, b_Q) \cap N_G(R)]C_G(Q) : C_G(Q)|$$

$$= |N_G(Q, b_Q) \cap N_G(R) : C_G(Q) \cap N_G(R)| \,\Big|\, |N_G(R) : C_G(R)| \,\Big|\, |\mathrm{Aut}(R)|.$$

Since $\mathrm{Aut}(R)$ is a 2-group, so is $N_G(Q, b_Q)/C_G(Q)$.

This shows that B is a nilpotent block. As before, Puig's theorem now implies that $\ell(B) = 1$; in particular, $Z_{2'}B$ is multiplicatively closed. □

In [14], H. Meyer proved that, for an odd prime p and a finite group G with a metacyclic Sylow p-subgroup, the p-regular subspace $Z_{p'}FG$ is multiplicatively closed. It is, at present, unclear to us whether this result generalizes to blocks as well.

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