VERTICES OF SMALL ORDER FOR SIMPLE MODULES OF FINITE SYMMETRIC GROUPS

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Abstract

Let F be a field of characteristic $p > 0$, let $\mathfrak{S}_n$ denote the symmetric group of degree $n < \infty$, and let M be a simple $F\mathfrak{S}_n$-module with vertex V belonging to a block B of $F\mathfrak{S}_n$ with defect group Δ. We prove that $|\Delta| \leq |V|!$, and we also investigate the embedding of V into $\mathfrak{S}_n$.
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1 Introduction

Let F be an algebraically closed field of characteristic $p > 0$. In [4] 2.8, Feit formulated the following conjecture:

Conjecture 1.1 (Feit)**.** For any finite p-group P, there are only finitely many isomorphism classes of indecomposable FP-modules occuring as sources of simple FG-modules for finite groups G containing P as a subgroup.

In [11], Puig argued that this conjecture holds if G is restricted to ranging over finite symmetric groups only. His proof made use of Morita equivalences between blocks of finite symmetric groups discovered by Scopes in [12]. It was also based on an inequality between the order $|V|$ of a vertex V of a simple module M and the order $|\Delta|$ of a defect group Δ of the block containing M. In fact, Puig claimed that $|\Delta| \leq |Z(V)|!$.
Unfortunately, this inequality is not correct. For instance, let P be a Sylow p-subgroup of $\mathfrak{S}_{p^2}$. Then P is both a defect group of the principal block B of $F\mathfrak{S}_{p^2}$ and a vertex of the trivial $F\mathfrak{S}_{p^2}$-module M which belongs to B. However, we have $|P| = p^{p+1} > p! = |Z(P)|!$. Nevertheless, one can find upper bounds for $|\Delta|$ depending only on $|V|$. In this paper, we will show that $|\Delta| \leq |V|!$. With this rather crude inequality, the rest of Puig's argument works. We will obtain our inequality as a consequence of a more detailed investigation of the embedding of V into the relevant symmetric group $\mathfrak{S}_n$. This will also make it possible to improve on the inequality under certain hypotheses. In particular, we will illustrate this for the cases where $|V| \in \{1, p, p^2, p^3, 16\}$. Here we will also make use of computational facts from [1] and [14].
The simple $F\mathfrak{S}_n$-modules are parametrized by the p-regular partitions of n. However, little is known concerning the vertices of the simple $F\mathfrak{S}_n$-modules (cf. [1], [5], [9] and [14]). We consider the results of this paper as a contribution to the problem of determining the vertices of the simple $F\mathfrak{S}_n$-modules in general.
We also mention that Puig had asked whether, for arbitrary finite groups G in odd characteristic p, the order of a defect group is bounded in terms of the order of a simple module (cf.

[13]), and that J. Zhang has reduced this question to finite quasisimple groups.
This paper elaborates on parts of the PhD thesis [1] of the first author, written under the direction of the second author. Both authors are grateful to L. Puig for discussions concerning [11]. The work on this paper was supported by the Deutsche Forschungsgemeinschaft.

2 Vertices of simple modules

In this section we recall some basic facts about simple modules over group algebras. Therefore, let F be an algebraically closed field of characteristic $p > 0$, let G be a finite group, and let M be an indecomposable FG-module. A subgroup V of G which is minimal with respect to the condition $M | \mathrm{Ind}_V^G(\mathrm{Res}_V^G(M))$ is called a *vertex* of M. Then the vertices of M form a G-conjugacy class of p-subgroups of G. If V is a vertex of M then there exist a defect group Δ of the block B containing M and a Sylow p-subgroup S of G such that $V \leq \Delta \leq S$ and $|S : V| \mid \dim(M)$. In this way one obtains both upper and lower bounds on the order of V.
If V is a vertex of M then there exists an indecomposable FV-module L with vertex V such that $M | \mathrm{Ind}_V^G(L)$ and $L | \mathrm{Res}_V^G(M)$. Moreover, L is determined up to isomorphism and conjugation with elements in $N_G(V)$. Then L is called a *source* of M.
If M is simple then the following results are important:

Theorem 2.1 (Knörr [8])**.** *Let M be a simple FG-module with vertex V belonging to a block B of FG. Then there exists a block b of $F[VC_G(V)]$ with defect group V such that $b^G = B$. Hence there is a defect group Δ of B such that $C_\Delta(V) \leq V \leq \Delta$.*

Theorem 2.2 (Erdmann [3])**.** *Let M be a simple FG-module with vertex V belonging to a block B of FG. If V is cyclic then V is a defect group of B.*

Thus simple FG-modules belonging to blocks with abelian defect group Δ have vertex Δ, and simple FG-modules belonging to blocks with noncyclic defect groups cannot have cyclic vertices.

3 Symmetric groups

In the following, we denote the symmetric group of degree $n < \infty$ by $\mathfrak{S}_n$. More generally, for any finite set Ω, we denote the symmetric group on Ω by $\mathfrak{S}(\Omega)$. If Ω is the disjoint union of subsets $\Omega_1, \ldots, \Omega_k$ then we identify $\mathfrak{S}(\Omega_1) \times \cdots \times \mathfrak{S}(\Omega_k)$ with a subgroup of $\mathfrak{S}(\Omega)$, in the usual way.
Let $\Lambda = \{1, \ldots, r\}$ for a positive integer r. Then we can view the wreath product

$$\mathfrak{S}(\Omega) \wr \mathfrak{S}(\Lambda) = \{(\sigma_1, \ldots, \sigma_r; \pi) | \sigma_1, \ldots, \sigma_r \in \mathfrak{S}(\Omega),\, \pi \in \mathfrak{S}(\Lambda)\}$$

as a subgroup of $\mathfrak{S}(\Omega \times \Lambda)$ by identifying an element $(\sigma_1, \ldots, \sigma_r; \pi) \in \mathfrak{S}(\Omega) \wr \mathfrak{S}(\Lambda)$ with the element $\overline{(\sigma_1, \ldots, \sigma_r; \pi)} \in \mathfrak{S}(\Omega \times \Lambda)$ defined as follows:

$$\overline{(\sigma_1, \ldots, \sigma_r; \pi)}(\alpha, i) := (\sigma_{\pi(i)}(\alpha), \pi(i)), \qquad \text{for } \alpha \in \Omega,\, i \in \Lambda.$$

If Λ is the empty set then we interpret $\mathfrak{S}(\Omega) \wr \mathfrak{S}(\Lambda)$ as the trivial group.
For a prime number p, the cyclic group $C_p := \langle (1, \ldots, p) \rangle$ of order p is a Sylow p-subgroup of $\mathfrak{S}_p$. We set $P_1 := 1$, $P_p := C_p$ and $P_{p^i} := P_{p^{i-1}} \wr C_p$ for $i \geq 2$. Then P_{p^i} becomes a Sylow

p-subgroup of $\mathfrak{S}(\{1, \ldots, p\}^i)$ which we identify with $\mathfrak{S}_{p^i}$.
Let $n = \sum_{i=0}^{t} n_i p^i$ be the p-adic expansion of n. Then $P_n := \prod_{i=0}^{t} (P_{p^i})^{n_i}$ is a Sylow p-subgroup of $\mathfrak{S}_n$. Hence $Q_n := P_n \cap \mathfrak{A}_n$ is a Sylow p-subgroup of the alternating group $\mathfrak{A}_n$ of degree n.

We briefly summarize some facts concerning the modular representation theory of symmetric groups. For more details, we refer the reader to [6] and [7]. For a p-regular partition λ of n, we denote the corresponding simple $F\mathfrak{S}_n$-module by D^λ. Given p-regular partitions λ and μ of n, the simple $F\mathfrak{S}_n$-modules D^λ and D^μ belong to the same block B of $F\mathfrak{S}_n$ if and only if λ and μ have the same p-core κ. In this case λ and μ then have the same p-weight w as well. We refer to κ and w as the p-core and p-weight, respectively, of the block B. In this situation, the defect groups of B are precisely the $\mathfrak{S}_n$-conjugates of the Sylow p-subgroup P_{pw} of $\mathfrak{S}_{pw}$, considered as a subgroup of $\mathfrak{S}_n$ as usual.

Remark 3.1. Let D be a simple $F\mathfrak{S}_n$-module belonging to a block B of weight w, let $\Delta \leq \mathfrak{S}_{pw}$ be a defect group of B, and let $V \leq \Delta$ be a vertex of D. Then Theorem 2.1 and [10], Proposition 1.4 imply that V has no fixed points on $\{1, \ldots, pw\}$ and that $C_{\mathfrak{S}_{pw}}(V) = Z(V)$ holds.

Motivated by Remark 3.1, we will investigate subgroups G of a finite symmetric group $\mathfrak{S}(\Omega)$ satisfying $C_{\mathfrak{S}(\Omega)}(G) = Z(G)$. In this case, we will call G a *selfcentralizing* subgroup of $\mathfrak{S}(\Omega)$. We note that, for an arbitrary subgroup H of $\mathfrak{S}(\Omega)$, the set Ω can be regarded as an H-set, and then $C_{\mathfrak{S}(\Omega)}(H)$ coincides with the automorphism group $\mathrm{Aut}_H(\Omega)$ of the H-set Ω.

4 Automorphism groups of finite G-sets

Let G be a finite group, and let Ω be a finite G-set. We denote the automorphism group of Ω by

$$\mathrm{Aut}_G(\Omega) := \{f \in \mathfrak{S}(\Omega) | f(g\alpha) = gf(\alpha), \text{ for all } g \in G, \alpha \in \Omega\}.$$

For any subgroup H of G, we set

$$\Omega_H := \{\alpha \in \Omega |\, \mathrm{Stb}_G(\alpha) \sim_G H\} \quad \text{and} \quad r_H := |\Omega_H|/|G : H|.$$

Thus Ω_H is the union of all orbits of G on Ω which are as G-sets isomorphic to G/H, and r_H is the number of these orbits.

Lemma 4.1. *Let $\mathfrak{H}$ be a transversal for the conjugacy classes of subgroups of G. Then the following holds:*

(i) $f(\Omega_H) = \Omega_H$, for $f \in \mathrm{Aut}_G(\Omega)$ and $H \in \mathfrak{H}$.

(ii) $\Omega = \biguplus_{H \in \mathfrak{H}} \Omega_H$ and $\mathrm{Aut}_G(\Omega) = \prod_{H \in \mathfrak{H}} \mathrm{Aut}_G(\Omega_H)$.

Proof. (i) Let $f \in \mathrm{Aut}_G(\Omega)$, $H \in \mathfrak{H}$, $\alpha \in \Omega_H$ and $g \in G$. Then $gf(\alpha) = f(\alpha)$ if and only if $g\alpha = \alpha$. Thus $\mathrm{Stb}_G(f(\alpha)) = \mathrm{Stb}_G(\alpha) \sim_G H$ and $f(\alpha) \in \Omega_H$.
(ii) The first assertion clearly holds. Thus, as in Section 3, we can identify $\prod_{H \in \mathfrak{H}} \mathfrak{S}(\Omega_H)$ with a subgroup of $\mathfrak{S}(\Omega)$. Under this identification, $\prod_{H \in \mathfrak{H}} \mathrm{Aut}_G(\Omega_H)$ corresponds to $\mathrm{Aut}_G(\Omega)$, by (i). □

Let Λ be a finite G-set, let r be a positive integer, and let $\Gamma := \{1, \ldots, r\}$. Then $\Lambda \times \Gamma$ becomes a G-set via

$$g(\alpha, i) := (g\alpha, i), \qquad \text{for } g \in G,\ \alpha \in \Lambda,\ i \in \Gamma.$$

Then the G-set $\Lambda \times \Gamma$ is isomorphic to a disjoint union of r copies of Λ. As in Section 3, we regard $\mathfrak{S}(\Lambda) \wr \mathfrak{S}(\Gamma)$ as a subgroup of $\mathfrak{S}(\Lambda \times \Gamma)$.

Lemma 4.2. *In the situation above, suppose that Λ is transitive. Then we have* $\mathrm{Aut}_G(\Lambda \times \Gamma) = \mathrm{Aut}_G(\Lambda) \wr \mathfrak{S}(\Gamma)$.

Proof. Let $(f_1, \ldots, f_r; \sigma) \in \mathrm{Aut}_G(\Lambda) \wr \mathfrak{S}(\Gamma)$, $g \in G$, $\alpha \in \Lambda$ and $i \in \Gamma$. Furthermore, let $\overline{(f_1, \ldots, f_r; \sigma)}$ denote the corresponding permutation in $\mathfrak{S}(\Lambda \times \Gamma)$. Then

$$\begin{aligned}\overline{(f_1, \ldots, f_r; \sigma)}[g(\alpha, i)] &= \overline{(f_1, \ldots, f_r; \sigma)}(g\alpha, i) = (f_{\sigma(i)}(g\alpha), \sigma(i)) = (gf_{\sigma(i)}(\alpha), \sigma(i)) \\ &= g(f_{\sigma(i)}(\alpha), \sigma(i)) = g[\overline{(f_1, \ldots, f_r; \sigma)}(\alpha, i)].\end{aligned}$$

This shows that $\mathrm{Aut}_G(\Lambda) \wr \mathfrak{S}(\Gamma) \leq \mathrm{Aut}_G(\Lambda \times \Gamma)$.

Conversely, let $f \in \mathrm{Aut}_G(\Lambda \times \Gamma)$, and let $(\alpha, i), (\beta, j) \in \Lambda \times \Gamma$ with $f(\alpha, i) = (\beta, j)$. Then

$$f(g\alpha, i) = f(g(\alpha, i)) = gf(\alpha, i) = g(\beta, j) = (g\beta, j), \qquad \text{for } g \in G.$$

Since Λ is transitive, we obtain $f(\Lambda \times \{i\}) = \Lambda \times \{j\}$. Thus, for every $i \in \Gamma$, there is one and only one $\sigma_f(i) \in \Gamma$ such that $f(\Lambda \times \{i\}) = \Lambda \times \{\sigma_f(i)\}$. This yields a permutation $\sigma_f \in \mathfrak{S}(\Gamma)$, and $(1, \ldots, 1; \sigma_f) \in \mathrm{Aut}_G(\Lambda) \wr \mathfrak{S}(\Gamma)$. Moreover, for $\alpha \in \Lambda$ and $i \in \Gamma$ we have

$$\overline{(1, \ldots, 1; \sigma_f)}(\alpha, i) = (\alpha, \sigma_f(i)),$$

so that $\overline{(1, \ldots, 1; \sigma_f)}(\Lambda \times \{i\}) = \Lambda \times \{\sigma_f(i)\}$. Hence we get $h := \overline{(1, \ldots, 1; \sigma_f)}^{-1} \circ f \in \mathrm{Aut}_G(\Lambda \times \Gamma)$ and

$$h(\Lambda \times \{i\}) = \overline{(1, \ldots, 1; \sigma_f)}^{-1}(\Lambda \times \{\sigma_f(i)\}) = \Lambda \times \{i\}.$$

For $\alpha \in \Lambda$ and $i \in \Gamma$, we may therefore write $h(\alpha, i) = (h_i(\alpha), i)$ with $h_i(\alpha) \in \Lambda$. Then $h_i \in \mathfrak{S}(\Lambda)$, and, for $g \in G$ and $\alpha \in \Lambda$, we have

$$(h_i(g\alpha), i) = h(g\alpha, i) = h(g(\alpha, i)) = gh(\alpha, i) = g(h_i(\alpha), i) = (gh_i(\alpha), i),$$

so that $h_i(g\alpha) = gh_i(\alpha)$. This shows that $h_i \in \mathrm{Aut}_G(\Lambda)$, for $i \in \Gamma$, and thus $h = \overline{(h_1, \ldots, h_r; 1)} \in \mathrm{Aut}_G(\Lambda) \wr \mathfrak{S}(\Gamma)$. Hence $f = \overline{(1, \ldots, 1; \sigma_f)} \circ h \in \mathrm{Aut}_G(\Lambda) \wr \mathfrak{S}(\Gamma)$, and the assertion follows. $\square$

Lemma 4.1 and Lemma 4.2 reduce the computation of $\mathrm{Aut}_G(\Omega)$ to the computation of automorphism groups of transitive G-sets. We therefore now consider $\mathrm{Aut}_G(\Lambda)$ for a transitive G-set Λ.

Lemma 4.3. *Let Λ be a transitive G-set and $H \leq G$ such that $\Lambda \cong G/H$. Then*

$$\mathrm{Aut}_G(\Lambda) \cong \mathrm{Aut}_G(G/H) \cong N_G(H)/H.$$

Proof. For $xH \in N_G(H)/H$, the map

$$f_{xH} : G/H \longrightarrow G/H,\ gH \longmapsto gHx^{-1} = gx^{-1}H,$$

belongs to $\mathrm{Aut}_G(G/H)$ since $g' f_{xH}(gH) = g'gx^{-1}H = f_{xH}(g'gH)$, for all $g, g' \in G$. Moreover, we have $f_{xH} \circ f_{yH} = f_{xyH}$, for all $xH, yH \in N_G(H)/H$. Thus the map

$$\Phi : N_G(H)/H \longrightarrow \mathrm{Aut}_G(G/H),\ xH \longmapsto f_{xH},$$

is a homomorphism of groups and certainly injective. In order to prove that Φ is surjective, let $f \in \mathrm{Aut}_G(G/H)$, and let $x \in G$ such that $f(H) = x^{-1}H$. Then $x^{-1}H = f(H) = f(hH) = hf(H) = hx^{-1}H$, for all $h \in H$, which implies that $x \in N_G(H)$, and $f(gH) = gf(H) = gx^{-1}H = f_{xH}(gH)$, for all $g \in G$. Therefore we get $f = f_{xH} = \Phi(xH)$, and Φ is an isomorphism of groups. □

Combining the previous results, we obtain the following

Theorem 4.4. *Let G be a finite group, let Ω be a finite G-set, and let $\mathfrak{H}$ be a transversal for the conjugacy classes of subgroups of G. Then*

$$\Omega = \biguplus_{H \in \mathfrak{H}} \Omega_H \qquad \textit{and} \qquad \mathrm{Aut}_G(\Omega) \cong \prod_{H \in \mathfrak{H}} (N_G(H)/H) \wr \mathfrak{S}_{r_H}.$$

Now we specialize to the situation where G is a subgroup of $\mathfrak{S}(\Omega)$, for a finite set Ω. Then Ω becomes a G-set in the usual way, and we have $\mathrm{Aut}_G(\Omega) = C_{\mathfrak{S}(\Omega)}(G)$, as we observed above.

Theorem 4.5. *Let Ω be a finite set, and let G be a subgroup of $\mathfrak{S}(\Omega)$. Then $C_{\mathfrak{S}(\Omega)}(G) = Z(G)$ if and only if there exist pairwise nonconjugate subgroups $H_1, \ldots, H_k$ of G such that*

$$\Omega \cong \biguplus_{i=1}^{k} G/H_i \qquad \textit{and} \qquad N_G(H_i) = H_i \times Z_i,$$

where $Z_i := Z(G) \cap \bigcap_{j \neq i} H_j$, for $i = 1, \ldots, k$. Moreover, in this case we have

$$Z(G) = \prod_{i=1}^{k} Z_i \cong \prod_{i=1}^{k} N_G(H_i)/H_i.$$

Proof. With the above notation, we have

$$\Omega = \biguplus_{H \in \mathfrak{H}} \Omega_H \qquad \text{and} \qquad C_{\mathfrak{S}(\Omega)}(G) \cong \prod_{H \in \mathfrak{H}} (N_G(H)/H) \wr \mathfrak{S}_{r_H}.$$

We suppose first that $C_{\mathfrak{S}(\Omega)}(G) = Z(G)$. Since the elements in $Z(G)$ fix every G-orbit, we conclude that $r_H \leq 1$, for $H \in \mathfrak{H}$. Let $\{H \in \mathfrak{H} | r_H = 1\} = \{H_1, \ldots, H_k\}$. Then

$$\Omega \cong \biguplus_{i=1}^{k} G/H_i \qquad \text{and} \qquad Z(G) = C_{\mathfrak{S}(\Omega)}(G) \cong \prod_{i=1}^{k} N_G(H_i)/H_i.$$

Moreover, the map

$$\varphi : Z(G) \longrightarrow \prod_{i=1}^{k} N_G(H_i)/H_i,\ z \longmapsto (zH_1, \ldots, zH_k),$$

is an isomorphism of groups. Hence $1 = \ker(\varphi) = Z(G) \cap H_1 \cap \ldots \cap H_k$, and for $i = 1, \ldots, k$ we have

$$\varphi(Z_i) = \prod_{l=1}^{k} K_l$$

where $K_i := N_G(H_i)/H_i$ and $K_l := 1$, for $l \neq i$. This implies that $Z(G) = \prod_{i=1}^{k} Z_i$ and $N_G(H_i) = H_i Z_i = H_i \times Z_i$, for $i = 1, \ldots, k$.

Conversely, suppose that $\Omega \cong \biguplus_{i=1}^{k} G/H_i$ with pairwise nonconjugate subgroups $H_1, \ldots, H_k \in \mathfrak{H}$ of G. Suppose further that $N_G(H_i) = H_i \times Z_i$ with Z_i as above, for $i = 1, \ldots, k$. Then $\Omega_{H_i} \cong G/H_i$ and $r_{H_i} = 1$, for $i = 1, \ldots, k$, whereas $\Omega_H = \emptyset$, for $H \in \mathfrak{H} \setminus \{H_1, \ldots, H_k\}$. Moreover, since Ω is a faithful G-set, we have $Z(G) \cap H_1 \cap \ldots \cap H_k = 1$. Thus $H_i \cap Z_i = 1$, for $i = 1, \ldots, k$, and

$$Z(G) \leq C_{\mathfrak{S}(\Omega)}(G) \cong \prod_{i=1}^{k} N_G(H_i)/H_i \cong \prod_{i=1}^{k} Z_i \leq Z(G),$$

and the result is proved. □

We now specialize to the case where G is abelian.

Corollary 4.6. *Let Ω be a finite set, and let G be an abelian subgroup of $\mathfrak{S}(\Omega)$ such that $C_{\mathfrak{S}(\Omega)}(G) = G$. Then*

$$\Omega = \biguplus_{i=1}^{k} \Omega_i \qquad \textit{and} \qquad G = \prod_{i=1}^{k} G_i,$$

where G_i is a regular permutation group on Ω_i, for $i = 1, \ldots, k$. In particular, G is the direct product of its transitive constituents.

Proof. By the theorem, we have $\Omega \cong \biguplus_{i=1}^{k} G/H_i$ with pairwise nonconjugate subgroups $H_1, \ldots, H_k$ of G satisfying $H_1 \cap \ldots \cap H_k = 1$. Moreover, we have $G = N_G(H_i) = H_i \times Z_i$ with $Z_i = \bigcap_{j \neq i} H_j$, for $i = 1, \ldots, k$, and $G = Z(G) = \prod_{i=1}^{k} Z_i$. Each Z_i acts regularly on G/H_i and trivially on G/H_j, for $j \neq i$. Setting $G_i := Z_i$, for $i = 1, \ldots, k$, the result follows. □

In contrast to the abelian case, a nonabelian group G, in the situation of Theorem 4.5, is not necessarily the direct product of its transitive constituents, as the following example shows.

Example 4.7. We consider the dihedral group D_8 of order 8 and the embeddings

$$G \hookrightarrow D_8 \times D_8 \hookrightarrow \mathfrak{S}_4 \times \mathfrak{S}_4 \hookrightarrow \mathfrak{S}_8,$$

where $G := \Delta(D_8)[1 \times Z(D_8)] = \Delta(D_8)[Z(D_8) \times 1]$, and $\Delta(D_8)$ denotes the diagonal subgroup of $D_8 \times D_8$. Then G has order 16 and is isomorphic to $D_8 \times C_2$. We now compute $C_{\mathfrak{S}_8}(G)$. We may assume that G contains the elements $x := (1) \cdots (4)(5,6)(7,8)$ and $y := (1,2)(3,4)(5) \cdots (8)$. Then $C_{\mathfrak{S}_8}(x) = \mathfrak{S}_4 \times D_8$ and $C_{\mathfrak{S}_8}(y) = D_8 \times \mathfrak{S}_4$. This shows that $C_{\mathfrak{S}_8}(G) \leq D_8 \times D_8$. Furthermore, we have

$$C_{\mathfrak{S}_8}(G) \leq C_{D_8 \times D_8}(\Delta(D_8)) \leq C_{D_8}(D_8) \times C_{D_8}(D_8) = Z(D_8) \times Z(D_8) \leq G$$

and thus $C_{\mathfrak{S}_8}(G) = Z(G)$. On the other hand, G acts on each of the two orbits of length 4 as a permutation group isomorphic to D_8. But $|G| = 16 < 64 = 8^2$, so that G is not the direct product of its transitive constituents.

Remark 4.8. We note that a part of Corollary 4.6 generalizes to the nonabelian case. In order to make this more precise, suppose that $H_i \trianglelefteq G$, for some $i \in \{1, \ldots, k\}$, in the situation of Theorem 4.5. Then $G = N_G(H_i) = Z_i \times H_i$ where Z_i acts regularly on $\Omega_i \cong G/H_i$ and trivially on G/H_j, for $j \neq i$. Moreover, H_i acts trivially on $G/H_i \cong \Omega_i$, and we have

$$Z_i \times Z(H_i) = Z(G) = C_{\mathfrak{S}(\Omega_i)}(Z_i) \times C_{\mathfrak{S}(\Omega \setminus \Omega_i)}(H_i).$$

We conclude that $C_{\mathfrak{S}(\Omega \setminus \Omega_i)}(H_i) = Z(H_i)$. Thus H_i, considered as a subgroup of $\mathfrak{S}(\Omega \setminus \Omega_i)$, satisfies the same conditions as the subgroup G of $\mathfrak{S}(\Omega)$.

Example 4.9. Let Ω be a finite set, and let G be a selfcentralizing subgroup of $\mathfrak{S}(\Omega)$. We assume that G is a (generalized) quaternion 2-group. Then, in the situation of Theorem 4.5, we have $Z(G) \cap H_1 \cap \ldots \cap H_k = 1$. Since $Z(G)$ is the only subgroup of G of order 2, this implies that $H_i = 1$, for some $i \in \{1, \ldots, k\}$. We conclude that $G = N_G(H_i) = H_i \times Z_i = Z_i$ is abelian which is impossible.

Example 4.9 shows that not every group can be realized as a selfcentralizing subgroup of a finite symmetric group. We will have to say more about this below.

5 Bounds

In the following, we specialize Theorem 4.5 to the case where G is a finite p-group, for some prime number p.

Theorem 5.1. *Let Ω be a finite set, and let G be a finite p-subgroup of $\mathfrak{S}(\Omega)$ where p is a prime number. Moreover, suppose that $C_{\mathfrak{S}(\Omega)}(G) = Z(G)$ and that G has no fixed points on Ω. Then the number k of orbits of G on Ω satisfies $k \leq \mathrm{rk}(Z(G))$, and*

$$|\Omega| \leq \frac{k|G|}{p^{k-1}} \leq |G|.$$

Proof. In the notation of Theorem 4.5, we have $H_i < G$ and thus $H_i < N_G(H_i)$, for $i = 1, \ldots, k$. Since $Z(G) \cong \prod_{i=1}^k N_G(H_i)/H_i$, we conclude that $k \leq \mathrm{rk}(Z(G))$. We may assume that $p^a := |H_1| \leq \ldots \leq |H_k|$. Then $Z_j = Z(G) \cap \bigcap_{l \neq j} H_l \leq H_1$, for $j = 2, \ldots, k$, so that $Z_2 \times \cdots \times Z_k \leq H_1$ and $p^a = |H_1| \geq p^{k-1} \geq k$. Hence

$$|\Omega| = \sum_{i=1}^{k} |G : H_i| \leq k|G : H_1| = \frac{k|G|}{p^a} \leq \frac{k|G|}{p^{k-1}} \leq |G|.$$

□

By Corollary 4.6, the case where G is abelian is well understood. Let us therefore analyze the nonabelian case in more detail.

Corollary 5.2. *In the situation of Theorem 5.1, suppose that G is nonabelian. Then*

$$|\Omega| \leq \frac{k|G|}{p^k}.$$

Proof. In the notation above, we assume that $H_1 = Z_2 \times \cdots \times Z_k$. Then $H_1 \trianglelefteq G$, so that $G = N_G(H_1) = Z_1 \times H_1 = Z_1 \times Z_2 \times \cdots \times Z_k = Z(G)$, a contradiction. Thus $|H_1| > |Z_2 \times \cdots \times Z_k| \geq p^{k-1}$ and $|H_1| \geq p^k$. We conclude that

$$|\Omega| = \sum_{i=1}^{k} |G : H_i| \leq k|G : H_1| \leq \frac{k|G|}{p^k}.$$

□

We now apply the above results to vertices of simple $F\mathfrak{S}_n$-modules. Therefore let D be a simple $F\mathfrak{S}_n$-module with vertex V belonging to a block B of $F\mathfrak{S}_n$ of weight w. By Remark 3.1, we may assume that V is a subgroup of $\mathfrak{S}_{pw} \leq \mathfrak{S}_n$. Then $C_{\mathfrak{S}_{pw}}(V) = Z(V)$, and V acts without fixed points on $\Omega := \{1, \ldots, pw\}$. Thus we can apply our results to $V = G$.

Proposition 5.3. *In the situation above, there exist pairwise nonconjugate proper subgroups $H_1, \ldots, H_k$ of V such that*

$$\Omega \cong \biguplus_{i=1}^{k} V/H_i \qquad \text{and} \qquad N_V(H_i) = H_i \times Z_i,$$

where $Z_i := Z(V) \cap \bigcap_{j \neq i} H_j$, for $i = 1, \ldots, k$. Moreover, we have

$$Z(V) = \prod_{i=1}^{k} Z_i \cong \prod_{i=1}^{k} N_V(H_i)/H_i,$$

so that $k \leq \mathrm{rk}(Z(V))$ and $w \leq k|V|/p^k \leq |V|/p$. Furthermore, the following holds:

(i) If V is abelian then $\Omega = \biguplus_{i=1}^{k} \Omega_i$ and $V = \prod_{i=1}^{k} V_i$ where V_i acts regularly on Ω_i and trivially on $\Omega \setminus \Omega_i$, for $i = 1, \ldots, k$.

(ii) If V is nonabelian then $w \leq \frac{k|V|}{p^{k+1}} \leq \frac{|V|}{p^2}$.

Since V is a subgroup of $\mathfrak{S}_{pw}$, we conclude that $|\Delta| \leq (pw)! \leq |V|!$, for any defect group Δ of the block B containing D. This is the inequality mentioned in the introduction.

6 Vertices of small orders

Let D be a simple $F\mathfrak{S}_n$-module with vertex V belonging to a block B of weight w. W.l.o.g. we assume that V is a subgroup of $\mathfrak{S}_{pw}$ and denote the number of orbits of V on $\Omega := \{1, \ldots, pw\}$ by k. In this section we illustrate the results of the previous sections by investigating the cases where $|V|$ is "small".

Proposition 6.1. *(i) If $|V| = 1$ then $V = 1$ and $w = 0$.*

(ii) If $|V| = p$ then $V \sim_{\mathfrak{S}_n} P_p$ and $w = 1$.

(iii) If $|V| = p^2$ then either $V \sim_{\mathfrak{S}_n} (P_p)^2$ and $w = 2$, or $V \sim_{\mathfrak{S}_n} E_{p^2}$ and $w = p$. Here, E_{p^2} denotes an elementary abelian group of order p^2 acting regularly on the set $\Omega := \{1, \dots, p^2\}$.

(iv) If $|V| = p^3$ then one of the following holds:

- *$V \sim_{\mathfrak{S}_n} (P_p)^3$, $k = 3$ and $w = 3$;*
- *$V \sim_{\mathfrak{S}_n} C_{p^2} \times P_p$, $k = 2$ and $w = p + 1$;*
- *$V \sim_{\mathfrak{S}_n} E_{p^2} \times P_p$, $k = 2$ and $w = p + 1$;*
- *$V \sim_{\mathfrak{S}_n} E_{p^3}$, $k = 1$ and $w = p^2$, where E_{p^3} denotes an elementary abelian group of order p^3 acting regularly on the set $\Omega := \{1, \dots, p^3\}$;*
- *$V \sim_{\mathfrak{S}_n} C_{p^2} \times C_p$, $k = 1$ and $w = p^2$;*
- *$V \sim_{\mathfrak{S}_n} P_4$, $p = 2$, $k = 1$ and $w = 2$;*
- *V is extraspecial of exponent p, $p > 2$, $k = 1$ and $w = p$;*
- *V is extraspecial of exponent p^2, $p > 2$, $k = 1$ and $w = p$.*

Proof. The assertions (i) and (ii) clearly hold. If $|V| = p^2$ then V is abelian. By Theorem 2.2, V cannot be cyclic. Thus (iii) follows immediately from Proposition 5.3 (i).
We may therefore assume that $|V| = p^3$. If V is abelian then $k \leq 3$, by Proposition 5.3. Since V cannot be cyclic, Proposition 5.3 (i) leads to the five possibilities in (iv).
It remains to deal with the case where G is nonabelian, so that $|Z(V)| = p$. In this case, Proposition 5.3 implies that $k = 1$. Hence V acts transitively on Ω. By Proposition 5.3 (ii), we must have $|\Omega| \in \{p, p^2\}$. But the case $|\Omega| = p$ cannot occur since $|P_p| = p$. Thus we get $|\Omega| = p^2$ and $w = p$. If $p = 2$ then $|V| = 8 = |P_4|$ and thus $V \sim_{\mathfrak{S}_n} P_4$. We may therefore assume that p is odd. This leads to the two remaining cases in (iv). ☐

Remark 6.2. (a) In each of the last two cases of (iv), V is unique up to conjugation in $\mathfrak{S}_{pw}$. This follows from the fact that, for any two noncentral subgroups V_1 and V_2 of order p in V, there exists an automorphism φ of V such that $\varphi(V_1) = V_2$.

(b) The observation (a) extends to extraspecial p-groups V of order p^{2r+1} and exponent p for $p > 2$. In fact, the automorphism group of V acts transitively on the set of noncentral subgroups of order p in V.

(c) By way of contrast, the automorphism group of an extraspecial p-group V of order $p^{2r+1} > p^3$ and exponent p^2, for $p > 2$, does not act transitively on the set of noncentral subgroups of order p in V. For instance, consider the extraspecial p-group V of order p^5 and exponent p^2, for $p > 2$. Then V is the central product of two extraspecial groups of order p^3, one of exponent p and one of exponent p^2. Hence there are noncentral elements $x, y \in V$ of order p such that $C_V(x)$ has exponent p and $C_V(y)$ has exponent p^2. Consequently, there cannot be an automorphism of V mapping $\langle x \rangle$ to $\langle y \rangle$.

By Proposition 6.1, p-groups V of order $|V| \leq p^3$ can only occur as vertices of simple $F\mathfrak{S}_n$-modules belonging to blocks of weight $w \leq p^2$. For $p = 2$, the vertices of all simple $F\mathfrak{S}_n$-modules belonging to blocks of weight $w \leq 4$ have been determined in [1] and [14]. Using these results we obtain:

Corollary 6.3. *Let $p = 2$ and $|V| \leq 8$.*

(i) If $|V| = 1$ then $V = 1$ and $w = 0$,

(ii) If $|V| = 2$ then $V \sim_{\mathfrak{S}_n} P_2$ and $w = 1$.

(iii) If $|V| = 4$ then either $V \sim_{\mathfrak{S}_n} (P_2)^2$ and $w = 2$, or $V \sim_{\mathfrak{S}_n} E_4 = Q_4$ and $w = 2$.

(iv) If $|V| = 8$ then one of the following holds:

- *$V \sim_{\mathfrak{S}_n} P_4$ and $w = 2$,*
- *$V \sim_{\mathfrak{S}_n} (P_2)^3$ and $w = 3$,*
- *$V \sim_{\mathfrak{S}_n} Q_4 \times P_2$ and $w = 3$,*
- *$V \sim_{\mathfrak{S}_n} E_8$ and $w = 4$.*

Moreover, each of these cases occurs.

Remark 6.4. (a) For $w \geq p > 2$, we do not know precisely which of the cases in Proposition 6.1 occur.

(b) For $p = 3$, most of the vertices of simple $F\mathfrak{S}_n$-modules belonging to blocks of weight $w = 3$ are known, by the results in [1] and [14]. Essentially, only the vertex of the simple $F\mathfrak{S}_{15}$-module $D^{(7,5,3)}$ of dimension 43497 is missing so far. This implies that at most one of the following three cases in Proposition 6.1 can occur, for $p = 3$ and $w = 3$:

- $V \sim_{\mathfrak{S}_n} E_9$,
- V extraspecial of order 27 and exponent 3,
- V extraspecial of order 27 and exponent 9.

Moreover, if one of these three cases occurs at all then it occurs for the simple $F\mathfrak{S}_{15}$-module $D^{(7,5,3)}$.

Next we turn to vertices of order 16.

Proposition 6.5. *Let $p = 2$ and $|V| = 16$.*

(i) If V is nonabelian then D belongs to a block of weight 3, and V is conjugate in $\mathfrak{S}_n$ to $P_6 = P_4 \times P_2$.

(ii) If V is abelian then one of the following holds:

- *$w = 8$, $k = 1$, and V is isomorphic to one of the following groups: $C_8 \times C_2$, $(C_4)^2$, $C_4 \times (C_2)^2$, $(C_2)^4$;*
- *$w = 5$, $k = 2$, and $V \cong V_1 \times V_2$ with transitive constituents $V_1 \in \{C_8, C_4 \times C_2, (C_2)^3\}$ and $V_2 = C_2$.*

Proof. As above, the simple $F\mathfrak{S}_n$-module D with vertex V belongs to a block B of weight w.

(i) If V is nonabelian then Proposition 5.3 (ii) implies that $w \leq 4$. The results in [1] show that the case $w = 4$ cannot occur since no simple $F\mathfrak{S}_n$-module in a block of weight 4 has a vertex of order 16. Thus we must have $w = 3$ and $V \sim_{\mathfrak{S}_n} P_6$.

(ii) Suppose that V is abelian. If V is transitive on $\Omega := \{1, \ldots, 2w\}$ then $w = 8$, and we are in the first case of (ii) since V cannot be cyclic.
Now suppose that V has two orbits on $\Omega := \{1, \ldots, 2w\}$. If these orbits have lengths 8 and 2 then we are in the second case of (ii). Assume that both orbits have length 4, so that $w = 4$. Then the results in [1] lead to a contradiction.
In a similar way, the assumption $k \geq 3$ leads to a contradiction. $\square$

Added in proof: The vertices of the simple $F\mathfrak{S}_{15}$-module $D^{(7,5,3)}$ in characteristic 3 are now known to be conjugate to the elementary abelian group E_9 of order 9 (see [2]). Therefore, we can also answer the question left open in Remark 6.4. Namely, if $p = 3$ then neither of the extraspecial groups of order 27 occurs as vertex of a simple $F\mathfrak{S}_n$-module.

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