The vertices and sources of the basic spin module for the symmetric group in characteristic 2

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Abstract

In this article we determine the vertices and sources of the basic spin module for the symmetric group $\mathfrak{S}_n$ of degree n over a field of characteristic 2.

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1 Introduction and result

This paper contributes to the general problem of determining the vertices of the simple modules for the symmetric group $\mathfrak{S}_n$ of degree $n \in \mathbb{N}$ over a field F of prime characteristic p. Introduced by J. A. Green 50 years ago [7], the notion of *vertices* provides a measure for the *relative projectivity* of indecomposable and, in particular, simple FG-modules. Here FG denotes the group algebra of a finite group G over the field F. Moreover, the concept of vertices in combination with the Green correspondence also enables us to connect representations of G and representations of p-subgroups of G and their normalizers which is an important issue in modular representation theory.

However, generally the vertices of simple FG-modules are still rather poorly understood, even when confining one's attention to specific classes of finite groups such as that consisting of the symmetric groups $\mathfrak{S}_n$ for $n \in \mathbb{N}$. Currently we are far from being able to give a description of the vertices of the simple $F\mathfrak{S}_n$-modules in general. Nevertheless, it turns out that for certain simple $F\mathfrak{S}_n$-modules this is indeed possible. Amongst these is the basic spin $F\mathfrak{S}_n$-module $D(n)$ in characteristic 2. That is, $D(n)$ is the simple $F\mathfrak{S}_n$-module parametrized by the partition $(m+1, m-1)$ if $n = 2m$, and by $(m+1, m)$ if $n = 2m+1$, for some $m \in \mathbb{N}$. Building on a series of computer calculations with the computer algebra system MAGMA [2], in [4] the authors and R. Zimmermann set up a conjecture what the vertices of $D(n)$ should look like. The goal of this paper now is to prove that conjecture. In fact, denoting the alternating group of degree $n \in \mathbb{N}$ by $\mathfrak{A}_n$, in Sections 4 and 5 we will show that the following holds:

Theorem 1.1. *Let $p = 2$, and let $n \geq 2$ with 2-adic expansion $n = \sum_{j=1}^{s} 2^{i_j}$, for some $s \in \mathbb{N}$ and $i_1 > \ldots > i_s \geq 0$. Suppose that $P \leq \mathfrak{S}_n$ is a vertex of the basic spin module $D(n)$.*

(i) If n is odd then P is conjugate to a Sylow 2-subgroup of $\prod_{j=1}^{s} \mathfrak{A}_{2^{i_j}}$.

(ii) If $n \equiv 2 \pmod 4$ then P is a Sylow 2-subgroup of $\mathfrak{S}_n$.

(iii) If $n \equiv 0 \pmod 4$, then P is a Sylow 2-subgroup of $\mathfrak{A}_n$.

The exposition of the present paper is as follows: we start by fixing the notation needed throughout, and by summarizing a series of preliminary results concerning the basic spin module for the symmetric group. After that, in Sections 4 and 5, we then subsequently give a proof of Theorem 1.1 above. The main ingredients of our proof are results of Nagai and Uno (cf. [15], [17], [19]) which give an explicit basis for the basic spin module and determine its restrictions to certain Young subgroups of $\mathfrak{S}_n$. In Section 6 we then characterize the sources of the basic spin $F\mathfrak{S}_n$-module.

By Clifford Theory, the restriction of the simple $F\mathfrak{S}_n$-module $D(n)$ to the alternating group $\mathfrak{A}_n$ is either again simple, or splits into the direct sum of two conjugate simple $F\mathfrak{A}_n$-modules. Benson's Theorem [1] enables us to decide which of the two cases occurs, depending on the congruence class of n modulo 4. In Section 7, we close with determining the vertices and sources of these simple $F\mathfrak{A}_n$-modules.

We assume the reader to be familiar with the theory of vertices and sources of indecomposable modules over group algebras, and refer to [16], Sec. 4.3 for an introduction to this topic. Moreover, as far as the representation theory of the symmetric groups is concerned, we refer to [11].

2 General notation and prerequisites

To begin with, we collect the notation needed in this note. In what follows all groups under consideration are supposed to be finite.

(1) Let G be a group, p a prime, and let $(\mathcal{O}, K, F)$ be a splitting p-modular system for G such that the field F is algebraically closed. The unique maximal ideal in $\mathcal{O}$ is denoted by (π). For $\Omega \in \{\mathcal{O}, K, F\}$, by an ΩG-module we always understand a finitely generated left ΩG-module. We consider a simple KG-module S with corresponding ordinary character χ. Furthermore, let S' be some $\mathcal{O}$-form of S, and $\overline{S'} := S'/\pi S'$ the respective FG-module with Brauer character $\overline{\chi} := \chi_{|G_{p'}}$. In case that $\overline{S'}$ is simple it is isomorphic to the reduction modulo (π) of any $\mathcal{O}$-form of S. We then also use the notation $\overline{S} := \overline{S'}$, and speak of *the* reduction of S modulo p.

(2) Loewy series. Let M be an FG-module with Loewy series $M = \operatorname{Rad}^0(M) \supset \operatorname{Rad}(M) \supset \ldots \supset \operatorname{Rad}^l(M) = 0$ such that

$$\operatorname{Rad}^{i-1}(M)/\operatorname{Rad}^i(M) = D_{i1} \oplus \cdots \oplus D_{ir_i},$$

for $i = 1, \ldots, l$, appropriate $r_i \in \mathbb{N}$, and simple FG-modules $D_{i1}, \ldots, D_{ir_i}$. Then we write

$$M \sim \begin{bmatrix} D_{11} \oplus \cdots \oplus D_{1r_1} \\ D_{21} \oplus \cdots \oplus D_{2r_2} \\ \vdots \\ D_{l1} \oplus \cdots \oplus D_{lr_l} \end{bmatrix}.$$

(3) Outer tensor products of modules. Consider groups G and H, an FG-module M and an FH-module N. Then we have an isomorphism of F-algebras

$$F[G \times H] \longrightarrow FG \otimes_F FH,$$

mapping (g, h), for $g \in G$ and $h \in H$, to $g \otimes h$. Via this isomorphism, the outer tensor product $M \otimes_F N$ becomes an $F[G \times H]$-module which we denote by $M \boxtimes N$. Furthermore, there exists an isomorphism of F-algebras

$$\Psi : \mathrm{End}_{FG}(M) \otimes_F \mathrm{End}_{FH}(N) \longrightarrow \mathrm{End}_{F[G\times H]}(M \boxtimes N),$$

such that $(\Psi(\varphi\otimes\psi))(m\otimes n) = \varphi(m)\otimes\psi(n)$, for all $\varphi \in \mathrm{End}_{FG}(M)$, $\psi \in \mathrm{End}_{FH}(N)$, $m \in M$ and $n \in N$. We also have the following:

Proposition 2.1 ([13], Prop. 1.2)**.** *If M and N are indecomposable with vertices P and Q, respectively, then $M \boxtimes N$ is indecomposable with vertex $P \times Q$. Moreover, if V is a source of M and if W is a source of N then $V \boxtimes W$ is an indecomposable $F[P \times Q]$-module, and is a source of $M \boxtimes N$.*

(4) Subgroups of the symmetric group. As far as the symmetric groups and their subgroups are concerned, we fix the following notation: let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n. Then the corresponding Young subgroup of $\mathfrak{S}_n$ will be denoted by

$$\mathfrak{S}_\lambda := \prod_{j=1}^{k} \mathfrak{S}_{\lambda_j},$$

with $\mathfrak{S}_{\lambda_j}$ acting on the set $\{(\sum_{i=1}^{j-1}\lambda_i) + 1, \ldots, \sum_{i=1}^{j}\lambda_i\}$, for $j = 1, \ldots, k$.

Let $C_p = \langle (1, \ldots, p) \rangle$ be the cyclic group of order p. Then we set $P_1 := 1$, $P_p := C_p$, and $P_{p^i} := P_{p^{i-1}} \wr P_p$ for $i \geq 2$. We can regard P_{p^i} as a subgroup of the symmetric group $\mathfrak{S}_{p^i}$: that is, P_{p^i} is the semidirect product of $P_{p^{i-1}} \times \cdots \times P_{p^{i-1}}$ with C_p. Here the p direct factors are acting on disjoint subsets of $\{1, \ldots, p^i\}$, of size p^{i-1} each, and the subgroup C_p permutes these direct factors in the obvious way. With this identification, P_{p^i} is a Sylow p-subgroup of $\mathfrak{S}_{p^i}$, by [11], 4.1.22 and 4.1.24.

Now consider the p-adic expansion $n = \sum_{j=1}^{s} \alpha_j p^{i_j}$, for appropriate $s \in \mathbb{N}$, $i_1 > \ldots > i_s \geq 0$, and $\alpha_j > 0$ for $j = 1, \ldots, s$. We set

$$\lambda := (\underbrace{p^{i_1}, \ldots, p^{i_1}}_{\alpha_1}, \underbrace{p^{i_2}, \ldots, p^{i_2}}_{\alpha_2}, \ldots, \underbrace{p^{i_s}, \ldots, p^{i_s}}_{\alpha_s}).$$

Then, by [11], 4.1.22, the Sylow p-subgroups of the Young subgroup $\mathfrak{S}_\lambda$ are also Sylow p-subgroups of $\mathfrak{S}_n$. Moreover, a fixed Sylow p-subgroup of $\mathfrak{S}_n$ will be denoted by

$$P_n = \prod_{j=1}^{s} (P_{p^{i_j}})^{\alpha_j}.$$

In particular, $Q_n := P_n \cap \mathfrak{A}_n$ is then a Sylow p-subgroup of the alternating group $\mathfrak{A}_n$.

3 Modules for the covering groups of $\mathfrak{S}_n$ and $\mathfrak{A}_n$

From now on, suppose that $p = 2$ and that $(\mathcal{O}, K, F)$ is a splitting 2-modular system for all groups under consideration, and suppose further that F is algebraically closed.

3.1 The basic spin module in characteristic 2

In the following we consider the *basic spin* $F\mathfrak{S}_n$*-module* $D(n)$, for $n \geq 2$. That is

$$D(n) \cong \begin{cases} D^{(m+1,m)}, & \text{if } n = 2m+1 \\ D^{(m+1,m-1)}, & \text{if } n = 2m, \end{cases}$$

for some $m \geq 1$. By [1], we have $\dim(D(n)) = 2^{\lfloor \frac{n-1}{2} \rfloor}$. Notice that $D(2)$ is the trivial $F\mathfrak{S}_2$-module F, and $D(3)$ is the projective simple $F\mathfrak{S}_3$-module $D^{(2,1)}$. Additionally we define $D(1)$ to be the trivial $F\mathfrak{S}_1$-module. Furthermore:

Theorem 3.1 (Benson [1])**.** *If* $n \equiv 2 \pmod 4$ *then* $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(D(n)) =: E(n)_0$ *is simple. Otherwise* $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(D(n)) = E(n)_+ \oplus E(n)_-$ *with non-isomorphic conjugate simple* $F\mathfrak{A}_n$*-modules* $E(n)_+$ *and* $E(n)_-$*.*

Thus, as an immediate consequence of this theorem and Green's Indecomposability Theorem [7], we get:

Corollary 3.2. *The basic spin* $F\mathfrak{S}_n$*-module* $D(n)$ *is relatively* $\mathfrak{A}_n$*-projective if and only if* $n \not\equiv 2 \pmod 4$.

For determining the vertices of $D(n)$ in the forthcoming sections, the following result will be important:

Proposition 3.3. *Let* $n \geq 3$.

(i) If n *is even then* $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D(n)) \cong D(n-1)$.

(ii) If n *is odd then* $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D(n)) \sim \begin{bmatrix} D(n-1) \\ D(n-1) \end{bmatrix}$.

(iii) If $n \equiv 3 \pmod 4$ *then* $D(n-2) \mid \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-2}}(D(n))$ *and* $D(n) \mid \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-2}}(D(n-2))$.

Proof. This follows immediately from [12], Thm. 11.2.10 and Thm. 11.2.11. □

Remark 3.4. In Section 5 we will make extensive use of a result due to T. Nagai and K. Uno [20] (see also [19] and [15]) which gives an explicit F-basis for $D(n)$. In order to state this result, we introduce the following notation: let $n = 2m+2$, for some $m \in \mathbb{N}$. For $i \in \{1, \ldots, n-1\}$ set $\sigma_i := (i, i+1) \in \mathfrak{S}_n$. Moreover, let

$$B := \langle \sigma_i \mid i \in \{1, \ldots, 2m\} \text{ odd } \rangle \leq \mathfrak{S}_{n-2},$$

and let $M := FB$ be the regular FB-module. That is, B is elementary abelian of order 2^m, and M has F-basis

$$B = \{\sigma_{i_1} \cdots \sigma_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq 2m-1,\ i_1, \ldots, i_k \text{ odd},\ k \in \{0, \ldots, m\}\}.$$

Now let $r \in \{1, \ldots, m-1\}$ and $i := 2r$. Then one can define an action of σ_i on M as follows:

$$\sigma_i(\sigma_{i_1}\cdots\sigma_{i_k}) := \begin{cases} \sigma_{i_1}\cdots\sigma_{i_k}, & \text{if } i-1, i+1 \in \{i_1,\ldots,i_k\} \\ (1+\sigma_{i+1})\sigma_{i_1}\cdots\sigma_{i_k}, & \text{if } i-1 \in \{i_1,\ldots,i_k\} \\ & \text{and } i+1 \notin \{i_1,\ldots,i_k\} \\ (1+\sigma_{i-1})\sigma_{i_1}\cdots\sigma_{i_k}, & \text{if } i-1 \notin \{i_1,\ldots,i_k\} \\ & \text{and } i+1 \in \{i_1,\ldots,i_k\} \\ (1+\sigma_{i-1}+\sigma_{i+1})\sigma_{i_1}\cdots\sigma_{i_k}, & \text{if } i-1, i+1 \notin \{i_1,\ldots,i_k\}. \end{cases}$$

Moreover, we set

$$\sigma_{2m}(\sigma_{i_1}\cdots\sigma_{i_k}) := \begin{cases} \sigma_{i_1}\cdots\sigma_{i_k}, & \text{if } 2m-1 \in \{i_1,\ldots,i_k\} \\ (1+\sigma_{2m-1})\sigma_{i_1}\cdots\sigma_{i_k}, & \text{if } 2m-1 \notin \{i_1,\ldots,i_k\}. \end{cases}$$

Finally, if m is even then we set

$$\sigma_{2m+1}(\sigma_{i_1}\cdots\sigma_{i_k}) := (1 + \sum_{\substack{j\in\{1,\ldots,2m-1\} \\ j \text{ odd}}} \sigma_j)\sigma_{i_1}\cdots\sigma_{i_k},$$

and if m is odd then we set

$$\sigma_{2m+1}(\sigma_{i_1}\cdots\sigma_{i_k}) := (\sum_{\substack{j\in\{1,\ldots,2m-1\} \\ j \text{ odd}}} \sigma_j)\sigma_{i_1}\cdots\sigma_{i_k}.$$

Now we can state the result by Nagai and Uno:

Theorem 3.5 (Nagai, Uno [20], [19], [15])**.** *With the definitions given in the previous remark, we obtain both an $F\mathfrak{S}_{n-1}$-module structure and an $F\mathfrak{S}_n$-module structure on M. As $F\mathfrak{S}_n$-module M is isomorphic to $D(n)$, and hence as $F\mathfrak{S}_{n-1}$-module M is isomorphic to $D(n-1)$.*

3.2 The basic spin module in characteristic 0

(1) Let $n \geq 4$, and let $\widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{S}}_n$ be the Schur double covers of $\mathfrak{S}_n$. If $\tilde{\nu} : \widetilde{\mathfrak{S}}_n \longrightarrow \mathfrak{S}_n$ and $\hat{\nu} : \widehat{\mathfrak{S}}_n \longrightarrow \mathfrak{S}_n$ are the canonical epimorphisms then, as usual, we choose notation such that the preimages under $\tilde{\nu}$ of the transpositions have order 4.

For $n \neq 6$, these double covers $\widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{S}}_n$ are non-isomorphic, whereas $\widetilde{\mathfrak{S}}_6 \cong \widehat{\mathfrak{S}}_6$ (cf. [8], Thm. 2.12).

(2) Furthermore, we set $\widehat{\mathfrak{A}}_n := \hat{\nu}^{-1}(\mathfrak{A}_n)$ and $\widetilde{\mathfrak{A}}_n := \tilde{\nu}^{-1}(\mathfrak{A}_n)$. Then $|\widehat{\mathfrak{A}}_n| = n! = |\widetilde{\mathfrak{A}}_n|$. If $6 \neq n \neq 7$ then $\widehat{\mathfrak{A}}_n \cong \widetilde{\mathfrak{A}}_n$ is the unique Schur cover of $\mathfrak{A}_n$.

(3) If M is a $K\widetilde{\mathfrak{S}}_n$-module then the *associate* module M^a is defined as follows: as K-vector space, M^a equals M. Moreover, for $m \in M^a$ and $g \in \widetilde{\mathfrak{S}}_n$, we set

$$g \cdot m := \mathbf{sgn}(\tilde{\nu}(g))gm.$$

In case that $M \cong M^a$, we call M *self-associate.*

(4) Let $\widetilde{D}$ be a simple $F\widetilde{\mathfrak{S}}_n$-module (respectively simple $F\widetilde{\mathfrak{A}}_n$-module). Since the centre of $\widetilde{\mathfrak{S}}_n$ has order 2, it acts trivially on $\widetilde{D}$. Therefore, $\widetilde{D}$ is the inflation of a simple $F\mathfrak{S}_n$-module (respectively simple $F\mathfrak{A}_n$-module) D. In what follows, we will then always identify $\widetilde{D}$ and D. Furthermore, we also identify the Brauer characters corresponding to $\widetilde{D}$ and D.

(5) From now on, by $R(n)$ we denote the basic spin $K\widetilde{\mathfrak{S}}_n$-module of dimension $2^{\lfloor \frac{n-1}{2} \rfloor}$ (cf. [8], Prop. 6.1, Thm. 6.2). The corresponding ordinary irreducible character of $\widetilde{\mathfrak{S}}_n$ will be denoted by χ_n.

The following will be important throughout:

Theorem 3.6. *The reduction modulo* 2 *of* $R(n)$ *is isomorphic to* $D(n)$*. In particular,* $\overline{\chi_n} = \varphi_n$ *where* φ_n *denotes the Brauer character corresponding to* $D(n)$*.*

Theorem 3.7 ([8], Thm. 6.8)**.** *Let* $g \in \widetilde{\mathfrak{S}}_n$ *be such that* $\tilde{\nu}(g)$ *has cycle type* μ *of length* l*.*

(i) If n *is odd then* χ_n *is self-associate, and*

$$\chi_n(g) = \begin{cases} \pm 2^{(l-1)/2}, & \text{if } \tilde{\nu}(g) \text{ has odd order} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If n *is even then* χ_n *is not self-associate, and*

$$\chi_n(g) = \begin{cases} \pm 2^{(l/2)-1}, & \text{if } \tilde{\nu}(g) \text{ has odd order} \\ \pm i^{(n/2)} \sqrt{n/2}, & \text{if } \mu = (n) \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the signs can be determined precisely. See for instance [8], Props. 6.6, 6.7. In combination with [8] Thm. 4.2, Theorem 3.7 above implies

Theorem 3.8. *If* n *is odd then* $\mathrm{Res}^{\widetilde{\mathfrak{S}}_n}_{\widetilde{\mathfrak{A}}_n}(R(n))$ *splits into the direct sum of two non-isomorphic conjugate simple* $K\widetilde{\mathfrak{A}}_n$*-modules* $S(n)_+$ *and* $S(n)_-$*. If* n *is even then* $\mathrm{Res}^{\widetilde{\mathfrak{S}}_n}_{\widetilde{\mathfrak{A}}_n}(R(n)) =: S(n)_0$ *is simple.*

The following result will also be useful:

Theorem 3.9 ([8], Thm. 6.9)**.** *Suppose that* $n = 2m + 1$*, for some* $m \geq 2$*. Then the ordinary characters* χ_n^+ *and* χ_n^- *corresponding to the simple* $K\widetilde{\mathfrak{A}}_n$*-modules* $S(n)_+$ *and* $S(n)_-$*, respectively, differ only on those elements* $g \in \widetilde{\mathfrak{A}}_n$ *such that* $\tilde{\nu}(g)$ *is an* n*-cycle. On these two* $\widetilde{\mathfrak{S}}_n$*-conjugacy classes the values of* χ_n^+ *and* χ_n^- *differ by* $\pm i^m \sqrt{n}$*.*

Remark 3.10. (a) As in the previous theorem, in the case that $n = 2m + 1$ for some $m \geq 2$, let $\chi_n^{\pm}$ be the ordinary irreducible character corresponding to the simple $K\widetilde{\mathfrak{A}}_n$-module $S(n)_{\pm}$. Then $\chi_n \downarrow_{\widetilde{\mathfrak{A}}_n} = \chi_n^+ + \chi_n^-$. We will use the following convention in order to make the labelling of these two characters precise: by [8], Thm. 3.8, the set $\mathcal{C} := \{g \in \widetilde{\mathfrak{S}}_n \mid \tilde{\nu}(g) \text{ is an } n\text{-cycle}\}$ splits into two conjugacy classes

$$\mathcal{C}_1 := \{g \in \mathcal{C} \mid |\langle g \rangle| \text{ odd}\} \quad \text{and} \quad \mathcal{C}_2 := \{g \in \mathcal{C} \mid |\langle g \rangle| \text{ even}\}$$

of $\widetilde{\mathfrak{S}}_n$. Moreover, in consequence of [8], Thm. 3.9, each of these in turn splits into two $\widetilde{\mathfrak{A}}_n$-conjugacy classes. That is, we get

$$\mathcal{C}_{1,+} := \{g \in \mathcal{C}_1 \mid \tilde{\nu}(g) \sim_{\mathfrak{A}_n} (1,\ldots,n)\}, \quad \mathcal{C}_{1,-} := \{g \in \mathcal{C}_1 \mid \tilde{\nu}(g) \not\sim_{\mathfrak{A}_n} (1,\ldots,n)\},$$
$$\mathcal{C}_{2,+} := \{g \in \mathcal{C}_2 \mid \tilde{\nu}(g) \sim_{\mathfrak{A}_n} (1,\ldots,n)\}, \quad \mathcal{C}_{2,-} := \{g \in \mathcal{C}_2 \mid \tilde{\nu}(g) \not\sim_{\mathfrak{A}_n} (1,\ldots,n)\}.$$

Now let $g_1 \in \mathcal{C}_{1,+}$ and $g_1' \in \mathcal{C}_{1,-}$ so that $g_2 := g_1 z \in \mathcal{C}_{2,+}$ and $g_2' := g_1' z \in \mathcal{C}_{2,-}$ where z denotes the central involution in $\widetilde{\mathfrak{S}}_n$. The proof of [8], Thm. 6.9 shows that we may choose notation such that

$$\chi_n^-(g_1) = \chi_n^+(g_1) + i^m\sqrt{n}.$$

Furthermore, we have $\chi_n^-(g_i') = \chi_n^+(g_i)$ and $\chi_n^-(g_i) = \chi_n^+(g_i')$ for $i \in \{1,2\}$, and $\chi_n^\pm(g_2) = -\chi_n^\pm(g_1)$ and $\chi_n^\pm(g_2') = -\chi_n^\pm(g_1')$. Consequently, we deduce that

$$\chi_n^-(g_1') = \chi_n^+(g_1') - i^m\sqrt{n}, \quad \chi_n^-(g_2) = \chi_n^+(g_2) - i^m\sqrt{n}, \quad \chi_n^-(g_2') = \chi_n^+(g_2') + i^m\sqrt{n}.$$

By Theorem 3.9, this determines the characters χ_n^+ and χ_n^- completely.

(b) Let further χ_n^0 be the ordinary irreducible character corresponding to the simple $K\widetilde{\mathfrak{A}}_n$-module $S(n)_0$ in the case that n is even. Furthermore, as in Theorem 3.6, let φ_n be the Brauer character corresponding to the simple $F\mathfrak{S}_n$-module $D(n)$, and denote the Brauer characters corresponding to the simple $F\mathfrak{A}_n$-modules $E(n)_\pm$ and $E(n)_0$, respectively, by $\varphi_n^\pm$ and φ_n^0, respectively.

(c) In view of Theorem 3.6 we may in fact choose notation such that $\overline{\chi_n^+} = \varphi_n^+$ and $\overline{\chi_n^-} = \varphi_n^-$ if $n > 3$ is odd. Hence, $E(n)_+$ is then supposed to be the simple $F\mathfrak{A}_n$-module with Brauer character $\overline{\chi_n^+}$, and $E(n)_-$ is the one with Brauer character $\overline{\chi_n^-}$. If $n = 3$ then the Brauer characters φ_3^+ and φ_3^- are of course known as well, and we may distinguish them via

$$\varphi_3^\pm((1,2,3)) := \frac{1}{2}(-1 \pm i\sqrt{3}) \quad \text{and} \quad \varphi_3^\pm((1,3,2)) := \frac{1}{2}(-1 \mp i\sqrt{3}),$$

so that also $\varphi_3^-((1,2,3)) = \varphi_3^+((1,2,3)) + i^3\sqrt{3}$ and $\varphi_3^-((1,3,2)) = \varphi_3^+((1,3,2)) - i^3\sqrt{3}$. If $n \equiv 0 \pmod 4$, then we know from Proposition 3.3 that $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D(n)) \cong D(n-1)$. Therefore, we also have

$$\mathrm{Res}^{\mathfrak{A}_n}_{\mathfrak{A}_{n-1}}(E(n)_+ \oplus E(n)_-) \cong \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_{n-1}}(D(n)) \cong \mathrm{Res}^{\mathfrak{S}_{n-1}}_{\mathfrak{A}_{n-1}}(D(n-1)) \cong E(n-1)_+ \oplus E(n-1)_-.$$

In this case notation shall be chosen such that $E(n-1)_+ \cong \mathrm{Res}^{\mathfrak{A}_n}_{\mathfrak{A}_{n-1}}(E(n)_+)$ and $E(n-1)_- \cong \mathrm{Res}^{\mathfrak{A}_n}_{\mathfrak{A}_{n-1}}(E(n)_-)$. With this we now obtain the following:

Corollary 3.11. *Let $n \geq 4$.*

(i) If n is odd then $\overline{\chi_n^\pm} = \varphi_n^\pm$.

(ii) If $n \equiv 2 \pmod 4$ then $\overline{\chi_n^0} = \varphi_n^0$.

(iii) If $n \equiv 0 \pmod 4$ then $\overline{\chi_n^0} = \varphi_n^+ + \varphi_n^-$.

Lemma 3.12. *Let $n \geq 2$.*

(i) If $n \equiv 2 \pmod 4$ then $E(n)_0$ is selfdual.

(ii) If $n \equiv 1 \pmod 4$ then $E(n)_+$ and $E(n)_-$ are selfdual.

(iii) If $n \equiv 3 \pmod 4$ or $n \equiv 0 \pmod 4$ then $E(n)_+$ and $E(n)_-$ are not selfdual. Moreover, we then have $E(n)_+ \cong E(n)_-^*$.

Proof. (i) is immediate from Theorem 3.1 and the fact that $D(n)$ is selfdual.

(ii) We show that φ_n^+ and φ_n^- are real which then implies the selfduality of $E(n)_+$ and $E(n)_-$. For this let $g \in \widetilde{\mathfrak{A}}_n$ such that $\tilde{\nu}(g)$ is of cycle type μ. Then, by Theorem 3.7 and Theorem 3.9 we have

$$\mathbb{R} \ni \chi_n(g) = \chi_n^+(g) + \chi_n^-(g) = \begin{cases} 2\chi_n^+(g), & \text{if } \mu \neq (n) \\ 2\chi_n^+(g) \pm \sqrt{n}, & \text{if } \mu = (n). \end{cases}$$

Consequently, χ_n^+ and χ_n^- are real and, by Corollary 3.11, also φ_n^+ and φ_n^- are real.

(iii) Next we suppose that $n \equiv 3 \pmod 4$. If $n = 3$ then φ_3^+ and φ_3^- are not real, by Remark 3.10. Thus let now $n > 3$. For an element $g \in \widetilde{\mathfrak{A}}_n$ such that $\tilde{\nu}(g)$ has cycle type μ we then obtain

$$\mathbb{R} \ni \chi_n(g) = \chi_n^+(g) + \chi_n^-(g) = \begin{cases} 2\chi_n^+(g), & \text{if } \mu \neq (n) \\ 2\chi_n^+(g) \pm i\sqrt{n}, & \text{if } \mu = (n), \end{cases}$$

by Theorem 3.7 and Theorem 3.9. Since n is odd, the n-cycles are both even and 2-regular. Hence, for the unique element $g \in \widetilde{\mathfrak{A}}_n$ of odd order with $\tilde{\nu}(g) = (1, \ldots, n)$, we have

$$\varphi_n^\pm((1, \ldots, n)) = \overline{\chi_n^\pm}((1, \ldots, n)) = \chi_n^\pm(g) \notin \mathbb{R}.$$

Thus φ_n^+ and φ_n^- are not real in this case so that $E(n)_+$ and $E(n)_-$ are not selfdual.

Finally let $n \equiv 0 \pmod 4$. Then, as mentioned in Remark 3.10, we have $\mathrm{Res}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n)_\pm) \cong E(n-1)_\pm$ and $\varphi_n^\pm \downarrow \mathfrak{A}_{n-1} = \varphi_{n-1}^\pm$. As we have just seen, φ_{n-1}^+ and φ_{n-1}^- are not real, and thus also φ_n^+ and φ_n^- are not real either. Consequently, $E(n)_+$ and $E(n)_-$ are not selfdual.

Since $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D(n))$ is selfdual, we deduce that $E(n)_+ \cong E(n)_-^*$, in the cases where $n \equiv 3 \pmod 4$ or $n \equiv 0 \pmod 4$. This proves the lemma. $\square$

4 Vertices of $D(n)$ for odd n

In the following, let $n \geq 3$ be odd, and we retain the notation introduced in the preceding sections. The goal of this section is to prove part (i) of Theorem 1.1. That is, supposing that n has 2-adic expansion

$$n = 1 + \sum_{j=1}^{s} 2^{i_j},$$

for appropriate $s \in \mathbb{N}$ and $i_1 > \ldots > i_s \geq 1$, we show that $Q := \prod_{j=1}^{s} Q_{2^{i_j}}$ is a vertex of $D(n)$.

4.1 The vertices of $D(n)$

Remark 4.1. We write $n = 2m+1$, for some $m \geq 1$. Then $D(n) \cong \mathrm{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(D(n+1))$, and $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D(n)) =: M(2m)$ is indecomposable of composition length 2 with $\mathrm{Hd}(M(2m)) \cong D(n-1) \cong \mathrm{Soc}(M(2m))$, by Proposition 3.3. In particular, $D(n)$ and $M(2m)$ have common vertices. Therefore, in the following we determine the vertices of $M(2m)$. For this, we first of all mention the following important result:

Theorem 4.2 (Nagai, Uno, cf. [17], Prop. 3.1). *Let $m \geq 1$ and $i \in \{1, \ldots, m\}$. Then*

$$\mathrm{Res}^{\mathfrak{S}_{2m}}_{\mathfrak{S}_{2i} \times \mathfrak{S}_{2(m-i)}}(M(2m)) \cong M(2i) \boxtimes M(2(m-i)).$$

Corollary 4.3. *Again let $2m+1 = n = 1 + \sum_{j=1}^{s} 2^{i_j}$, with $s \in \mathbb{N}$ and $i_1 > \ldots > i_s \geq 1$, be the 2-adic expansion of n. Let further $\lambda := (2^{i_1}, \ldots, 2^{i_s})$. Then*

$$\mathrm{Res}^{\mathfrak{S}_{2m}}_{\mathfrak{S}_\lambda}(M(2m)) \cong M(2^{i_1}) \boxtimes \cdots \boxtimes M(2^{i_s}).$$

In particular, if V_j is a vertex of $M(2^{i_j})$, for $j = 1, \ldots, s$, then $\prod_{j=1}^{s} V_j$ is a vertex of $M(2m)$, and thus a vertex of $D(n)$.

4.2 The vertices of $M(2^r)$

Due to Corollary 4.3 above, it suffices to determine the vertices of $M(2^r)$, for $r \in \mathbb{N}$. This will be done in the course of this subsection. Namely, we aim to show that the following holds:

Proposition 4.4. *For $r \in \mathbb{N}$, the Sylow 2-subgroups of $\mathfrak{A}_{2^r}$ are vertices of $M(2^r)$.*

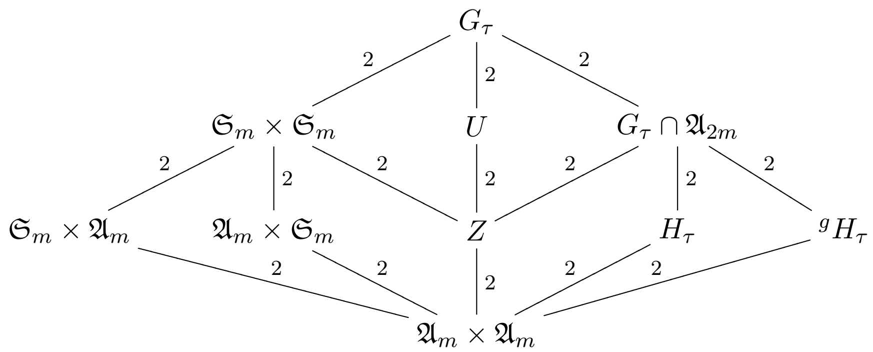
Proof. We argue by induction on r, and for this set $m := 2^{r-1}$ so that $2^r = 2m$. We know that every vertex of $M(2m)$ is also a vertex of $D(2m+1)$. In the cases where $r \in \{1, 2\}$ the assertion thus follows from [4], so we may from now on assume that $r \geq 3$. Then, by Corollary 3.2, every vertex of $D(2m+1)$ is contained in $\mathfrak{A}_{2m+1}$ and therefore in $\mathfrak{A}_{2m}$. Hence every vertex of $M(2m)$ is contained in a Sylow 2-subgroup of $\mathfrak{A}_{2m}$, and in order to show equality, we set

$$G_\tau := (\mathfrak{S}_m \times \mathfrak{S}_m)\langle\tau\rangle \quad \text{where } \tau := (1, m+1)(2, m+2) \cdots (m, 2m).$$

Then G_τ is a subgroup of $\mathfrak{S}_{2m}$ isomorphic to the wreath product $\mathfrak{S}_m \wr \mathfrak{S}_2$. Moreover, $|\mathfrak{S}_{2m} : G_\tau|$ is odd, and $X := \mathrm{Res}^{\mathfrak{S}_{2m}}_{G_\tau}(M(2m))$ is indecomposable, since $\mathrm{Res}^{G_\tau}_{\mathfrak{S}_m \times \mathfrak{S}_m}(X) = \mathrm{Res}^{\mathfrak{S}_{2m}}_{\mathfrak{S}_m \times \mathfrak{S}_m}(M(2m)) \cong M(m) \boxtimes M(m)$, by Theorem 4.2. Thus every vertex of X is also a vertex of $M(2m)$, and is therefore contained in $G_\tau \cap \mathfrak{A}_{2m}$. By induction, $M(m)$ has vertex $Q_m \in \mathrm{Syl}_2(\mathfrak{A}_m)$, so that $M(m) \boxtimes M(m)$ has vertex $Q_m \times Q_m \in \mathrm{Syl}_2(\mathfrak{A}_m \times \mathfrak{A}_m)$, by Proposition 2.1. Consequently X has a vertex P such that

$$Q_m \times Q_m \leq P \leq G_\tau \cap \mathfrak{A}_{2m}.$$

Note that $\mathfrak{A}_m \times \mathfrak{A}_m$ is a normal subgroup of G_τ such that $G_\tau/(\mathfrak{A}_m \times \mathfrak{A}_m)$ is a dihedral group of order 8. We illustrate the subgroups of G_τ containing $\mathfrak{A}_m \times \mathfrak{A}_m$ as follows:



Here $Z = (\mathfrak{A}_m \times \mathfrak{A}_m)\langle(1,2)(m+1,m+2)\rangle$, $H_\tau = (\mathfrak{A}_m \times \mathfrak{A}_m)\langle\tau\rangle$, and g is a suitable element in G_τ. Since $\mathrm{Res}^{G_\tau}_{\mathfrak{S}_m \times \mathfrak{S}_m}(X)$ is indecomposable, X cannot be relatively $\mathfrak{S}_m \times \mathfrak{S}_m$-projective, by Green's Indecomposability Theorem. Since X is relatively $P(\mathfrak{A}_m \times \mathfrak{A}_m)$-projective, this implies that $P(\mathfrak{A}_m \times \mathfrak{A}_m) \not\leq \mathfrak{S}_m \times \mathfrak{S}_m$. On the other hand, we have $P(\mathfrak{A}_m \times \mathfrak{A}_m) \leq G_\tau \cap \mathfrak{A}_{2m}$. We claim that it suffices to show that X is not relatively H_τ-projective. For then X is not relatively ${}^gH_\tau$-projective either, so we must have $P(\mathfrak{A}_m \times \mathfrak{A}_m) = G_\tau \cap \mathfrak{A}_{2m}$. This implies

$$\begin{aligned} |P : Q_m \times Q_m| &= |P : P \cap (\mathfrak{A}_m \times \mathfrak{A}_m)| = |P(\mathfrak{A}_m \times \mathfrak{A}_m) : \mathfrak{A}_m \times \mathfrak{A}_m| \\ &= |G_\tau \cap \mathfrak{A}_{2m} : \mathfrak{A}_m \times \mathfrak{A}_m| = 4, \end{aligned}$$

so that $P \in \mathrm{Syl}_2(G_\tau \cap \mathfrak{A}_{2m}) \subseteq \mathrm{Syl}_2(\mathfrak{A}_{2m})$. Since P is a vertex of X and $M(2m)$, the result will then follow.

Thus what remains to prove is that $X = \mathrm{Res}^{\mathfrak{S}_{2m}}_{G_\tau}(M(2m))$ is not relatively H_τ-projective. This will be done below, by computations in the G_τ-algebra $\mathrm{End}_{F[\mathfrak{A}_m \times \mathfrak{A}_m]}(M(2m))$. □

For the remainder of this subsection, we suppose that $2m = 2^r$, for some $r \geq 3$, and show that the following holds:

Proposition 4.5. *Let $X := \mathrm{Res}^{\mathfrak{S}_{2m}}_{G_\tau}(M(2m)) \cong \mathrm{Res}^{\mathfrak{S}_{2m+2}}_{G_\tau}(D(2m+2))$ where $G_\tau := (\mathfrak{S}_m \times \mathfrak{S}_m)\langle\tau\rangle$ and $\tau := (1, m+1)(2, m+2)\cdots(m, 2m)$ as in Proposition 4.4. Moreover, let $H_\tau := (\mathfrak{A}_m \times \mathfrak{A}_m)\langle\tau\rangle$. Then X is not relatively H_τ-projective.*

This will complete the proof of Proposition 4.4, and together with Corollary 4.3 we will then immediately deduce assertion (i) of Theorem 1.1. In order to prove Proposition 4.5, we proceed as follows: we investigate the endomorphism algebra

$$\begin{aligned} \mathrm{End}_{F[\mathfrak{A}_m \times \mathfrak{A}_m]}(M(2m)) &\cong \mathrm{End}_{F[\mathfrak{A}_m \times \mathfrak{A}_m]}(M(m) \boxtimes M(m)) \\ &\cong \mathrm{End}_{F\mathfrak{A}_m}(M(m)) \otimes_F \mathrm{End}_{F\mathfrak{A}_m}(M(m)) \end{aligned}$$

and its structure as a G_τ-algebra. For the fixed points under the G_τ-action on the endomorphism algebra $\mathrm{End}_{F[\mathfrak{A}_m \times \mathfrak{A}_m]}(M(2m))$ are precisely the elements in $\mathrm{End}_{FG_\tau}(M(2m)) = \mathrm{End}_{FG_\tau}(D(2m+2))$. Having determined $\mathrm{End}_{FG_\tau}(D(2m+2))$, we will then also be able to show that the relative trace map

$$\mathrm{Tr}^{G_\tau}_{H_\tau} : \mathrm{End}_{FH_\tau}(D(2m+2)) \longrightarrow \mathrm{End}_{FG_\tau}(D(2m+2))$$

is not surjective so that $D(2m+2)$ cannot be relatively H_τ-projective, by Higman's Criterion (cf. [16], Thm. 4.2.2).

Now we begin our investigation of $\mathrm{End}_{F[\mathfrak{A}_m\times\mathfrak{A}_m]}(M(2m))$ with the following:

Proposition 4.6. *In the situation above, we have* $\mathrm{Res}^{\mathfrak{S}_m}_{\mathfrak{A}_m}(M(m)) = N_+ \oplus N_-$ *where*

$$N_+ \sim \begin{bmatrix} E(m)_- \\ E(m)_+ \end{bmatrix} \qquad \text{and} \qquad N_- \sim \begin{bmatrix} E(m)_+ \\ E(m)_- \end{bmatrix}$$

are indecomposable and selfdual $F\mathfrak{A}_m$-modules which are conjugate in $\mathfrak{S}_m$.

Proof. We know that

$$\mathrm{Res}^{\mathfrak{S}_{m+1}}_{\mathfrak{S}_m}(D(m+1)) = M(m) \sim \begin{bmatrix} D(m) \\ D(m) \end{bmatrix},$$

and that $\mathrm{Res}^{\mathfrak{S}_m}_{\mathfrak{A}_m}(D(m)) \cong E(m)_+ \oplus E(m)_-$. Since $D(m+1)$ is relatively $\mathfrak{A}_{m+1}$-projective, $M(m)$ is relatively $\mathfrak{A}_m$-projective. Thus

$$\mathrm{Res}^{\mathfrak{S}_m}_{\mathfrak{A}_m}(M(m)) = N_+ \oplus N_-$$

with indecomposable conjugate modules N_+ and N_- of composition length 2. We can choose notation such that $\mathrm{Soc}(N_\pm) \cong E(m)_\pm$. On the other hand, Theorem 3.1 implies that

$$\begin{aligned} \mathrm{Res}^{\mathfrak{S}_m}_{\mathfrak{A}_m}(M(m)) &\cong \mathrm{Res}^{\mathfrak{S}_{m+1}}_{\mathfrak{A}_m}(D(m+1)) \cong \mathrm{Res}^{\mathfrak{A}_{m+1}}_{\mathfrak{A}_m}(E(m+1)_+ \oplus E(m+1)_-) \\ &\cong \mathrm{Res}^{\mathfrak{A}_{m+1}}_{\mathfrak{A}_m}(E(m+1)_+) \oplus \mathrm{Res}^{\mathfrak{A}_{m+1}}_{\mathfrak{A}_m}(E(m+1)_-). \end{aligned}$$

That is $N_\pm \cong \mathrm{Res}^{\mathfrak{A}_{m+1}}_{\mathfrak{A}_m}(E(m+1)_\pm)$ or $N_\pm \cong \mathrm{Res}^{\mathfrak{A}_{m+1}}_{\mathfrak{A}_m}(E(m+1)_\mp)$. Since $m+1 \equiv 1 \pmod 4$, the modules $E(m+1)_+$ and $E(m+1)_-$ are selfdual, by Lemma 3.12, and hence also N_+ and N_- are selfdual. Consequently, we deduce $\mathrm{Hd}(N_+) \cong \mathrm{Soc}(N_+)^* \cong E(m)_+^* \cong E(m)_-$ and $\mathrm{Hd}(N_-) \cong \mathrm{Soc}(N_-)^* \cong E(m)_-^* \cong E(m)_+$, by Lemma 3.12. □

Remark 4.7. (The endomorphism algebra $\mathrm{End}_{F[\mathfrak{A}_m\times\mathfrak{A}_m]}(M(2m))$.)
The proposition above allows us to compute the $\mathfrak{S}_m$-algebra $\mathrm{End}_{F\mathfrak{A}_m}(M(m))$. Obviously, $\mathrm{End}_{F\mathfrak{A}_m}(M(m))$ is 4-dimensional with basis $\{e, f, x, y\}$ where e and f are the projections onto N_+ and N_-, respectively. Furthermore, x annihilates N_+ and maps N_- onto $\mathrm{Soc}(N_+)$, whereas y annihilates N_- and maps N_+ onto $\mathrm{Soc}(N_-)$. Then the following relations hold:

$$\begin{aligned} &e^2 = e, && ef = 0 = fe, && f^2 = f, \\ &ex = x = xf, && fx = 0 = xe, && \\ &ey = 0 = yf, && fy = y = ye, && \\ &x^2 = 0, && xy = 0 = yx, && y^2 = 0. \end{aligned}$$

Moreover, an element $t \in \mathfrak{S}_m \setminus \mathfrak{A}_m$ acts on $\mathrm{End}_{F\mathfrak{A}_m}(M(m))$ by interchanging e and f, and x and y, provided that y is suitably normalized.

Next we compute the $\mathfrak{S}_m \times \mathfrak{S}_m$-algebra structure of

$$\begin{aligned}\mathrm{End}_{F[\mathfrak{A}_m\times\mathfrak{A}_m]}(M(2m)) &\cong \mathrm{End}_{F[\mathfrak{A}_m\times\mathfrak{A}_m]}(M(m)\boxtimes M(m))\\ &\cong \mathrm{End}_{F\mathfrak{A}_m}(M(m))\otimes_F \mathrm{End}_{F\mathfrak{A}_m}(M(m)).\end{aligned}$$

For simplicity of notation, we read all these isomorphisms as equalities. Then the algebra $\mathrm{End}_{F[\mathfrak{A}_m\times\mathfrak{A}_m]}(M(2m))$ has dimension 16 and basis:

$$\mathfrak{B} := \{e\otimes e, e\otimes f, f\otimes e, f\otimes f, e\otimes x, x\otimes e, e\otimes y, y\otimes e, f\otimes x, x\otimes f, f\otimes y, y\otimes f, x\otimes x, x\otimes y, y\otimes x, y\otimes y\}.$$

Moreover, the $\mathfrak{S}_m\times\mathfrak{S}_m$-algebra structure of $\mathrm{End}_{F[\mathfrak{A}_m\times\mathfrak{A}_m]}(M(2m))$ is the obvious one, coming from the action of $\mathfrak{S}_m \times \mathfrak{S}_m$ on $\mathrm{End}_{F\mathfrak{A}_m}(M(m))\otimes_F \mathrm{End}_{F\mathfrak{A}_m}(M(m))$.

Remark 4.8. (The endomorphism algebra $\mathrm{End}_{FG_\tau}(M(2m))$.)
In order to complete the proof of Proposition 4.4, we need to know the G_τ-algebra structure of $\mathrm{End}_{F[\mathfrak{A}_m\times\mathfrak{A}_m]}(M(2m))$ where $G_\tau = (\mathfrak{S}_m\times\mathfrak{S}_m)\langle\tau\rangle \cong \mathfrak{S}_m \wr \mathfrak{S}_2$, as before. Thus the only thing which is still missing at this point is the action of τ on $\mathrm{End}_{F[\mathfrak{A}_m\times\mathfrak{A}_m]}(M(2m))$. Of course, this action is induced by the action of τ on the $F[\mathfrak{S}_m\times\mathfrak{S}_m]$-module $M(2m)\cong M(m)\boxtimes M(m)$. Now there is a canonical extension of the $F[\mathfrak{S}_m\times\mathfrak{S}_m]$-module $M(m)\boxtimes M(m)$ to $\mathfrak{S}_m\wr\mathfrak{S}_2$ obtained by letting the subgroup $\mathfrak{S}_2$ of $\mathfrak{S}_m\wr\mathfrak{S}_2$ interchange the two factors of the outer tensor product. However, the action of τ on $M(2m) = M(m)\boxtimes M(m)$ does **not** just permute the two factors of the outer tensor product. This follows from the fact, stated in [17], that the restriction of $M(2m)$ to $F\langle\tau\rangle$ is a free module.

In order to overcome the difficulty that the action of τ on $M(m)\boxtimes M(m)$ is not the obvious one, we need to take a little detour. Recall from Proposition 3.3 that

$$M(2m) \cong \mathrm{Res}^{\mathfrak{S}_{2m+2}}_{\mathfrak{S}_{2m}}(D(2m+2)).$$

From [17], Prop. 3.1 we deduce that

$$\mathrm{Res}^{\mathfrak{S}_{2m+2}}_{\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1}}(D(2m+2)) \cong D(m+1)\boxtimes D(m+1).$$

Now $\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1} \le (\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1})\langle\sigma\rangle \cong \mathfrak{S}_{m+1}\wr\mathfrak{S}_2$ where

$$\sigma := (1, m+2)(2, m+3)\cdots(m+1, 2m+2) \in \mathfrak{S}_{2m+2}.$$

Since $D(m+1)\boxtimes D(m+1)$ is a simple $F[\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1}]$-module, [10], Cor. VII. 9.13 implies that $D(m+1)\boxtimes D(m+1)$ has a unique extension to $(\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1})\langle\sigma\rangle$, up to isomorphism. Thus $\mathrm{Res}^{\mathfrak{S}_{2m+2}}_{(\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1})\langle\sigma\rangle}(D(2m+2))$ must coincide with the canonical extension of the simple $F[\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1}]$-module $D(m+1)\boxtimes D(m+1)$ to $(\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1})\langle\sigma\rangle \cong \mathfrak{S}_{m+1}\wr\mathfrak{S}_2$.

Remark 4.9. (The endomorphism algebra $\mathrm{End}_{FG_\sigma}(M(2m))$.)
Now we view $\mathfrak{S}_m$ as a subgroup of $\mathfrak{S}_{m+1}$, and $\mathfrak{S}_m\times\mathfrak{S}_m$ as a subgroup of $\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1}$. Then $G_\sigma := (\mathfrak{S}_m\times\mathfrak{S}_m)\langle\sigma\rangle$ becomes a subgroup of $(\mathfrak{S}_{m+1}\times\mathfrak{S}_{m+1})\langle\sigma\rangle$, and $G_\sigma\cong\mathfrak{S}_m\wr\mathfrak{S}_2$. Moreover, $\mathrm{Res}^{\mathfrak{S}_{2m+2}}_{G_\sigma}(D(2m+2))$ is now the canonical extension of the $F[\mathfrak{S}_m\times\mathfrak{S}_m]$-module $M(m)\boxtimes M(m)$ to $G_\sigma\cong\mathfrak{S}_m\wr\mathfrak{S}_2$. It will be important to not confuse the two isomorphic copies of $\mathfrak{S}_m\wr\mathfrak{S}_2$ in $\mathfrak{S}_{2m+2}$. Whereas G_τ fixes two points in $\{1,\ldots,2m+2\}$, this is not the case for G_σ; in particular, G_τ and G_σ are **not** conjugate in $\mathfrak{S}_{2m+2}$. In fact, in order to

understand the difference between G_τ and G_σ better, we now view the subgroup $\mathfrak{S}_m \times \mathfrak{S}_m$ of G_τ as acting on $\{1, \ldots, m\} \cup \{m+2, \ldots, 2m+1\}$. Then τ becomes the permutation

$$(1, m+2)(2, m+3) \cdots (m, 2m+1)$$

which we denote by τ again. With this convention, G_σ and G_τ have the common subgroup $\mathfrak{S}_m \times \mathfrak{S}_m$ of index 2, but

$$\sigma = \tau(m+1, 2m+2).$$

The remarks above imply that the G_σ-algebra structure of

$$\mathrm{End}_{F[\mathfrak{A}_m \times \mathfrak{A}_m]}(M(2m)) = \mathrm{End}_{F\mathfrak{A}_m}(M(m)) \otimes_F \mathrm{End}_{F\mathfrak{A}_m}(M(m))$$

is the obvious one: σ interchanges the two factors of the tensor product, and $\mathfrak{S}_m \times \mathfrak{S}_m$ acts as before. The basis $\mathfrak{B}$ of $\mathrm{End}_{F[\mathfrak{A}_m \times \mathfrak{A}_m]}(M(2m))$ is permuted under this action, and the corresponding orbits are:

$$\begin{aligned}
\mathfrak{B}_1 &:= \{e \otimes e, e \otimes f, f \otimes e, f \otimes f\},\\
\mathfrak{B}_2 &:= \{e \otimes x, x \otimes e, f \otimes x, x \otimes f, e \otimes y, y \otimes e, f \otimes y, y \otimes f\},\\
\mathfrak{B}_3 &:= \{x \otimes x, x \otimes y, y \otimes x, y \otimes y\}.
\end{aligned}$$

Thus the orbit sums $\mathfrak{B}_1^+ = 1 = \mathrm{id}_{M(2m)}$, $\mathfrak{B}_2^+$, $\mathfrak{B}_3^+$ form a basis of $\mathrm{End}_{FG_\sigma}(M(2m))$. One checks easily that $(F\mathfrak{B}_2^+ + F\mathfrak{B}_3^+)^2 = 0$, so that $\mathbf{J}(\mathrm{End}_{FG_\sigma}(M(2m))) = F\mathfrak{B}_2^+ + F\mathfrak{B}_3^+$. Since the F-algebra $\mathrm{End}_{FG_\sigma}(M(2m))$ is local, its group of units is

$$F^\times \times (1 + \mathbf{J}(\mathrm{End}_{FG_\sigma}(M(2m)))),$$

and all non-trivial elements in $1 + \mathbf{J}(\mathrm{End}_{FG_\sigma}(M(2m)))$ have order 2.

Remark 4.10. (The endomorphism algebra $\mathrm{End}_{FG_\tau}(M(2m))$ – continued.)
We are really interested in the G_τ-algebra structure of $\mathrm{End}_{F[\mathfrak{A}_m \times \mathfrak{A}_m]}(M(2m))$ (not in the G_σ-algebra structure). In the following, we denote by Δ the representation of $\mathfrak{S}_{2m+2}$ over F afforded by $D(2m+2)$. Note that

$$\Delta(\tau) = \Delta(\sigma(m+1, 2m+2)) = \Delta(\sigma)\Gamma = \Gamma\Delta(\sigma)$$

where $\Gamma = \Delta((m+1, 2m+2)) \in \mathrm{End}_{FG_\sigma}(M(2m))$. Since $\Gamma^2 = 1$, we can write

$$\Gamma = 1 + \beta\mathfrak{B}_2^+ + \gamma\mathfrak{B}_3^+,$$

with β, $\gamma \in F$.

The elements in $\mathrm{End}_{FH_\tau}(M(2m))$ are precisely those elements $a \in \mathrm{End}_{F[\mathfrak{A}_m \times \mathfrak{A}_m]}(M(2m))$ satisfying $\Delta(\tau)a = a\Delta(\tau)$. This last condition is equivalent to

$$\Gamma \cdot {}^\sigma a = a\Gamma. \tag{1}$$

Writing

$$\begin{aligned}
a = {} & \alpha_1(e \otimes e) + \alpha_2(e \otimes f) + \alpha_3(f \otimes e) + \alpha_4(f \otimes f) + \alpha_5(x \otimes e) + \alpha_6(e \otimes x) + \alpha_7(f \otimes x)\\
& + \alpha_8(x \otimes f) + \alpha_9(y \otimes e) + \alpha_{10}(e \otimes y) + \alpha_{11}(f \otimes y) + \alpha_{12}(y \otimes f) + \alpha_{13}(x \otimes x)\\
& + \alpha_{14}(x \otimes y) + \alpha_{15}(y \otimes x) + \alpha_{16}(y \otimes y),
\end{aligned}$$

with $\alpha_1, \ldots, \alpha_{16} \in F$, a routine calculation shows that (1) holds if and only if the following equations are satisfied:

$$\begin{aligned}
\alpha_2 &= \alpha_3, \\
\alpha_5 + \alpha_6 &= \beta(\alpha_1 + \alpha_2), \\
\alpha_7 + \alpha_8 &= \beta(\alpha_2 + \alpha_4), \\
\alpha_9 + \alpha_{10} &= \alpha_5 + \alpha_6, \\
\alpha_{11} + \alpha_{12} &= \alpha_7 + \alpha_8, \\
\alpha_{14} + \alpha_{15} &= \beta(\alpha_6 + \alpha_8 + \alpha_{10} + \alpha_{12}), \\
\gamma(\alpha_1 + \alpha_4) &= \beta^2(\alpha_1 + \alpha_4).
\end{aligned}$$

We distinguish between four cases.

Case 1: $\beta = 0 = \gamma$. Then $\Gamma = 1$, and

$$\begin{aligned}
a = {} & \alpha_1(e \otimes e) + \alpha_2(e \otimes f + f \otimes e) + \alpha_4(f \otimes f) + \alpha_5(x \otimes e + e \otimes x) + \alpha_7(f \otimes x + x \otimes f) \\
& + \alpha_9(y \otimes e + e \otimes y) + \alpha_{11}(f \otimes y + y \otimes f) + \alpha_{13}(x \otimes x) + \alpha_{14}(x \otimes y + y \otimes x) + \alpha_{16}(y \otimes y).
\end{aligned}$$

The coefficients α_i can be chosen arbitrarily, for $i \in \{1, 2, 4, 5, 7, 9, 11, 13, 14, 16\}$.

Case 2: $\beta = 0 \neq \gamma$. Then

$$\begin{aligned}
a = {} & \alpha_1(e \otimes e + f \otimes f) + \alpha_2(e \otimes f + f \otimes e) + \alpha_5(x \otimes e + e \otimes x) + \alpha_7(f \otimes x + x \otimes f) \\
& + \alpha_9(y \otimes e + e \otimes y) + \alpha_{11}(f \otimes y + y \otimes f) + \alpha_{13}(x \otimes x) + \alpha_{14}(x \otimes y + y \otimes x) + \alpha_{16}(y \otimes y).
\end{aligned}$$

Here the coefficients α_i, for $i \in \{1, 2, 5, 7, 9, 11, 13, 14, 16\}$, can be chosen arbitrarily.

Case 3: $\beta \neq 0$ and $\gamma \neq \beta^2$. Then

$$\begin{aligned}
a = {} & \alpha_1(e \otimes e + f \otimes f) + \alpha_2(e \otimes f + f \otimes e) + \alpha_5(x \otimes e + e \otimes x) \\
& + \alpha_1\beta(e \otimes x + x \otimes f + e \otimes y + y \otimes f) + \alpha_2\beta(e \otimes x + x \otimes f + e \otimes y + y \otimes f) \\
& + \alpha_7(f \otimes x + x \otimes f) + \alpha_9(y \otimes e + e \otimes y) + \alpha_{11}(f \otimes y + y \otimes f) \\
& + \alpha_{13}(x \otimes x) + \alpha_{14}(x \otimes y + y \otimes x) + \beta(\alpha_5 + \alpha_7 + \alpha_9 + \alpha_{11})(y \otimes x) + \alpha_{16}(y \otimes y),
\end{aligned}$$

and the coefficients α_i can be chosen arbitrarily, for $i \in \{1, 2, 5, 7, 9, 11, 13, 14, 16\}$.

Case 4: $0 \neq \gamma = \beta^2$. Then

$$\begin{aligned}
a = {} & \alpha_1(e \otimes e) + \alpha_2(e \otimes f + f \otimes e) + \alpha_4(f \otimes f) + \alpha_5(x \otimes e + e \otimes x) + \alpha_1\beta(e \otimes x + e \otimes y) \\
& + \alpha_2\beta(e \otimes x + x \otimes f + e \otimes y + y \otimes f) + \alpha_4\beta(x \otimes f + y \otimes f) + \alpha_7(f \otimes x + x \otimes f) \\
& + \alpha_9(y \otimes e + e \otimes y) + \alpha_{11}(f \otimes y + y \otimes f) + \alpha_{13}(x \otimes x) + \alpha_{14}(x \otimes y + y \otimes x) \\
& + \alpha_{16}(y \otimes y) + \beta(\alpha_5 + \alpha_7 + \alpha_9 + \alpha_{11})(y \otimes x),
\end{aligned}$$

and the coefficients α_i can be chosen arbitrarily, for $i \in \{1, 2, 4, 5, 7, 9, 11, 13, 14, 16\}$.

These computations now lead to the following:

Proposition 4.11. *Keep the notation from the previous remark. Then $\Delta(\tau) = \Delta(\sigma)\Gamma$, for some $\Gamma = 1 + \beta\mathfrak{B}_2^+ + \gamma\mathfrak{B}_3^+ \in \mathrm{End}_{FG_\sigma}(D(2m+2))$. Moreover, $\gamma \neq \beta^2$, and the relative trace map*

$$\mathrm{Tr}_{H_\tau}^{G_\tau} : \mathrm{End}_{FH_\tau}(D(2m+2)) \longrightarrow \mathrm{End}_{FG_\tau}(D(2m+2))$$

is not surjective.

Proof. We again consider the indecomposable $F[\mathfrak{S}_m \times \mathfrak{S}_m]$-module

$$M(m) \boxtimes M(m) \cong \mathrm{Res}_{\mathfrak{S}_m \times \mathfrak{S}_m}^{\mathfrak{S}_{2m+2}}(D(2m+2)).$$

Then $M(m) \boxtimes M(m)$ canonically becomes an FG_τ-module, by letting τ permute the two factors of the outer tensor product. We denote this module by $M(m)_\tau^{\otimes 2}$. We now assume that $\gamma = \beta^2$, and show that then

$$M(m)_\tau^{\otimes 2} \cong \mathrm{Res}_{G_\tau}^{\mathfrak{S}_{2m+2}}(D(2m+2)) = \mathrm{Res}_{G_\tau}^{\mathfrak{S}_{2m}}(M(2m)). \tag{2}$$

For this, let Δ_τ be the representation of G_τ over F afforded by $M(m)_\tau^{\otimes 2}$, and as before let Δ be the representation of $\mathfrak{S}_{2m+2}$ over F afforded by $D(2m+2)$. Furthermore, we set $b := 1 + \beta(e \otimes x + f \otimes x + e \otimes y + f \otimes y) \in \mathrm{End}_{F[\mathfrak{S}_m \times \mathfrak{S}_m]}(D(2m+2))$. Then $b^2 = 1$, and

$$\Delta(g)b = b\Delta(g), \text{ for } g \in \mathfrak{S}_m \times \mathfrak{S}_m.$$

One also checks easily that $b = {}^{\sigma}b\Gamma$, or equivalently,

$$\Delta(\sigma)b = b\Delta(\sigma)\Gamma.$$

Since $\Delta_\tau(g) = \Delta(g)$, for $g \in \mathfrak{S}_m \times \mathfrak{S}_m$, and $\Delta(\tau) = \Delta(\sigma)\Gamma = \Delta_\tau(\tau)\Gamma$, this implies the claimed equation (2).

By [17], L. 3.2, $\mathrm{Res}_{\langle\tau\rangle}^{\mathfrak{S}_{2m}}(M(2m))$ is projective, whereas $\mathrm{Res}_{\langle\tau\rangle}^{G_\tau}(M(m)_\tau^{\otimes 2})$ is, by construction, not projective, a contradiction.

Therefore, we deduce that $\gamma \neq \beta^2$, and we are then either in case 2 or case 3 of the previous remark. That is either $\beta = 0 \neq \gamma$, or $\beta \neq 0$ and $\gamma \neq \beta^2$. We now take $a \in \mathrm{End}_{FH_\tau}(D(2m+2))$. Then the calculations in Remark 4.10 imply that

$$\mathrm{Tr}_{H_\tau}^{G_\tau}(a) = a + {}^{(1\times t)}a + {}^{(t\times 1)}a + {}^{(t\times t)}a \in \mathrm{span}_F(\{\mathfrak{B}_2^+, \mathfrak{B}_3^+\})$$

where again t is an element in $\mathfrak{S}_m \setminus \mathfrak{A}_m$. That is,

$$\mathrm{Tr}_{H_\tau}^{G_\tau}(\mathrm{End}_{FH_\tau}(D(2m+2))) \subseteq \mathbf{J}(\mathrm{End}_{FG_\tau}(D(2m+2))),$$

and the assertion of the proposition follows. □

Remark 4.12. The previous proposition now finishes the proof of Theorem 1.1 (i).

5 Vertices of $D(n)$ for even n

In this section we investigate the case where n is even, and prove parts (ii) and (iii) of Theorem 1.1. That is, we show that $D(n)$ has vertex P_n if $n \equiv 2 \pmod 4$, and vertex Q_n if $n \equiv 0 \pmod 4$. In the proof we will make use of part (i) of Theorem 1.1 which has already been established.

5.1 The case $n \equiv 2 \pmod 4$

For the following we suppose that $n \geq 2$ such that $n \equiv 2 \pmod 4$, and write $n = 2m+2$, for some $m \equiv 0 \pmod 2$. Furthermore, we consider the 2-adic expansion of n:

$$n = 2 + \sum_{j=1}^{s} 2^{i_j},$$

for appropriate $s \in \mathbb{N}_0$ and $i_1 > i_2 > \ldots > i_s \geq 2$. Then we obtain the following:

Proposition 5.1. *If $n \equiv 2 \pmod 4$ then $D(n)$ has vertex P_n.*

Proof. For $n = 2$ the assertion trivially holds, since then $D(n) = D(2) = F$. Therefore, we may now suppose that $n > 2$. By Proposition 3.3, we know that $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D(n)) \cong D(n-1)$. Moreover, as we have just proved in the previous section,

$$Q := \prod_{j=1}^{s} Q_{2^{i_j}}$$

is a vertex of $D(n-1)$. Let $Q \leq P_n \in \mathrm{Syl}_2(\mathfrak{S}_n)$. Then each orbit of P_n on $\{1, \ldots, n\}$ is a union of orbits of Q. Thus

$$P_n = \prod_{j=1}^{s} P_{2^{i_j}} \times P_2.$$

In particular, P_n is the only Sylow 2-subgroup of $\mathfrak{S}_n$ containing Q. Hence $D(n)$ has a vertex P such that $Q < P \leq P_n$. We assume that $P \neq P_n$. Then P is contained in a maximal subgroup $\widetilde{P}$ of P_n, so that $D(n)$ is relatively $\widetilde{P}$-projective. Moreover, $\widetilde{P}/Q$ is a maximal subgroup of the elementary abelian 2-group

$$P_n/Q \cong \prod_{j=1}^{s} (P_{2^{i_j}}/Q_{2^{i_j}}) \times P_2. \tag{3}$$

Hence $\widetilde{P}/Q$ is the kernel of a non-trivial homomorphism $\bar{\lambda} : P_n/Q \longrightarrow \mathbb{C}^{\times}$. Identifying both sides of (3), $\bar{\lambda}$ has the form

$$\bar{\lambda} = \bar{\lambda}_1 \times \cdots \times \bar{\lambda}_s \times \bar{\lambda}_{s+1}$$

with homomorphisms $\bar{\lambda}_j : P_{2^{i_j}}/Q_{2^{i_j}} \longrightarrow \mathbb{C}^{\times}$ for $j = 1, \ldots, s$, and $\bar{\lambda}_{s+1} : P_2 \longrightarrow \mathbb{C}^{\times}$. For each $j \in \{1, \ldots, s+1\}$, the inflation λ_j of $\bar{\lambda}_j$ to $P_{2^{i_j}}$ is either trivial or the restriction of the sign character of $\mathfrak{S}_{2^{i_j}}$. We set

$$\mathcal{L} := \{ j \mid 1 \leq j \leq s+1,\ \lambda_j \text{ non-trivial } \} \neq \emptyset.$$

Then

$$\widetilde{P} \leq \prod_{j \notin \mathcal{L}} \mathfrak{S}_{2^{i_j}} \times (\mathfrak{A}_n \cap \prod_{j \in \mathcal{L}} \mathfrak{S}_{2^{i_j}}). \tag{4}$$

We write $\sum_{j \notin \mathcal{L}} 2^{i_j} = 2k$ so that $2k < n$. Then the group on the right hand side of (4) is conjugate to a subgroup of $\mathfrak{S}_{2k} \times \mathfrak{A}_{n-2k}$. We conclude that $D(n)$ is relatively $(\mathfrak{S}_{2k} \times \mathfrak{A}_{n-2k})$-projective. In particular, $2k \neq 0$, since $D(n)$ is not relatively $\mathfrak{A}_n$-projective. In the following, we abbreviate $G := \mathfrak{S}_{2k} \times \mathfrak{S}_{n-2k}$, $H := \mathfrak{S}_{2k} \times \mathfrak{A}_{n-2k}$ and $X := \mathrm{Res}^{\mathfrak{S}_n}_{G}(D(n))$. By [17], Prop.

3.1, X is indecomposable. Since $|\mathfrak{S}_n : G|$ is odd, $D(n)$ and X have a common vertex. It suffices to show that $\mathrm{Res}_H^G(X) = \mathrm{Res}_H^{\mathfrak{S}_n}(D(n))$ is indecomposable, for then Green's Indecomposability Theorem will imply that X is not relatively H-projective. So $D(n)$ is not relatively H-projective either, and we have a contradiction.

In order to show that $\mathrm{Res}_H^G(X)$ is indecomposable, we set $t_i := \sigma_{2i-1} = (2i-1, 2i)$ for $i = 1, \ldots, m+1$, in the notation of Remark 3.4. Then we consider the following elementary abelian 2-subgroups of G:

$$\begin{aligned}
A &:= \langle t_1, \ldots, t_{m+1} \rangle, \\
B &:= \langle t_1, \ldots, t_m \rangle, \\
C &:= \langle t_1, \ldots, t_k, t_{k+1}t_{k+2}, t_{k+2}t_{k+3}, \ldots, t_m t_{m+1} \rangle \leq H, \\
E &:= B \cap C = \langle t_1, \ldots, t_k, t_{k+1}t_{k+2}, \ldots, t_{m-1}t_m \rangle.
\end{aligned}$$

That is B and C are maximal subgroups of A, and it suffices to show that $\mathrm{Res}_C^{\mathfrak{S}_n}(D(n))$ is indecomposable. By Theorem 3.5, we may identify $\mathrm{Res}_B^{\mathfrak{S}_n}(D(n))$ with FB. In particular, if $2k = n-2$ then $C = B$, so that $\mathrm{Res}_C^{\mathfrak{S}_n}(D(n))$ is then indecomposable. Hence we may from now on suppose that $2 \leq 2k < n-2$. We have

$$t_{m+1}z = (1 + t_1 + \cdots + t_m)z, \qquad \text{for } z \in D(n),$$

again by Theorem 3.5. This provides an explicit description of $\mathrm{Res}_A^{\mathfrak{S}_n}(D(n))$. In order to show that $\mathrm{Res}_C^{\mathfrak{S}_n}(D(n))$ is indecomposable, we will prove that the F-algebra $\mathrm{End}_{FC}(D(n))$ is local. Observe that $\mathrm{End}_{FE}(D(n))$ is an A-algebra with respect to the conjugation action, and that $\mathrm{End}_{FC}(D(n))$ consists of the C-fixed points in $\mathrm{End}_{FE}(D(n))$. Since $\mathrm{Res}_E^{\mathfrak{S}_n}(D(n)) \cong \mathrm{Res}_E^B(FB) = FE \oplus FEt_m \cong (FE)^2$, we have

$$\mathrm{End}_{FE}(D(n)) \cong \mathrm{Mat}(2, \mathrm{End}_{FE}(FE)) \cong \mathrm{Mat}(2, FE).$$

Under the natural embedding $FA \longrightarrow \mathrm{End}_{FE}(D(n)) \cong \mathrm{Mat}(2, FE)$, the element t_i, for $i \in \{1, \ldots, m+1\}$, corresponds to the matrix $T_i \in \mathrm{Mat}(2, FE)$, defined by

$$T_i = \begin{cases}
\begin{pmatrix} t_i & 0 \\ 0 & t_i \end{pmatrix}, & \text{for } i = 1, \ldots, k, \\[2ex]
\begin{pmatrix} 0 & t_i t_m \\ t_i t_m & 0 \end{pmatrix}, & \text{for } i = k+1, \ldots, m, \\[2ex]
\begin{pmatrix} 1 + t_1 + \cdots + t_k & 1 + t_m(t_{k+1} + \cdots + t_{m-1}) \\ 1 + t_m(t_{k+1} + \cdots + t_{m-1}) & 1 + t_1 + \cdots + t_k \end{pmatrix}, & \text{for } i = m+1.
\end{cases}$$

Now an element $\varphi = \begin{pmatrix} u & v \\ x & y \end{pmatrix} \in \mathrm{Mat}(2, FE)$ corresponds to an element in $\mathrm{End}_{FC}(D(n))$ if and only if φ commutes with

$$T_m T_{m+1} = \begin{pmatrix} 1 + t_m(t_{k+1} + \cdots + t_{m-1}) & 1 + t_1 + \cdots + t_k \\ 1 + t_1 + \cdots + t_k & 1 + t_m(t_{k+1} + \cdots + t_{m-1}) \end{pmatrix}.$$

An easy calculation shows that this is equivalent to

$$(u+y)(1+t_1+\cdots+t_k)=0=(x+v)(1+t_1+\cdots+t_k),$$

that is to $u+y,\, x+v\in I$ where I denotes the annihilator of $1+t_1+\cdots+t_k$, which is a proper ideal in the local F-algebra FE. Thus

$$\mathrm{End}_{FC}(D(n))\cong\left\{\begin{pmatrix} u & v\\ x & y\end{pmatrix}\in\mathrm{Mat}(2,FE)\mid u+y,\ x+v\in I\right\}=:L,$$

and it suffices to show that the F-algebra L is local. Since L is a subalgebra of $\mathrm{Mat}(2,FE)$, the intersection $L\cap\mathbf{J}(\mathrm{Mat}(2,FE))$ is a nilpotent ideal of L such that

$$\begin{aligned} L/(L\cap\mathbf{J}(\mathrm{Mat}(2,FE))) &\cong L+\mathbf{J}(\mathrm{Mat}(2,FE))/\mathbf{J}(\mathrm{Mat}(2,FE))\\ &\subseteq \mathrm{Mat}(2,FE)/\mathbf{J}(\mathrm{Mat}(2,FE)).\end{aligned}$$

The canonical isomorphism $\mathrm{Mat}(2,FE)/\mathbf{J}(\mathrm{Mat}(2,FE))\cong\mathrm{Mat}(2,F)$ maps the subalgebra $L+\mathbf{J}(\mathrm{Mat}(2,FE))/\mathbf{J}(\mathrm{Mat}(2,FE))$ onto the subalgebra

$$\left\{\begin{pmatrix} \bar u & \bar v\\ \bar x & \bar y\end{pmatrix}\in\mathrm{Mat}(2,F)\mid \bar u=\bar y,\ \bar x=\bar v\right\}\cong FC_2;$$

here we use the fact that $u+y,\, x+v\in I\subseteq\mathbf{J}(FE)$. This finally shows that L is a local F-algebra, and finishes the proof of the proposition. □

5.2 The case $n\equiv 0 \pmod 4$

In this subsection, we prove case (iii) of Theorem 1.1. That is, we show that $D(n)$ has vertex Q_n, provided that $n\equiv 0\pmod 4$, and for this we suppose that n has 2-adic expansion

$$n=\sum_{j=1}^{s}2^{i_j},$$

for some $s\geq 1$ and some $i_1>\ldots>i_s\geq 2$. Recall from Corollary 3.2 that in this case $D(n)$ is indeed relatively Q_n-projective, and from Theorem 3.1 that $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(D(n))=E(n)_+\oplus E(n)_-$. Hence we only need to show that $D(n)$ is not relatively R-projective for any maximal subgroup R of Q_n. In order to do this, we will show that if $E(n)_+$ were relatively projective with respect to some maximal subgroup R of Q_n then R would contain an elementary abelian 2-group $\overline{C}$ such that $E(n)_+$ restricts indecomposably to $\overline{C}$. From Green's Indecomposability Theorem we will then derive a contradiction so that $E(n)_+$, $E(n)_-$ and hence also $D(n)$ must have vertex Q_n. We first of all mention the following:

Proposition 5.2. *Let $n=2m+2$, for some odd integer $m>1$. Let further $2\leq k\leq m-1$, and set*

$$\overline{C}:=\overline{C}_k:=\langle t_it_{i+1}\mid i\in\{1,\ldots,m\}\setminus\{k\}\rangle\leq Q_n,$$

where $t_i:=(2i-1,2i)$, for $i=1,\ldots,m+1$. Then the restrictions of $E(n)_+$ and $E(n)_-$ to $\overline{C}$ are indecomposable.

Proof. We will prove the assertion by showing that the endomorphism algebras $\mathrm{End}_{F\overline{C}}(E(n)_+)$ and $\mathrm{End}_{F\overline{C}}(E(n)_-)$ are local.

As in the proof of Proposition 5.1, we consider the elementary abelian 2-subgroups

$$A := \langle t_1, \ldots, t_{m+1} \rangle \quad \text{and} \quad B := \langle t_1, \ldots, t_m \rangle$$

of $\mathfrak{S}_n$. Moreover, we set

$$\begin{aligned}
\overline{A} &:= A \cap \mathfrak{A}_n = \langle t_1t_2, t_2t_3, \ldots, t_mt_{m+1} \rangle, \\
\overline{B} &:= B \cap \mathfrak{A}_n = \langle t_1t_2, t_2t_3, \ldots, t_{m-1}t_m \rangle, \\
\overline{E} &:= \overline{B} \cap \overline{C} = \langle t_1t_2, t_2t_3, \ldots, t_{k-1}t_k, t_{k+1}t_{k+2}, t_{k+2}t_{k+3}, \ldots, t_{m-1}t_m \rangle.
\end{aligned}$$

By Theorem 3.5 we know that

$$\begin{aligned}
\mathrm{Res}_B^{\mathfrak{S}_n}(D(n)) \cong FB &= F\overline{B} \oplus F\overline{B}t_1 \\
&= F\overline{E} \oplus F\overline{E}t_kt_{k+1} \oplus F\overline{E}t_1 \oplus F\overline{E}t_1t_kt_{k+1}.
\end{aligned}$$

In the following we will again identify $\mathrm{Res}_B^{\mathfrak{S}_n}(D(n))$ with FB so that

$$(F\overline{B})^2 \cong \mathrm{Res}_{\overline{B}}^{\mathfrak{S}_n}(D(n)) = X_1 \oplus X_2,$$

where the submodule X_1 has basis $\overline{B}$ and X_2 has basis $\overline{B}t_1$. We observe that X_1 is in fact invariant under $\overline{A}$. Namely, for $i = 1, \ldots, m-1$, the element t_it_{i+1} belongs to $\overline{B}$, and therefore clearly leaves X_1 invariant. Furthermore, by Remark 3.4, we also have

$$t_mt_{m+1}z = (1 + t_1t_m + t_2t_m + \cdots + t_{m-1}t_m)z,$$

for $z \in D(n)$. Since $1 + t_1t_m + t_2t_m + \cdots + t_{m-1}t_m \in F\overline{B}$, also t_mt_{m+1} leaves X_1 invariant. Analogously, we deduce that X_2 is an $F\overline{A}$-invariant submodule of $\mathrm{Res}_{\overline{B}}^{\mathfrak{S}_n}(D(n))$ as well. Consequently, $\mathrm{Res}_{\overline{A}}^{\mathfrak{S}_n}(D(n)) = X_1 \oplus X_2$.

On the other hand, we have

$$\mathrm{Res}_{\overline{A}}^{\mathfrak{S}_n}(D(n)) \cong \mathrm{Res}_{\overline{A}}^{\mathfrak{A}_n}(E(n)_+) \oplus \mathrm{Res}_{\overline{A}}^{\mathfrak{A}_n}(E(n)_-).$$

Therefore X_1 and X_2 are isomorphic to the restrictions to $\overline{A}$ of $E(n)_+$ and $E(n)_-$. We show that the endomorphism algebras $\mathrm{End}_{F\overline{C}}(X_1)$ and $\mathrm{End}_{F\overline{C}}(X_2)$ are local, thereby proving the indecomposability of $\mathrm{Res}_{\overline{C}}^{\mathfrak{A}_n}(E(n)_+)$ and $\mathrm{Res}_{\overline{C}}^{\mathfrak{A}_n}(E(n)_-)$.

Via the identification of X_1 with $F\overline{B} = F\overline{E} \oplus F\overline{E}t_kt_{k+1}$, we obtain

$$\mathrm{End}_{F\overline{E}}(X_1) \cong \mathrm{Mat}(2, \mathrm{End}_{F\overline{E}}(F\overline{E})) \cong \mathrm{Mat}(2, F\overline{E}),$$

and $\mathrm{End}_{F\overline{E}}(X_1)$ carries an $\overline{A}$-algebra structure with respect to the conjugation action. This $\overline{A}$-algebra structure comes from the natural embedding

$$F\overline{A} \longrightarrow \mathrm{End}_{F\overline{E}}(X_1) \cong \mathrm{Mat}(2, F\overline{E}),$$

which, for $i, j \in \{1, \dots, m+1\}$, maps the element $t_i t_j \in \overline{A}$ to a matrix $T_{ij} \in \mathrm{Mat}(2, F\overline{E})$. For $j \in \{1, \dots, m+1\}$, from Remark 3.4 we deduce that the matrix T_{jm} has the following form:

$$T_{jm} = \begin{cases} \begin{pmatrix} t_j t_m & 0 \\ 0 & t_j t_m \end{pmatrix}, & \text{if } k < j \leq m, \\ \\ \begin{pmatrix} 0 & t_j t_k \cdot t_{k+1} t_m \\ t_j t_k \cdot t_{k+1} t_m & 0 \end{pmatrix}, & \text{if } 1 \leq j \leq k, \\ \\ \begin{pmatrix} 1 + t_m(t_{k+1} + \cdots + t_{m-1}) & t_{k+1} t_m (1 + t_1 t_k + \cdots + t_{k-1} t_k) \\ t_{k+1} t_m (1 + t_1 t_k + \cdots + t_{k-1} t_k) & 1 + t_m(t_{k+1} + \cdots + t_{m-1}) \end{pmatrix}, & \text{if } j = m+1. \end{cases}$$

We need to determine the $\overline{C}$-fixed points in $\mathrm{End}_{F\overline{E}}(X_1)$ under the $\overline{A}$-action, since these are precisely the elements in $\mathrm{End}_{F\overline{C}}(X_1)$. For this, let $\varphi := \begin{pmatrix} u & v \\ x & y \end{pmatrix} \in \mathrm{Mat}(2, F\overline{E})$. Then φ belongs to $\mathrm{End}_{F\overline{C}}(X_1)$ if and only if it commutes with the matrix $T_{m+1,m}$. One easily computes that this is equivalent to

$$(u + y)(1 + t_1 t_k + \cdots + t_{k-1} t_k) = 0 = (x + v)(1 + t_1 t_k + \cdots + t_{k-1} t_k).$$

This means that $u + y$ and $x + v$ belong to the annihilator of $1 + t_1 t_k + \cdots + t_{k-1} t_k \neq 0$ which is a proper ideal in the local algebra $F\overline{E}$. Hence we can now argue as in the proof of Proposition 5.1 to conclude that $\mathrm{End}_{F\overline{C}}(X_1)$ is a local F-algebra so that $\mathrm{Res}^{\overline{A}}_{\overline{C}}(X_1)$ is indecomposable. Similarly, also $\mathrm{Res}^{\overline{A}}_{\overline{C}}(X_2)$ has to be indecomposable. Therefore, both $\mathrm{Res}^{\mathfrak{A}_n}_{\overline{C}}(E(n)_+)$ and $\mathrm{Res}^{\mathfrak{A}_n}_{\overline{C}}(E(n)_-)$ are indecomposable, and the proposition is proved. □

We now distinguish between the cases $s = 1$ and $s \geq 2$. The reason for this is that we will make use of the structure of the Frattini subgroups of Q_n and P_n, respectively, and we have $\Phi(Q_n) = \Phi(P_n)$ if $s \geq 2$, and $\Phi(Q_n) < \Phi(P_n)$ if $s = 1$. We examine the case $s = 1$ first. Then the following holds:

Proposition 5.3. *Let $n = 2^r$, for some $r \geq 2$. Then*

(i) $t_i t_{i+1} \in \Phi(P_n)$, *for* $i \in \{1, \dots, 2^{r-1} - 1\}$,

(ii) $t_i t_{i+1} \in \Phi(Q_n)$, *for* $i \in \{1, \dots, 2^{r-1} - 1\} \setminus \{2^{r-2}\}$.

Proof. As in Section 2 (4), we again identify P_{2^r} with the wreath product $P_{2^{r-1}} \wr C_2 = \{(x_1, x_2; \rho) \mid x_1, x_2 \in P_{2^{r-1}},\ \rho \in C_2\}$. Since P_{2^r} is generated by elements of order 2, also the factor group P_{2^r}/P'_{2^r} is generated by elements of order 2, and is hence elementary abelian. Thus $P'_{2^r} = \Phi(P_{2^r})$, and

$$\Phi(P_{2^r}) = \{(x_1, x_2; 1) \mid x_1 x_2 \in \Phi(P_{2^{r-1}})\}, \tag{5}$$

by [18], L. 1.4. First of all, we show that $t_i t_{i+1} \in \Phi(P_{2^r})$, for all $i \in \{1, \dots, 2^{r-1} - 1\}$. For $r = 2$ this is obviously true. Therefore, we may now suppose that $r \geq 3$, and argue by induction on r. If $i < 2^{r-2}$ or $i > 2^{r-2}$ then $t_i t_{i+1}$ is contained in one of the two direct factors of the base group $P_{2^{r-1}} \times P_{2^{r-1}}$ of P_{2^r}, and hence $t_it_{i+1} \in \Phi(P_{2^{r-1}}) \times \Phi(P_{2^{r-1}}) \leq \Phi(P_{2^r})$, by induction. Now suppose that $i = 2^{r-2}$. Then t_it_{i+1} corresponds to the element $(t_i, t_1; 1) \in P_{2^{r-1}} \wr C_2$. Furthermore, $t_it_1 = t_1t_i = t_1t_2 \cdot t_2t_3 \cdots t_{i-1}t_i \in \Phi(P_{2^{r-1}})$, by induction. Thus (5) implies $t_it_{i+1} \in \Phi(P_{2^r})$ also in this case.

Next we prove assertion (ii). For $r = 2$ it trivially holds, since then $\{1, \ldots, 2^{r-1}-1\} \setminus \{2^{r-2}\} = \emptyset$. Let now $r \geq 3$, and $i \in \{1, \ldots, 2^{r-1} - 1\} \setminus \{2^{r-2}\}$. From what we have just proved above we deduce that

$$t_it_{i+1} \in \Phi(P_{2^{r-1}}) \times \Phi(P_{2^{r-1}}) \leq \Phi(P_{2^r}),$$

and it thus suffices to show that $\Phi(P_{2^{r-1}}) \times \Phi(P_{2^{r-1}}) \leq \Phi(Q_{2^r})$. For this, let $g \in P_{2^{r-1}}$. Since, by [9], Satz III.3.14, $\Phi(P_{2^{r-1}})$ is generated by the squares in $P_{2^{r-1}}$, we need to show that $g^2 \in \Phi(Q_{2^r})$. If $g \in Q_{2^{r-1}}$ then $g \in Q_{2^r}$ as well, and so $g^2 \in \Phi(Q_{2^r})$. Otherwise $\bar{g} := g \cdot t_{2^{r-2}+1} \in Q_{2^r}$ such that $g^2 = \bar{g}^2 \in \Phi(Q_{2^r})$. Consequently,

$$\Phi(P_{2^{r-1}}) \times \Phi(P_{2^{r-1}}) \leq \Phi(Q_{2^r}),$$

and the proposition is proved. □

With this we deduce:

Corollary 5.4. *Let $n = 2^r$ for some $r \geq 2$. Then Q_n is a vertex of $D(n)$.*

Proof. By Green's Indecomposability Theorem, it suffices to show that $E(n)_+$ and $E(n)_-$ restrict indecomposably to every maximal subgroup of Q_n. For then these modules are not relatively projective with respect to any maximal subgroup of Q_n, and so neither is $D(n)$. This then clearly forces $D(n)$ to have vertex Q_n. For $r = 2$ this is of course true, since $\dim(E(4)_+) = 1 = \dim(E(4)_-)$. For $r > 2$, in consequence of Propositions 5.3 and 5.2 above, we know further that $E(n)_+$ and $E(n)_-$ restrict indecomposably to the elementary abelian 2-subgroup $\overline{C} := \langle t_it_{i+1} \mid i \in \{1, \ldots, 2^{r-1} - 1\} \setminus \{2^{r-2}\}\rangle$ of the Frattini subgroup $\Phi(Q_n)$ of Q_n, and thus the assertion of the corollary follows. □

Remark 5.5. To close this section, we now settle the case where $n = \sum_{i=1}^{s} 2^{i_j}$, for some $s \geq 2$ and appropriate $i_1 > \ldots > i_s \geq 2$. We introduce some further notation: for $j = 1, \ldots, s$, we set $n_j := 2^{i_j}$ so that $P_n = P_{n_1} \times \cdots \times P_{n_s}$, and P_{n_j} is acting on the set $\Omega_j := \{\sum_{l=1}^{j-1} n_l + 1, \ldots, \sum_{l=1}^{j} n_l\}$. More precisely, P_{n_j} shall be generated by the elements $w_{2,j}, w_{4,j}, \ldots, w_{2^{i_j},j}$ where

$$w_{2^i,j} := \prod_{k=1}^{2^{i-1}} (k_j + k, k_j + k + 2^{i-1}),$$

and $k_j := \sum_{l=1}^{j-1} n_l$, for $j = 1, \ldots, s$ and $i = 1, \ldots, i_j$. For instance, if $n = 12$ then $P_{12} = P_8 \times P_4$ is generated by $w_{2,1} = (1, 2)$, $w_{4,1} = (1, 3)(2, 4)$, $w_{8,1} = (1, 5)(2, 6)(3, 7)(4, 8)$, $w_{2,2} = (9, 10)$ and $w_{4,2} = (9, 11)(10, 12)$.

Moreover, when identifying $P_{n_j} = P_{2^{i_j}}$ with the wreath product $P_{2^{i_j-1}} \wr C_2$ as in Section 2 (4), the base group of $P_{2^{i_j-1}} \wr C_2$ corresponds to a normal subgroup of P_{n_j} isomorphic to $P_{2^{i_j-1}} \times P_{2^{i_j-1}}$. From now on this subgroup is denoted by B_{n_j}, and we also set $\overline{B}_{n_j} := B_{n_j} \cap Q_n \trianglelefteq P_{n_j}$ for $j = 1, \ldots, s$. Lastly, for $j = 1, \ldots, s$ we define $y_{n_j} := w_{2,j}w_{4,j} \cdots w_{2^{i_j},j}$. Then, by [14], L.3.3,

y_{n_j} is an n_j-cycle. Furthermore $y_{n_j}^2 = w_{2,j} \cdots w_{2^{i_j-1},j} \cdot w_{2^{i_j},j} w_{2,j} \cdots w_{2^{i_j-1},j} w_{2^{i_j},j} \in \overline{B}_{n_j}$, for $j = 1, \ldots, s$. Again consider the example $n = 12$. Then $y_{n_1} = (1, 5, 3, 7, 2, 6, 4, 8)$ and $y_{n_2} = (9, 11, 10, 12)$.

Similar to the case $s = 1$ just treated, we aim to show that, assuming R is a maximal subgroup of Q_n such that $E(n)_+$ or $E(n)_-$ is relatively R-projective, then R has to contain an elementary abelian 2-group $\overline{C}_k$, for some $k \in \{2, \ldots, n/2 - 2\}$ as defined in Proposition 5.2. Again we will then derive a contradiction by applying Proposition 5.2 and Green's Indecomposability Theorem. As a consequence this will show that $E(n)_+$ and $E(n)_-$ are not relatively projective with respect to any maximal subgroup of Q_n, and so neither is $D(n)$, yielding assertion (iii) of Theorem 1.1. We will describe the structure of these maximal subgroups of Q_n in detail, and the next Proposition will play an important role for this.

Proposition 5.6. *With the above notation, let $m := n - n_s$, and let S be a maximal subgroup of $P_m \times P_{n_s}$ containing $Q_m \times \overline{B}_{n_s}$. Then S is one of the following seven groups:*

$S_1 := P_m \times B_{n_s}$, $S_2 := P_m \times Q_{n_s}$,
$S_3 := P_m \times U_{n_s}$, $S_4 := Q_{m+n_s}$,
$S_5 := Q_m \times P_{n_s}$, $S_6 := (Q_m \times B_{n_s})\langle (m-1, m)(m+1, m+2) \rangle$,
$S_7 := (Q_m \times U_{n_s})\langle (m-1, m)(m+1, m+2) \rangle$.

Here $U_{n_s} := \overline{B}_{n_s}\langle y_{n_s} \rangle \leq P_{n_s}$.

Proof. First of all notice that $P_{n_s}/\overline{B}_{n_s}$ is elementary abelian of order 4, and that the elements $1, w_{2,s}, w_{n_s,s}, y_{n_s}$ form a transversal for $P_{n_s}/\overline{B}_{n_s}$. For all of these four elements clearly belong to different left cosets of $\overline{B}_{n_s}$ in P_{n_s}, and $w_{2,s}^2 = 1 = w_{n_s,s}^2 \in \overline{B}_{n_s}$ as well as $y_{n_s}^2 \in \overline{B}_{n_s}$. Therefore, there are three maximal subgroups of P_{n_s} containing $\overline{B}_{n_s}$. These are:

$$B_{n_s} = \overline{B}_{n_s}\langle w_{2,s} \rangle, \; Q_{n_s} = \overline{B}_{n_s}\langle w_{n_s,s} \rangle, \; U_{n_s} = \overline{B}_{n_s}\langle y_{n_s} \rangle.$$

This also yields that $S_1, \ldots, S_7$ are maximal subgroups of $P_m \times P_{n_s}$ containing the normal subgroup $Q_m \times \overline{B}_{n_s}$, and that $S_i \neq S_j$ for $i \neq j$. Since $(P_m \times P_{n_s})/(Q_m \times \overline{B}_{n_s})$ is elementary abelian of order $8 = 2^3$, there are precisely $2^3 - 1 = 7$ maximal subgroups of $P_m \times P_{n_s}$ containing $Q_m \times \overline{B}_{n_s}$, and thus $S_1, \ldots, S_7$ are all of these. □

Proposition 5.7. *Let $n \equiv 0 \pmod 4$. Then $D(n)$ has vertex Q_n.*

Proof. If $s = 1$ then the assertion follows from Corollary 5.4. Thus, for the remainder of the proof we may suppose that $s \geq 2$, and keep the notation introduced in Remark 5.5. By Proposition 5.2 we know that $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E(n)_+)$ and $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E(n)_-)$ are indecomposable. Hence any vertex Q of $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E(n)_+)$ is also a vertex of $E(n)_+$ and $D(n)$. Assume that $Q < Q_n$, and let R be a maximal subgroup of Q_n such that $Q \leq R < Q_n$. By Proposition 3.3, $\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D(n)) \cong D(n-1)$ and, as we have proved in the previous section, $D(n-1)$ has vertex

$$\overline{Q} := \prod_{j=1}^{s-1} Q_{2^{i_j}} \times \prod_{i=0}^{i_s-2} Q_{2^{i_s-1-i}} \leq Q_{n-4}.$$

Therefore, there is also a source L of $D(n-1)$ such that $L \mid \mathrm{Res}^{\mathfrak{S}_n}_{Q}(D(n)) \cong \mathrm{Res}^{\mathfrak{A}_n}_{Q}(E(n)_+) \oplus \mathrm{Res}^{\mathfrak{A}_n}_{Q}(E(n)_-)$. We may suppose that $L \mid \mathrm{Res}^{\mathfrak{A}_n}_{Q}(E(n)_+) = \mathrm{Res}^{Q_n}_{Q}(\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E(n)_+))$. If $L \mid \mathrm{Res}^{\mathfrak{A}_n}_{Q}(E(n)_-)$ then we can argue analogously. That is $\overline{Q} \leq_{Q_n} Q \leq R < Q_n$. Since R is normal in Q_n, we get $\overline{Q} \leq R$. On the other hand we clearly have $\Phi(Q_n) \leq R$, and $\Phi(Q_n) = \Phi(P_n) = \Phi(P_{n_1}) \times \cdots \times \Phi(P_{n_s})$, since $s \geq 2$. A proof for this can for example be found in [5]. By construction, $\overline{Q}$, in particular, contains $Q_{2^{i_s-1}} \leq P_{2^{i_s}} = P_{n_s}$. Consequently, we get $\overline{B}_{n_s} = Q_{2^{i_s-1}}\Phi(P_{n_s}) \leq R$, by [5], and thus

$$\overline{R} := Q_{n_1} \times \cdots \times Q_{n_{s-1}} \times \overline{B}_{n_s} \leq R < Q_n < P_n.$$

Obviously, $\overline{R}$ is normal in P_n, and $P_n/\overline{R}$ and $Q_n/\overline{R}$ are elementary abelian of order 2^{s+1} and 2^s, respectively. Regarding $P_n/\overline{R}$ as $\mathbb{F}_2$-vector space, the maximal subgroup $R/\overline{R}$ of $Q_n/\overline{R}$ is a subspace of $P_n/\overline{R}$ of codimension 2, and $Q_n/\overline{R}$ is a subspace of codimension 1. Hence there is a subspace $S/\overline{R} \neq Q_n/\overline{R}$ of $P_n/\overline{R}$ of codimension 1 such that $R/\overline{R} = S/\overline{R} \cap Q_n/\overline{R}$. That is $R = S \cap Q_n$, for some maximal subgroup $S \geq \overline{R}$ of P_n. Now

$$S/\overline{R} \leq P_n/\overline{R} \cong P_{n_1}/Q_{n_1} \times \cdots \times P_{n_{s-1}}/Q_{n_{s-1}} \times P_{n_s}/\overline{B}_{n_s},$$

so that $S/\overline{R}$ is the kernel of a non-trivial homomorphism $\bar{\lambda} : P_n/\overline{R} \longrightarrow \mathbb{C}^\times$, and we may write $\bar{\lambda} = \bar{\lambda}_1 \times \cdots \times \bar{\lambda}_s$ where $\bar{\lambda}_j : P_{n_j}/Q_{n_j} \longrightarrow \mathbb{C}^\times$, for $j = 1, \ldots, s-1$, and $\bar{\lambda}_s : P_{n_s}/\overline{B}_{n_s} \longrightarrow \mathbb{C}^\times$. The inflation of $\bar{\lambda}_j$ to P_{n_j} shall be denoted by λ_j, for $j = 1, \ldots, s$. Then $\lambda := \lambda_1 \times \cdots \times \lambda_s : P_n \longrightarrow \mathbb{C}^\times$ with kernel S. As in the proof of Proposition 5.1, for $j = 1, \ldots, s-1$, the homomorphism λ_j is either trivial or the restriction of the sign character of $\mathfrak{S}_{n_j}$ to P_{n_j}. We set

$$\mathcal{L} := \{1 \leq j \leq s-1 \mid \lambda_j \neq 1\}.$$

Then

$$\ker(\lambda_1 \times \cdots \times \lambda_{s-1}) = \prod_{j \notin \mathcal{L} \cup \{s\}} P_{n_j} \times (Q_n \cap \prod_{j \in \mathcal{L}} P_{n_j}).$$

Hence $\prod_{j \notin \mathcal{L} \cup \{s\}} P_{n_j}$ is a Sylow 2-subgroup of $\mathfrak{S}(\bigcup_{j \notin \mathcal{L} \cup \{s\}} \Omega_j)$, and $Q_n \cap \prod_{j \in \mathcal{L}} P_{n_j}$ is a Sylow 2-subgroup of $\mathfrak{A}(\bigcup_{j \in \mathcal{L}} \Omega_j)$. For convenience, we set $2k := \sum_{j \notin \mathcal{L} \cup \{s\}} n_j$, and simply denote $\prod_{j \notin \mathcal{L} \cup \{s\}} P_{n_j}$ by P_{2k} and $Q_n \cap \prod_{j \in \mathcal{L}} P_{n_j}$ by Q_{n-n_s-2k}. Then

$$P_{2k} \times Q_{n-n_s-2k} \times \overline{B}_{n_s} \leq \ker(\lambda_1 \times \cdots \times \lambda_{s-1}) \times \ker(\lambda_s) \leq S < P_{2k} \times P_{n-n_s-2k} \times P_{n_s} = P_n,$$

so that $S = P_{2k} \times \overline{S}$, for some maximal subgroup $\overline{S}$ of $P_{n-n_s-2k} \times P_{n_s}$ containing $Q_{n-n_s-2k} \times \overline{B}_{n_s}$. If $2k = n - n_s$ or $2k = 0$ then S contains the elementary abelian 2-subgroup $\overline{C}_{\frac{n-n_s}{2}}$, occuring in Proposition 5.2. Notice that indeed $3 \leq (n-n_s)/2 \leq n/2 - 2$. Therefore, we may now suppose that $0 < 2k < n - n_s$. Then from Proposition 5.6 we deduce that $S = P_{2k} \times S_i$ for some $i \in \{1, \ldots, 7\}$, where $S_1, \ldots, S_7$ are the maximal subgroups of $P_{n-n_s-2k} \times P_{n_s}$ determined in Proposition 5.6. But this shows that S contains a subgroup which is in $\mathfrak{A}_n$ conjugate to $\overline{C}_l$ where

$$l = \begin{cases} \frac{n-n_s}{2}, & \text{if } i \in \{1,2,3\}, \\ k, & \text{if } i \in \{4,6,7\}, \\ k + \frac{n_s}{2}, & \text{if } i = 5. \end{cases}$$

Consequently, $\overline{C}_l \leq_{\mathfrak{A}_n} S \cap Q_n = R < Q_n$, and we may suppose that $\overline{C}_l \leq R$. For otherwise we can replace Q_n by gQ_ng^{-1} and R by gRg^{-1}, for some appropriate $g \in \mathfrak{A}_n$. In all these cases $2 \leq l \leq n/2-2$. Thus Proposition 5.2 yields the indecomposability of $\mathrm{Res}^{\mathfrak{A}_n}_R(E(n)_+)$, and from Green's Indecomposability Theorem [7] we finally derive the desired contradiction that $E(n)_+$ cannot be relatively R-projective. □

We have now also completed the entire proof of Theorem 1.1.

6 Sources of the basic spin modules

In this section we again suppose that $n \geq 2$, and give a description of the sources of the basic spin $F\mathfrak{S}_n$-module $D(n)$. For this recall from Proposition 4.6 that, for any $r \geq 2$, we have $\mathrm{Res}^{\mathfrak{S}_{2^r}}_{\mathfrak{A}_{2^r}}(M(2^r)) = N_+ \oplus N_-$ where $M(2^r) = \mathrm{Res}^{\mathfrak{S}_{2^r+1}}_{\mathfrak{S}_{2^r}}(D(2^r+1))$, and N_+ and N_- are indecomposable, selfdual, and in $\mathfrak{S}_{2^r}$ conjugate. In this situation, $N(2^r)$ is now always supposed to be either N_+ or N_-.

Theorem 6.1. *Let $n \geq 2$ with 2-adic expansion $n = \sum_{j=1}^{s} 2^{i_j}$, for some $s \in \mathbb{N}$ and some $i_1 > \ldots > i_s \geq 0$.*

(i) If n is odd then let $j \in \{1, \ldots, s\}$ be minimal with $i_j > 1$. Then

$$\mathrm{Res}^{\mathfrak{A}_{2^{i_1}}}_{Q_{2^{i_1}}}(N(2^{i_1})) \boxtimes \cdots \boxtimes \mathrm{Res}^{\mathfrak{A}_{2^{i_j}}}_{Q_{2^{i_j}}}(N(2^{i_j}))$$

of dimension $2^{(2^{i_1-1}+\cdots+2^{i_j-1})-j}$ is a source of $D(n)$.

(ii) If $n \equiv 2 \pmod 4$ then $\mathrm{Res}^{\mathfrak{S}_n}_{P_n}(D(n))$ is a source of $D(n)$.

(iii) If $n \equiv 0 \pmod 4$ then $\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E(n)_+)$ and $\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E(n)_-)$ are sources of $D(n)$.

Proof. Assertion (ii) is clear from Remark 3.4, and (iii) follows from the proofs of Propositions 5.1 and 5.7. We prove (i). If $n \equiv 3 \pmod 4$ then, by Proposition 3.3, we know that $D(n)$ and $D(n-2)$ have common vertices and common sources. Thus we may henceforth suppose that $n \equiv 1 \pmod 4$ and that $n > 1$. That is, $n = 1 + \sum_{j=1}^{s} 2^{i_j}$, for some $s \geq 1$ and $i_1 > \ldots > i_s \geq 2$. Then $D(n)$ and $M(2^{i_1}) \boxtimes \cdots \boxtimes M(2^{i_s})$ have common vertex $\prod_{j=1}^{s} Q_{2^{i_j}}$ and common sources. Hence we only need to determine the sources of $M(2^{i_j})$, for $j = 1, \ldots, s$. For this, consider $r \geq 2$, so that

$$M(2^r) = \mathrm{Res}^{\mathfrak{S}_{2^r+1}}_{\mathfrak{S}_{2^r}}(D(2^r+1)) \cong \mathrm{Res}^{\mathfrak{S}_{2^r+2}}_{\mathfrak{S}_{2^r}}(D(2^r+2)).$$

By Remark 3.4, we thus have $\mathrm{Res}^{\mathfrak{S}_{2^r}}_{B}(M(2^r)) \cong FB$, where again B denotes the elementary abelian 2-group $\langle t_1, \ldots, t_{2^{r-1}} \rangle \leq P_{2^r}$. Therefore, $\overline{B} := B \cap \mathfrak{A}_{2^r} \leq Q_{2^r}$, and $\mathrm{Res}^{\mathfrak{S}_{2^r}}_{\overline{B}}(M(2^r)) \cong F\overline{B} \oplus F\overline{B}$. Since $\mathrm{Res}^{\mathfrak{S}_{2^r}}_{\mathfrak{A}_{2^r}}(M(2^r)) = N_+ \oplus N_-$, this shows that

$$\mathrm{Res}^{\mathfrak{A}_{2^r}}_{\overline{B}}(N_+) \cong F\overline{B} \cong \mathrm{Res}^{\mathfrak{A}_{2^r}}_{\overline{B}}(N_-).$$

In particular, $\mathrm{Res}^{\mathfrak{A}_{2^r}}_{Q_{2^r}}(N_+)$ and $\mathrm{Res}^{\mathfrak{A}_{2^r}}_{Q_{2^r}}(N_-)$ are indecomposable of dimension $|B|/2 = 2^{2^{r-1}-1}$ each, and hence sources of $M(2^r)$. Consequently, $\mathrm{Res}^{\mathfrak{A}_{2^{i_j}}}_{Q_{2^{i_j}}}(N(2^{i_j}))$ has dimension $2^{2^{i_j-1}-1}$ and is a source of $M(2^{i_j})$, for $j = 1, \ldots, s$. From this and Proposition 2.1 we now deduce assertion (i), and the theorem is proved. □

7 Spin modules for the alternating groups

Remark 7.1. In the following, let $n \geq 3$. Once knowing the vertices of the basic spin $F\mathfrak{S}_n$-module $D(n)$, it is natural to ask what the vertices of the simple $F\mathfrak{A}_n$-modules $E(n)_\pm$ and $E(n)_0$, respectively, look like. Recall from Theorem 3.1 that $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(D(n)) = E(n)_0$ is simple for $n \equiv 2 \pmod 4$, and $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(D(n)) = E(n)_+ \oplus E(n)_-$ with non-isomorphic conjugate simple $F\mathfrak{A}_n$-modules $E(n)_+$ and $E(n)_-$ otherwise. This shows, in particular, that $D(n)$ and the $F\mathfrak{A}_n$-modules $E(n)_+$ and $E(n)_-$ have common vertices and common sources, for $n \not\equiv 2 \pmod 4$. In the theorem below, we will also determine the vertices and sources of the simple $F\mathfrak{A}_n$-module $E(n)_0$, for $n \equiv 2 \pmod 4$.

Theorem 7.2. *Let $n \geq 3$ with 2-adic expansion $n = \sum_{j=1}^{s} 2^{i_j}$, for some $s \in \mathbb{N}$ and $i_1 > \ldots > i_s \geq 0$. Then the following hold:*

(i) If n is odd then $E(n)_+$ and $E(n)_-$ have common vertices and common sources with $D(n)$. That is, the Sylow 2-subgroups of $\prod_{j=1}^{s} \mathfrak{A}_{2^{i_j}}$ are vertices of $E(n)_+$ and $E(n)_-$. Moreover, if $j \in \{1, \ldots, s\}$ is minimal with $i_j > 1$ then

$$\mathrm{Res}^{\mathfrak{A}_{2^{i_1}}}_{Q_{2^{i_1}}}(N(2^{i_1})) \boxtimes \cdots \boxtimes \mathrm{Res}^{\mathfrak{A}_{2^{i_j}}}_{Q_{2^{i_j}}}(N(2^{i_j}))$$

is a source of $E(n)_+$ or $E(n)_-$. Here, $N(2^{i_1}), \ldots, N(2^{i_j})$ are as in Theorem 6.1.

(ii) If $n \equiv 0 \pmod 4$ then $E(n)_+$ and $E(n)_-$ have common vertices and common sources with $D(n)$. That is, the Sylow 2-subgroups of $\mathfrak{A}_n$ are vertices of $E(n)_+$ and $E(n)_-$. Furthermore, $\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E(n))_+$ is source of $E(n)_+$, and $\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E(n))_-$ is source of $E(n)_-$.

(iii) If $n \equiv 2 \pmod 4$ then $E(n)_0$ has common vertices and common sources with both $E(n-1)_+$ and $E(n-1)_-$, and thus with $D(n-1)$.

Proof. If n is odd, or if $n \equiv 0 \pmod 4$ then the vertices of $E(n)_+$ and the vertices of $E(n)_-$ are in $\mathfrak{S}_n$ conjugate to any vertex Q of $D(n)$. Since, by Theorem 1.1, $N_{\mathfrak{S}_n}(Q) \not\leq \mathfrak{A}_n$, the $\mathfrak{S}_n$-conjugates of Q are precisely the $\mathfrak{A}_n$-conjugates of Q. Thus $E(n)_+$ and $E(n)_-$ have indeed the same vertices, and assertions (i) and (ii) are now immediate from Theorem 1.1 and Theorem 6.1.

Therefore, we may from now on suppose that $n \equiv 2 \pmod 4$ so that then $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(D(n)) = E(n)_0$ is simple.

By Proposition 3.3, we have $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D(n)) \cong D(n-1)$ so that

$$\mathrm{Res}^{\mathfrak{A}_n}_{\mathfrak{A}_{n-1}}(E(n)_0) \cong \mathrm{Res}^{\mathfrak{S}_{n-1}}_{\mathfrak{A}_{n-1}}(D(n-1)) \cong E(n-1)_+ \oplus E(n-1)_-. \tag{6}$$

On the other hand, by [3], L.4.1 (ii), we have

$$\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D(n-1)) \cong D^{(\frac{n}{2}, \frac{n-2}{2}, 1)} \oplus X,$$

where X is indecomposable of composition length 2 with $\mathrm{Hd}(X) \cong D(n) \cong \mathrm{Soc}(X)$. (In fact, L. 4.1 in [3] is stated for $n \geq 8$, but is also true when assuming $n \geq 5$.) By Mackey's

Decomposition Theorem, we thus obtain

$$\begin{aligned}\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n-1)_+)\oplus\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n-1)_-) &\cong \mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(\mathrm{Res}_{\mathfrak{A}_{n-1}}^{\mathfrak{S}_{n-1}}(D(n-1))\\ &\cong \mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D(n-1))\\ &\cong \mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D^{(\frac{n}{2},\frac{n-2}{2},1)})\oplus\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(X).\end{aligned}$$

Since X is of composition length 2 with $\mathrm{Hd}(X)\cong D(n)\cong\mathrm{Soc}(X)$, the restriction $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(X)$ has precisely two composition factors both of which are isomorphic to $E(n)_0$. Thus either $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(X)$ remains indecomposable or is isomorphic to $E(n)_0\oplus E(n)_0$. By [1], we also know that $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D^{(\frac{n}{2},\frac{n-2}{2},1)})$ splits into the direct sum of two conjugate simple modules $E_+^{(\frac{n}{2},\frac{n-2}{2},1)}$ and $E_-^{(\frac{n}{2},\frac{n-2}{2},1)}$. Therefore

$$\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n-1)_+)\oplus\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n-1)_-)\cong E_+^{(\frac{n}{2},\frac{n-2}{2},1)}\oplus E_-^{(\frac{n}{2},\frac{n-2}{2},1)}\oplus\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(X).$$

The modules $\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n-1)_+)$ and $\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n-1)_-)$ are in $\mathfrak{S}_n$ conjugate and have thus, in particular, the same number of indecomposable direct summands. This forces $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(X)\cong E(n)_0\oplus E(n)_0$. Furthermore, we see that $\dim(D^{(\frac{n}{2},\frac{n-2}{2},1)})=(n-2)2^{\frac{n-2}{2}}$, that is

$$\dim(E_+^{(\frac{n}{2},\frac{n-2}{2},1)})=(n-2)2^{\frac{n-4}{2}}=\dim(E_-^{(\frac{n}{2},\frac{n-2}{2},1)}).$$

Since $\dim(E(n)_0)=2^{\frac{n-2}{2}}\neq(n-2)2^{\frac{n-4}{2}}$, $E(n)_0$ is not conjugate to $E_+^{(\frac{n}{2},\frac{n-2}{2},1)}$ or $E_-^{(\frac{n}{2},\frac{n-2}{2},1)}$. Together with (6) we therefore deduce

$$E(n)_0\mid\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n-1)_+)\quad\text{and}\quad E(n-1)_+\mid\mathrm{Res}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n)_0)$$

and

$$E(n)_0\mid\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n-1)_-)\quad\text{and}\quad E(n-1)_-\mid\mathrm{Res}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E(n)_0).$$

This proves the proposition. □

Remark 7.3. Notice that, for $n\equiv 2\pmod 4$, Theorems 1.1 and 7.2 show that $D(n)$ has vertex P_n whereas $E(n)_0=\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D(n))$ has some vertex $Q<Q_n=P_n\cap\mathfrak{A}_n$. Consequently, a vertex of a simple $F\mathfrak{A}_n$-module is generally not just the intersection of $\mathfrak{A}_n$ with some vertex of the corresponding simple $F\mathfrak{S}_n$-module.

8 Closing Remarks

To close, we mention some more general problems related to the results of this article. For this we now allow p to be an arbitrary prime and $n\in\mathbb{N}$.

Remark 8.1. Suppose that $p=2$, and consider an arbitrary simple $F\mathfrak{S}_n$-module D^λ. Then Benson's Theorem [1] mentioned at the beginning of Subsection 3.1, enables us to decide combinatorially whether $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D^\lambda)$ splits into the direct sum of two non-isomorphic conjugate simple modules, or is simple. Namely, $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D^\lambda)$ splits if and only if the labelling partition λ is a so-called *S-partition* (see [1] for a definition). Furthermore, a result of Ford [6] gives the respective combinatorial criterion in the case where p is odd.

In any case, if $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(D^\lambda)$ does split into the direct sum of two simple modules then, as for the basic spin module, these shall be denoted by E^λ_+ and E^λ_-. In the case of the basic spin module in characteristic 2, we have been able to identify $E(n)_+$ and $E(n)_-$ via their Brauer characters. However, in general, the Brauer characters of E^λ_+ and E^λ_- are not known, and there also seems to be no criterion to distinguish the modules E^λ_+ and E^λ_- explicitly.

Remark 8.2. Suppose that $p = 2$ and that $n \geq 3$ with $n \equiv 2 \pmod 4$. We write $n = 2m$, for an appropriate $m \in \mathbb{N}$. Then the branching rules [12], Thms. 11.2.10 and 11.2.11 immediately yield that $D^{(m+1,m-1,1)} \mid \mathrm{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(D(n))$ and also $D(n) \mid \mathrm{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(D^{(m+1,m-1,1)})$. Thus, in consequence of Theorem 1.1 and Theorem 6.1, the simple $F\mathfrak{S}_{n+1}$-module $D^{(m+1,m-1,1)}$ and the basic spin $F\mathfrak{S}_n$-module $D(n)$ have then common vertex P_n and common source $\mathrm{Res}^{\mathfrak{S}_n}_{P_n}(D(n))$.

Remark 8.3. Once again we emphasize that it is not known what the vertices of the simple $F\mathfrak{S}_n$-modules generally look like, neither in characteristic 2 nor in odd characteristic. Nevertheless, in [4] the authors and R. Zimmermann set up a few conjectures on vertices and sources of simple $F\mathfrak{S}_n$-modules. For instance, suppose that $p \mid n$. Then the vertices of the simple $F\mathfrak{S}_n$-modules belonging to the principal block should be the Sylow p-subgroups of $\mathfrak{S}_n$ or $\mathfrak{A}_n$. If n is in fact a p-power then any simple $F\mathfrak{S}_n$-module D^λ belonging to the principal block should further restrict indecomposably to the Sylow p-subgroups of $\mathfrak{S}_n$. In particular, if p is odd, or if $p = 2$ and λ is not an S-partition, then $\mathrm{Res}^{\mathfrak{S}_n}_{P_n}(D^\lambda)$ should be a source of D^λ. Thus Theorem 1.1 and Theorem 6.1 provide some evidence for these conjectures.

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