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VERTICES OF SIMPLE MODULES FOR SYMMETRIC GROUPS: A SURVEY

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ABSTRACT. In this paper we give a survey on some recent results and open questions concerning the vertices of the simple modules for symmetric groups.

1. Introduction

Let F be a field, G a finite group and FG the group algebra of G over F. The vertex of an indecomposable FG-module M, first defined by J. A. Green [33], is an important invariant of M. It is an essentially unique subgroup of G which measures the (relative) projectivity of M. Vertices of simple FG-modules enjoy a number of properties not necessarily valid for vertices of arbitrary indecomposable FG-modules. It is an open problem to determine the vertices of the simple $F\mathfrak{S}_n$-modules where $\mathfrak{S}_n$ denotes the symmetric group of degree n. At this time not even a reasonable general conjecture exists.

In the present paper, we will describe some of the positive results which have been obtained in recent years by several mathematicians including B. Fotsing, J. Müller, M. Wildon, R. Zimmermann and the authors of this paper. As is common in mathematics, these positive results have led to a number of further questions, and we take this opportunity to mention those which we consider to be most interesting. It was impossible to include all relevant material in this survey; so we had to make choices. Also, we decided to omit all proofs.

2. Relative projectivity

As above, let F be a field of characteristic $p \geq 0$, and let FG denote the group algebra of a finite group G over F. When we speak of an FG-module then we will always mean a finitely generated left FG-module. For FG-modules M, N, we write $M \mid N$ if M is isomorphic to a direct summand of N. For a subgroup H of G, we denote the *restriction* of an FG-module M to FH by $\mathrm{Res}_H^G(M)$. Similarly, we define the *induction* of an FH-module N to FG by $\mathrm{Ind}_H^G(N) = FG \otimes_{FH} N$. An FG-module M is called *relatively H-projective* if

$$M \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M)).$$

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This notion goes back to the work of D. G. Higman [37] and G. Hochschild [38]. If $H = 1$ is the trivial subgroup of G then M is relatively 1-projective if and only if it is projective, i.e. if and only if $M \mid L$ for a free FG-module L. In this way, the notion of "relative projectivity" generalizes the notion of "projectivity".

If an FG-module M is relatively H-projective for a subgroup H of G then so is $M \otimes N$, for any FG-module N. Thus, for every positive integer i, the i-th *tensor power* $\bigotimes^i M$ is relatively H-projective. Also, the i-th *symmetric power* $\mathrm{S}_i(M)$ is relatively H-projective whenever p does not divide i, as shown by P. Fleischmann [28]. A similar result holds for the i-th *exterior power* $\bigwedge^i M$ (cf. [45]). If $\mathrm{char}(F) = 0$ then, by Maschke's Theorem, every FG-module is projective. For this reason, we will from now on assume that $\mathrm{char}(F) = p > 0$.

By the Krull-Schmidt Theorem, every FG-module M can be decomposed as a direct sum of indecomposable submodules $M_1, \ldots, M_n$:

$$M = M_1 \oplus \cdots \oplus M_n;$$

moreover, such a decomposition is essentially unique. Thus, for many purposes, it suffices to deal with indecomposable FG-modules.

3. Vertices of indecomposable modules

If M is an indecomposable FG-module then, following J. A. Green [33], a *vertex* of M is a subgroup Q of G which is minimal subject to the condition that M is relatively Q-projective. In this case Q is a p-subgroup of G and unique up to conjugation within G.

Let M be an indecomposable FG-module with vertex Q. Then Q is also a vertex of the *dual* FG-module $M^* := \mathrm{Hom}_F(M, F)$. Moreover, Q is also a vertex of the *Heller translate* ΩM where ΩM denotes the kernel of a projective cover

$$0 \longrightarrow \Omega M \longrightarrow P \longrightarrow M \longrightarrow 0$$

of M (provided that $Q \neq 1$).

Let M be an indecomposable FG-module with vertex Q, and let $E|F$ be a field extension. Then Q is also a vertex of every indecomposable direct summand of the EG-module $E \otimes_F M$ obtained from M by an *extension of scalars*. For this reason, we will from now on assume that F is algebraically closed.

Let again M be an indecomposable FG-module with vertex Q, and let S be a Sylow p-subgroup of G containing Q. Then Green's Indecomposability Theorem [33] implies that

$$|S : Q| \;\Big|\; \dim M. \tag{1}$$

This divisibility property is very useful for the actual computation of vertices. It can also be generalized in various directions. For example, C.

Bessenrodt and W. Willems [6] have proved that, in fact,

$$p^{r-c}|S:Q| \;\Big|\; \dim M; \tag{2}$$

here r denotes the *rank* of Q (i.e. p^r is the maximal order of an elementary abelian subgroup of Q), and c denotes the *complexity* of M (i.e. c is the minimal integer s such that

$$\lim_{n\to\infty} \frac{\dim P_n}{n^s} = 0$$

where $\ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ denotes a minimal projective resolution of M. We mention that always $r \geq c$; and $c = 0$ if and only if M is projective, while $c \leq 1$ if and only if M is periodic. Moreover, the complexity of M is also the dimension of the *variety* of M, which can be defined in terms of the cohomology ring $\mathrm{H}^*(G,k)$ (cf. [11]).

The divisibility property (2) has been improved further by C. Bessenrodt [5]: Suppose again that M is indecomposable with vertex Q, and let X be a subgroup of Q such that $X \cong \mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_s}\mathbb{Z}$ and $n_1 \geq \ldots \geq n_s$. Then

$$p^{n_{c+1}} \ldots p^{n_s}|S:Q| \;\Big|\; \dim M \tag{3}$$

where c denotes the complexity of $\mathrm{Res}^G_X(M)$, and S is as before.

4. Indecomposable modules and the Grothendieck ring

In the following, we denote the isomorphism class of a simple FG-module E by $[E]$. Moreover, we denote by $\mathcal{R}(FG)$ the free abelian group generated by the isomorphism classes of the simple FG-modules. Then $\mathcal{R}(FG)$ is a free abelian group of rank ℓ where ℓ stands for the number of p-regular conjugacy classes of G.

For an FG-module M with composition factors $E_1, \ldots, E_r$, we set

$$[M] := [E_1] + \cdots + [E_r] \in \mathcal{R}(FG).$$

A multiplication in $\mathcal{R}(FG)$ can be defined in such a way that

$$[M] \cdot [N] = [M \otimes_F N]$$

for FG-modules M, N. Then $\mathcal{R}(FG)$ becomes a commutative ring called the *Grothendieck ring* of FG. The identity element of $\mathcal{R}(FG)$ is the isomorphism class of the *trivial* FG-module which is the field F with

$$g\alpha = \alpha \quad \text{for} \quad g \in G,\ \alpha \in F.$$

It is easy to see that $\mathcal{P}(FG)$, the subgroup of $\mathcal{R}(FG)$ generated by

$$\{[P] : P \text{ projective } FG\text{-module}\},$$

is an ideal of $\mathcal{R}(FG)$. The quotient $\mathcal{R}(FG)/\mathcal{P}(FG)$ is sometimes called the *Cartan ring* of FG. A result of R. Brauer [9] determines the structure of $\mathcal{R}(FG)/\mathcal{P}(FG)$ as an abelian group: it is a finite abelian p-group; its order is the determinant of the *Cartan matrix* C of FG. (Recall that the entries of C are the multiplicities of the simple FG-modules as composition factors of the indecomposable projective FG-modules.) More precisely, we have

$$\mathcal{R}(FG)/\mathcal{P}(FG) \cong \prod_{i=1}^{\ell}(\mathbb{Z}/|S_i|\mathbb{Z})$$

where $g_1, \dots, g_\ell$ are representatives for the p-regular conjugacy classes of G, and S_i is a Sylow p-subgroup of the centralizer $C_G(g_i)$ for $i = 1, \dots, \ell$. (The structure of $\mathcal{R}(FG)/\mathcal{P}(FG)$ as a ring was determined by R. Boltje and B. Külshammer [7].)

For an indecomposable FG-module M with vertex Q, the order of Q is related to the order of the coset $[M] + \mathcal{P}(FG)$ in $\mathcal{R}(FG)/\mathcal{P}(FG)$ in the following way:

$$|Q| \cdot [M] \in \mathcal{P}(FG); \tag{4}$$

this is a result of M. Broué [10]. Since, by Dickson's Theorem, the dimension of a projective FG-module is divisible by the order of a Sylow p-subgroup of G, (4) is another strengthening of (1).

5. Sources and multiplicity modules

If M is an indecomposable FG-module with vertex Q then there exists an indecomposable FQ-module V such that $M \mid \mathrm{Ind}_Q^G(V)$. As J. A. Green [33] showed, V is unique up to isomorphism and conjugation with elements in $N_G(Q)$, and $V \mid \mathrm{Res}_Q^G(M)$. Moreover, V has vertex Q and is called a *source* of M. A. Watanabe [58] showed the following:

$$p \,\Big|\, \dim V \Longrightarrow p \,\Big|\, \frac{\dim M}{|S : Q|} \tag{5}$$

where S is a Sylow p-subgroup of G containing Q. (She also remarked that this fact is implicit in earlier work of R. Knörr.) Note that (5) also strengthens (1), in favourable situations.

We mention that, besides the vertex Q and the source V, there is another interesting invariant which can be attached to any indecomposable FG-module M. This is its *multiplicity module*; its definition goes back to L. Puig [53]. The multiplicity module of M is an indecomposable projective $F_\gamma[N_G(Q,V)/Q]$-module where

$$N_G(Q,V) := \{g \in N_G(Q) : g \otimes V \cong V\}$$

is the *inertia group* of V in $N_G(Q)$, and where $F_\gamma[N_G(Q,V)/Q]$ denotes a twisted group algebra of $N_G(Q,V)/Q$. This twisted group algebra can be constructed by regarding the endomorphism algebra

$$E := \mathrm{End}_{FN_G(Q,V)}(\mathrm{Ind}_Q^{N_G(Q,V)}(V))$$

as an $N_G(Q,V)/Q$-graded algebra and factoring out the graded radical of E. We note that the dimension of the multiplicity module of M is precisely the multiplicity of V as a direct summand of $\mathrm{Res}_Q^G(M)$; this explains the terminology.

In this way every indecomposable FG-module M determines an essentially unique triple (Q, V, P) where the p-subgroup Q of G is a vertex of M, the indecomposable FQ-module V is a source of M, and the indecomposable projective $F_\gamma[N_G(Q, V)/Q]$-module P is a multiplicity module of M. L. Puig [54] has shown that this leads to a bijection between the isomorphism classes of indecomposable FG-modules and the equivalence classes of such triples (Q, V, P) (with respect to an appropriate equivalence relation). We omit the details and refer the reader to the original articles [54] and [57].

It is a general theme of modern representation theory to describe the modules of a finite group G by "local" data; these "local" data are concerned with the non-trivial p-subgroups of G and their normalizers in G. For (non-projective) indecomposable FG-modules, Puig's parametrization provides such a local description.

Using the notion of multiplicity modules, L. Barker [1] has formulated a further strengthening of (1): Let M be an indecomposable FG-module, and let (Q, V, P) be a corresponding triple as considered above. Then

$$\dim M \equiv |G : N_G(Q, V)| \cdot \dim P \cdot \dim V \pmod{|S : Q| \mathrm{spr}_G(Q)}; \tag{6}$$

here S is again a Sylow p-subgroup of G containing Q, and

$$\mathrm{spr}_G(Q) := \min\{|Q : Q \cap gQg^{-1}| : g \in G \setminus N_G(Q)\}$$

is called the *spire* of Q in G. Moreover, $\mathrm{spr}_G(Q)$ is interpreted as 0 if Q is normal in G, and then the congruence (6) is interpreted as an equality.

We also note that, by (3), we have

$$p^{n_{c+1}} \cdots p^{n_s} \;\Big|\; \dim V$$

where c is the complexity of the restriction of V to an abelian subgroup $X \cong \mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_s}\mathbb{Z}$ of Q, and $n_1 \geq \ldots \geq n_s$.

6. Blocks and defect groups

The concept of a block was first defined by R. Brauer more than 60 years ago. Here we follow the later (equivalent) approach by J. A. Green [34]. The group algebra FG can be considered as an $F[G \times G]$-module where

$$(g, h) \cdot x = gxh^{-1} \quad \text{for} \quad g, h \in G,\ x \in FG.$$

Then there is a decomposition into indecomposable $F[G \times G]$-modules,

$$FG = B_1 \oplus \cdots \oplus B_r,$$

which is unique up to the order of $B_1, \ldots, B_r$. Then $B_1, \ldots, B_r$ are called the *blocks* of FG. By definition, each B_i is an ideal of FG and an F-algebra in its own right; its identity element e_i is called the *block idempotent* of B_i. Then

$$1_G = e_1 + \cdots + e_r \quad \text{with} \quad e_1, \ldots, e_r \in Z(FG),$$

and $B_i B_j = 0$ whenever $i \neq j$; in particular, $e_i e_j = 0$ whenever $i \neq j$.

We fix a block $B = B_i$ of FG. Since B is an indecomposable $F[G \times G]$-module, it has a vertex which is a p-subgroup of $G \times G$. It is easy to see that B has a vertex of the form $\Delta(R)$ where

$$\Delta : G \longrightarrow G \times G, \quad g \longmapsto (g, g),$$

denotes the diagonal map. Then R is a p-subgroup of G which is called a *defect group* of B. It was proved already by R. Brauer (following his original approach to block theory) that the defect groups of B are all conjugate in G. J. A. Green [34] showed that every defect group of a block is an intersection of (at most) two Sylow p-subgroups; in combination with results of R. Brauer, this implies that every defect group of a block is a radical p-subgroup. (Recall that a p-subgroup Q of G is called *radical* in G if $Q = O_p(N_G(Q))$.)

7. Modules and blocks

Let M be an indecomposable FG-module. Then there is a unique block B of FG such that $BM \neq 0$. Then $BM = M$, so that M can be considered as an indecomposable B-module. We say that *M belongs to B*. Then M is relatively R-projective where R denotes a defect group of B, as shown by J. A. Green [33]. Thus the defect group R of B contains a vertex Q of M. This means that the defect groups of a block B of FG are "upper bounds" for the vertices of the indecomposable FG-modules belonging to B.

The trivial FG-module F is, of course, indecomposable (even simple). Thus it belongs to a block B_0 of FG. Then B_0 is called the *principal block* of FG. By (1), the vertices of F are Sylow p-subgroups of G. Hence, by the preceding paragraph, the defect groups of the principal block B_0 of FG are also Sylow p-subgroups of G.

8. Vertices of simple modules

In the following, we specialize from indecomposable modules to simple modules. One reason for this is that, for a finite group G and an algebraically closed field F of characteristic $p > 0$, the number of isomorphism classes of indecomposable FG-modules is often infinite whereas the number of isomorphism classes of simple FG-modules is always finite. Also, vertices of simple FG-modules have certain special properties not shared by vertices of arbitrary indecomposable FG-modules. One of the most useful results in this direction is the following one:

Theorem 8.1. *(Knörr [44]) Let M be a simple FG-module with vertex Q belonging to a block B of FG. Then there exists a defect group R of B such that*

$$Z(R) \leq C_R(Q) \leq Q \leq R.$$

This result can be interpreted as giving "lower bounds" for the vertices of simple FG-modules. In the special case where B has an abelian defect group R, the lower bound and the upper bound coincide, so that then all simple FG-modules belonging to B have vertex R.

A result related to Knörr's Theorem above is the following one:

Theorem 8.2. *(Erdmann [25]) Let M be a simple FG-module with cyclic vertex Q belonging to a block B of FG. Then Q is a defect group of B.*

A generalization of Erdmann's Theorem can be found in [46]. Also, for every block B of FG with defect group R, there is always at least one simple FG-module that belongs to B and has vertex R.

Conversely, let us start with a simple FG-module M with vertex Q belonging to a block B of FG. Motivated by the theorems above, one may ask for consequences on the structure of a defect group R of B. A specific question, attributed to Puig, is asked in a paper by J. Zhang [60]:

Question 8.3. *Is $|R|$ bounded in terms of $|Q|$?*

Having said this, we must point out immediately that there are counterexamples for $p = 2$: In these examples Q is elementary abelian of order 4, and R is dihedral of arbitrarily large order [26]. Apparently, similar examples in the case where p is odd are not known at present.

Motivated by this, J. Zhang [60] has proved a reduction theorem for Question 8.3 to quasi-simple groups. On the other hand, Question 8.3 has a positive answer for p-solvable groups. This follows from an unpublished result of L. Puig; a proof can be found in [6]. Question 8.3 also has a positive answer for symmetric groups, as we will see below.

9. Sources of simple modules

Puig's Question 8.3 is closely related to the following conjecture:

Conjecture 9.1. *(Feit [27]) Let Q be a finite p-group. Then there are only finitely many indecomposable FQ-modules, up to isomorphism, which are sources of simple FG-modules, for finite overgroups G of Q.*

A weak form of this conjecture was proved by Dade [13]: he showed that, for any fixed positive integer d, there are only finitely many indecomposable FQ-modules of dimension d, satisfying the conclusion of Conjecture 9.1. Dade's proof uses methods of algebraic geometry.

It is also known that Feit's conjecture holds in the context of p-solvable groups; this is an unpublished result of Puig. A proof can be obtained by using the fact that, for a p-solvable group G, the sources of the simple FG-modules are algebraic endo-permutation modules. (We recall that an FG-module M is called *algebraic* if its tensor powers $M, M \otimes M, M \otimes M \otimes M, \ldots$ have only finitely many indecomposable direct summands, up to isomorphism; and an FG-module M is called an *endo-permutation* module if $M^* \otimes M = \mathrm{End}_F(M)$ is a permutation FG-module where $M^* = \mathrm{Hom}_F(M, F)$ denotes the dual module of M.)

Suppose again that G is an arbitrary finite group. R. Knörr [44] has also proved that, for a simple FG-module M with vertex Q and source V, the converse of (5) holds:

$$p \mid \dim V \Longleftrightarrow p \mid \frac{\dim M}{|S:Q|} \tag{7}$$

where S is a Sylow p-subgroup of G containing Q.

This fact has been made more precise by L. Barker [1], by making use of (6). He shows the following:

$$(\dim M)_p \equiv |S:Q|(\dim V)_p \pmod{|S:Q|\mathrm{spr}_G(Q)}; \tag{8}$$

here n_p denotes the p-part of an integer n, i.e. the highest power of p dividing n. The essential result behind (8) is a consequence of the proof of Knörr's Theorem (8.1): The multiplicity modules of simple FG-modules are again simple (and projective). Again, the remarks following (6) can provide further powers of p dividing $\dim V$, in favourable situations.

10. Other classes of groups

In order to put the results on symmetric groups which we are going to describe below into perspective, we briefly recall what is known for other classes of finite groups.

Suppose first that G is p-solvable, and let M be a simple FG-module with vertex Q contained in a Sylow p-subgroup S of G. Then, as shown by W. Hamernik and G. O. Michler [35], the p-part of $\dim M$ coincides with $|S:Q|$ (cf. (1)); they also showed that then the projective cover of M has dimension $|Q|\dim M$. Analogous results are not true for symmetric groups, in general.

An unpublished result of T. Okuyama [51] implies that, for a p-solvable group G, every vertex of a simple FG-module is an intersection of two Sylow p-subgroups of G. His paper also shows that, for a p-solvable group G, every vertex of a simple FG-module is a radical p-subgroup of G. Similar results have also been proved by C. Bessenrodt [4]. Again, analogous results do not hold for symmetric groups.

Now suppose that G is a finite group of Lie type in characteristic p. Then, as shown by R. Dipper [22, 23], the vertices of all non-projective simple FG-modules are Sylow p-subgroups of G. Again, a corresponding result for symmetric groups does not hold.

11. Partitions

Now we turn our attention to a finite symmetric group $\mathfrak{S}_n$ of degree n. It is well-known that the isomorphism classes of simple $F\mathfrak{S}_n$-modules are labelled by p-regular partitions of n. Recall that a *partition* of n is a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers $\lambda_1 \geq \ldots \geq \lambda_k$ such that $\lambda_1 + \cdots + \lambda_k = n$. We write $\lambda \vdash n$. Such a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is called *p-regular* if each part λ_i appears at most $p-1$ times in λ. (In particular, for $p = 2$ we must have $\lambda_1 > \ldots > \lambda_k$.)

For a p-regular partition λ of n, the corresponding simple $F\mathfrak{S}_n$-module will be denoted by D^λ, as usual [40]. The module D^λ can be defined as the unique simple quotient of the *Specht module* S^λ. One of the main open problems in the representation theory of symmetric groups is the determination of the dimension of D^λ. At present there are complete results for $n \leq 17$ only. A related open problem is the determination of the vertices and sources of D^λ; here we now have complete results for $n \leq 14$ [20].

Many of these calculations require heavy computer assistance. One of the most complicated examples computed so far has been the simple $F\mathfrak{S}_{15}$-module $D^{(7,5,3)}$, of dimension 43497, in characteristic $p = 3$. It turns out that it has a vertex Q which is an elementary abelian group of order 9 acting regularly on $1, \ldots, 9$ (and fixing $10, \ldots, 15$). The defect groups of the corresponding block B of $F\mathfrak{S}_{15}$ are nonabelian of order 81.

We may add that also the vertices of all simple $F\mathfrak{S}_n$-modules (n arbitrary) of dimension $d \leq 1000$ have been determined [16].

12. Blocks and defect groups in symmetric groups

The distribution of the simple $F\mathfrak{S}_n$-modules D^λ into blocks is well-known. The corresponding result is still known as the Nakayama Conjecture, although it was proved by R. Brauer and G. de B. Robinson about 60 years ago (cf. [40]). An elegant proof of the Nakayama conjecture was given by L. Puig [52].

Theorem 12.1. *Let $\lambda, \mu \vdash n$ be p-regular. Then D^λ and D^μ belong to the same block B of $F\mathfrak{S}_n$ if and only if λ and μ have the same p-core.*

Here the *p-core* of λ is a p-regular partition κ of $n - pw$, for some nonnegative integer w called the *p-weight* of λ (and of the corresponding block of $F\mathfrak{S}_n$). The partition κ is constructed from the partition λ by a combinatorial algorithm called the *removal* of *p-hooks*. Details can be found in [40]. It is important that then the Sylow p-subgroups of the subgroup $\mathfrak{S}_{pw}$ of $\mathfrak{S}_n$ are defect groups of B.

The results above give a combinatorial description of the blocks of symmetric groups and their defect groups. It is our hope that some day a similar (but perhaps more involved) combinatorial description of the vertices of the simple $F\mathfrak{S}_n$-modules will exist.

Important progress on the block structure of symmetric groups was achieved by J. Scopes [56]. She proved that the blocks of all symmetric groups of a fixed weight w fall into finitely many *families*. Blocks in the same family are Morita equivalent, and for blocks in the same family, there is a natural bijection between the isomorphism classes of simple modules preserving vertices and sources. Let us illustrate her results by considering the example $p = 3$ and $w = 3$:

In this example the families are represented by blocks of $F\mathfrak{S}_n$ where $n \in \{9, 10, 11, 13, 14, 15, 19, 25\}$. Each of these blocks has 10 simple modules. The corresponding vertices have order 81, 27 or 9. The following table gives details:

$n \setminus \|Q\|$	81	27	9
9	10	-	-
10	8	2	-
11	7	3	-
13, 14	6	4	-
15, 19	5	4	1
25	4	4	2

It is a general fact that the number of simple modules belonging to a block B of $F\mathfrak{S}_n$ depends only on the characteristic p of F and the weight w of B. On the other hand, the table above shows that the vertices of the simple modules belonging to B do not just depend on p and w. It is perhaps of interest that in the first row of the table (which corresponds to the principal block of $F\mathfrak{S}_9$) all simple modules have the defect groups of the relevant block as vertices. A similar phenomenon appears in other examples, with slight modification for $p = 2$; but at present we cannot prove a general result. Moreover, when going down in the table, the average order of a vertex appears to become smaller. This can also be observed in other examples, but again, we cannot offer an explanation.

J. Chuang and R. Rouquier [12] have recently proved the important result that any two blocks A and B of finite symmetric groups having the same weight are derived equivalent; however, it is not clear at present how to use this fact for the determination of vertices of simple modules.

13. Puig's question for symmetric groups

In [55], L. Puig stated that his question (Question 8.3 above) has a positive answer in the context of finite symmetric groups. More precisely, he claimed that the defect group R of a block B of $F\mathfrak{S}_n$ and the vertex Q of a simple $F\mathfrak{S}_n$-module D^λ belonging to B are related by the following inequality:

$$|R| \leq |Z(Q)|!; \tag{9}$$

however, it was observed by the first author of this paper that there are counterexamples to this inequality. A correct inequality can be given as follows [18]:

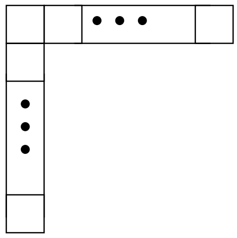
$$|R| \leq |Q|!. \tag{10}$$

This means that Puig's Question has indeed a positive answer for symmetric groups. A consequence of this and of Scopes' results mentioned above is that also Feit's Conjecture 9.1 on sources of simple modules holds for symmetric groups.

We note that (10) is not really a sharp bound; for example, it is possible to show that $|R| \leq p^{p+1}$ if $|Q| = p^2$.

14. Hook partitions

A *hook partition* is a partition of the form $\lambda = (n-r, 1^r) = (n-r, 1, \dots, 1)$. It is p-regular if and only if $r < p$ (provided that $n > p$). The *Young diagram* (cf. [40]) $[\lambda]$ of a hook partition λ of n has the following shape:



We believe that the vertices of the simple $F\mathfrak{S}_n$-module D^λ corresponding to a p-regular hook partition are always defect groups of the corresponding block B, except when $p = 2$, $n = 4$ and $r = 1$; in this exceptional case the unique Sylow 2-subgroup V_4 of the alternating group $\mathfrak{A}_4$ is a vertex of $D^{(3,1)}$ while the defect groups of the corresponding block are Sylow 2-subgroups of $\mathfrak{S}_4$ [49]. This conjecture is now known to hold in most cases, as follows from papers by M. Wildon [59], by J. Müller and R. Zimmermann [49], and by S. Danz [14]. The only case which has remained open so far is the following one:

$$p > 2, \quad n = pw, \quad w \equiv 1 \pmod{p}, \quad r = p - 1.$$

In this case D^λ belongs to the principal block of $F\mathfrak{S}_n$, so we expect that the vertices of D^λ are Sylow p-subgroups of $\mathfrak{S}_n$.

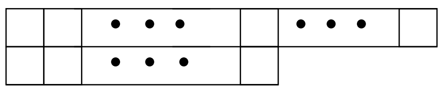
It is known that

$$D^{(n-r,1^r)} \cong \bigwedge^r D \quad \text{for} \quad r < p$$

where $D = D^{(n-1,1)}$ denotes the *natural* simple $F\mathfrak{S}_n$-module, i.e. the only nontrivial composition factor of the permutation module of $F\mathfrak{S}_n$ on the points $1, \dots, n$ (provided that it exists). Then $\dim D = n - 1$ if $p \nmid n$, and $\dim D = n - 2$ if $p \mid n$. Moreover, it is known that, in odd characteristic, $\bigwedge^r D$ is a simple $F\mathfrak{S}_n$-module whenever $r \leq \dim D$. We note that in the case where $r \geq p$ there are only partial results concerning the vertices of $\bigwedge^r D$ (see [14] and [59]).

15. Two-part partitions

In this section we consider p-regular partitions of the form $\lambda = (n-m, m)$ where $n \geq 2m$. The Young diagram of such a partition has the form:



The vertices of D^λ were determined by S. Danz [17] in the case where $p > 2$ and $m < p(p+1)/2$. They are either defect groups of the corresponding block or Sylow p-subgroups of the Young subgroup $\mathfrak{S}_{n-2m} \times \mathfrak{S}_m \times \mathfrak{S}_m$ of $\mathfrak{S}_n$. Moreover, one can say precisely when the second alternative occurs. The same result is true for arbitrary two-part partitions whenever $p > 2$ and $n < 2p^2$.

On the other hand, not much is known about the vertices of D^λ, for $\lambda = (n-m, m)$ a two-part partition, in the case when $p = 2$; an exception is the case when $m = 1$ where λ is also a hook partition, and another exception is the case where λ corresponds to a basic spin module which will be discussed below.

16. Completely splittable modules

Suppose that $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ is p-regular. Then D^λ is called *completely splittable* if the restriction of D^λ to $\mathfrak{S}_m$ is semisimple for $m = 1, \ldots, n-1$. A. Kleshchev [41] has given a combinatorial characterization of such modules: The simple $F\mathfrak{S}_n$-module D^λ is completely splittable if and only if $\lambda_1 - \lambda_k + k \leq p$. It was shown by S. Danz [17] that the vertices of the completely splittable $F\mathfrak{S}_n$-modules are always defect groups of the corresponding blocks.

17. Basic spin modules

Suppose that $p = 2$. The *basic spin module* of $F\mathfrak{S}_n$ comes from the action of $\mathfrak{S}_n$ on certain Clifford algebras. It is labelled by the partition $\lambda = (m+1, m)$ if $n = 2m+1$ is odd, and by the partition $\lambda = (m+1, m-1)$ if $n = 2m$ is even. Its dimension is always a power of 2: $\dim D^\lambda = 2^m$ if n is odd, and $\dim D^\lambda = 2^{m-1}$ if n is even. The vertices of the basic spin module were recently determined by the authors [19]: Let Q be a Sylow 2-subgroup of

- $\mathfrak{A}_n$ if $n \equiv 0 \pmod 4$,
- $\mathfrak{S}_n$ if $n \equiv 2 \pmod 4$,
- $\prod_{i=0}^{s} (\mathfrak{A}_{2^i})^{\alpha_i}$ if $n \equiv 1 \pmod 2$;

here $n = \sum_{i=0}^{s} \alpha_i 2^i$ is the 2-adic expansion of n. Then Q is a vertex of the basic spin module D^λ of $F\mathfrak{S}_n$. This result confirms an earlier conjecture in [20] and [61].

18. Some open questions

A number of open questions were already mentioned. In addition, we are interested in the following:

Question 18.1. *Let $p = 2$, and let $\lambda \vdash n$ be 2-regular. Moreover, let V be a source of D^λ such that V is not the trivial module. Is* $\dim V$ *always even?*

This seems to be the case in all examples that have been computed so far. We note that the sources of the simple $F\mathfrak{S}_n$-modules need not be self-dual although the simple $F\mathfrak{S}_n$-modules D^λ certainly are. (Recall that a module M is called *self-dual* if it is isomorphic to its dual M^*.) We have seen above that $\dim V$ is even if and only if $\frac{\dim D^\lambda}{|S:Q|}$ is even where Q is a vertex of D^λ and S is a Sylow 2-subgroup of $\mathfrak{S}_n$ containing Q.

Now suppose again that p is arbitrary, and let M be an indecomposable FG-module, for a finite group G, with vertex Q and source V. Then D. J. Benson and J. F. Carlson have proved [3] that p does not divide $\dim V$ if and only if the *Scott module* $\mathrm{Sc}_G(Q)$ is a direct summand of $M^* \otimes M$; here $\mathrm{Sc}_G(Q)$ is the only indecomposable direct summand (up to isomorphism) of $\mathrm{Ind}_Q^G(F)$ with a trivial submodule. This criterion is also related to Question 18.1; but there seems to be no easy way in order to apply it in our situation. We note that J. Murray [50] has proved a variant of the Benson-Carlson criterion.

We said before that at present there is no general conjecture concerning the structure of the vertices of the simple $F\mathfrak{S}_n$-modules. However, in all examples that have been computed the structure of these vertices is rather restricted. They appear to be of the following form:

$$Q = Q_1 \times \cdots \times Q_r$$

where Q_i

- is a Sylow p-subgroup of $\mathfrak{S}_{n_i}$,
- or a Sylow p-subgroup of $\mathfrak{A}_{n_i}$,
- or a p-subgroup of $\mathfrak{S}_{n_i}$ acting regularly,

for $i = 1, \ldots, r$ and $n_1 + \cdots + n_r = n$. (If p is odd then the first two alternatives are, of course, equivalent.)

Question 18.2. *Are these the only possibilities?*

19. Methods

Let us try to explain some of the methods that have been used in order to determine vertices. We already mentioned that (1) and the related results (2) - (10) can be applied to get bounds for the vertex of an indecomposable module. These are mainly used in order to show that the vertices of certain simple modules are defect groups of the corresponding blocks. Also the Theorems of Knörr (8.1) and Erdmann (8.2) often give useful lower bounds.

Another standard method is to apply induction and restriction. Here the branching rules of A. Kleshchev are the essential tools; they are described, for example, in [42]. Also the Scopes equivalence (cf. Section 12) is very useful whenever it can be applied.

Furthermore, indecomposable $F\mathfrak{S}_n$-modules with a trivial source are important. For $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, let

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}$$

denote the *Young subgroup* of $\mathfrak{S}_n$ corresponding to λ. Then the indecomposable direct summands of the permutation module

$$M^\lambda := \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(F)$$

are called *Young modules*. They were investigated by G. D. James [39], A. A. Klyachko [43], J. Grabmeier [32] and others. It turns out that there is a bijection between the isomorphism classes of Young modules and partitions of n. For $\lambda \vdash n$, a corresponding Young module is usually denoted by Y^λ. It can be characterized as the unique (up to isomorphism) indecomposable direct summand of the permutation module M^λ containing the Specht module S^λ. There is an easy combinatorial description of the vertices of Y^λ (cf. [32]).

S. Donkin [24] has defined a generalization of Young modules called *signed Young modules*. These are the indecomposable direct summands of

$$\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(L)$$

where L is an $F\mathfrak{S}_\lambda$-module of dimension 1, and $\lambda \vdash n$; such modules also have a trivial source. S. Donkin showed that the isomorphism classes of signed Young modules are in bijection with certain pairs of partitions. Again, their vertices have an easy combinatorial description.

The connection with simple $F\mathfrak{S}_n$-modules comes via a result of D. Hemmer [36]. He proved that every simple Specht module for $F\mathfrak{S}_n$ is in fact a signed Young module; in particular, it has a trivial source. Moreover, it is not difficult to see that every simple $F\mathfrak{S}_n$-module which has a trivial source arises in this way.

Also, the Mullineux conjecture, proved by B. Ford and A. Kleshchev [29] can often be applied. The Mullineux conjecture gives a combinatorial description of the p-regular partition λ^{M} satisfying $D^{\lambda^{\mathrm{M}}} \cong \mathbf{sgn} \otimes D^\lambda$, for a p-regular partition λ of n; here $\mathbf{sgn}$ denotes the sign representation of $\mathfrak{S}_n$. The reason behind the usefulness of the Mullineux conjecture is the standard fact that, for an indecomposable FG-module M and a 1-dimensional FG-module L the vertices of M are also vertices of the indecomposable FG-module $L \otimes M$; here G denotes an arbitrary finite group.

Another useful tool, in characteristic 2, is a combinatorial description, by D. Benson [2], of the 2-regular partitions λ of n with the property that the simple $F\mathfrak{S}_n$-module D^λ is relatively $\mathfrak{A}_n$-projective. Apart from these general methods, often ad hoc arguments have to be used.

20. Computational aspects

As we mentioned above, many of the results for symmetric groups of small degrees and for modules of small dimension have been obtained with computer assistance. The relevant algorithms were written in the computer algebra systems GAP [31] and MAGMA [8], by R. Zimmermann and S. Danz. Our computer calculations of vertices make use of a test for relative projectivity which is based on Higman's Criterion:

An FG-module M is relatively H-projective, for a subgroup H of a finite group G, if and only if the *relative trace map*

$$\mathrm{Tr}_H^G : \mathrm{End}_{FH}(M) \longrightarrow \mathrm{End}_{FG}(M)$$

is surjective; here Tr_H^G assigns to an FH-endomorphism $f : M \longrightarrow M$ the FG-endomorphism $\mathrm{Tr}_H^G(f) : M \longrightarrow M$ defined by

$$(\mathrm{Tr}_H^G(f))(m) = \sum_{gH \in G/H} gf(g^{-1}m) \quad \text{for} \quad m \in M.$$

In order to apply this criterion, one has to compute the endomorphism ring $\mathrm{End}_{FH}(M)$.

Computations of endomorphism rings are also required when decomposing a module into indecomposable direct summands (cf. [47] and [48]). The calculation of endomorphism rings is usually time and memory consuming (and thus impossible for modules of large dimension). Therefore one often tries to speed up the computations by splitting off irrelevant direct summands. For example, in the case of a finite p-group G it is easy to split off projective summands, by using a randomized algorithm. This algorithm can be generalized to split off indecomposable direct summands with a cyclic vertex [20]; note that simple modules with a cyclic vertex are taken care of by Erdmann's Theorem (8.2) above.

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