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Explicit and Canonical Dress Induction

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Abstract. We prove a strengthened statement of Dress's induction theorem for the Green ring of modular representations of a finite group using the theory of the multiplicity module of an indecomposable modular representation. Moreover, we construct an integral canonical induction formula for the Green ring for indecomposable representations with normal vertex.

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Introduction

In this paper we deal with induction theorems for the Green ring $a(FG)$ of a finite group G over an algebraically closed field F of characteristic $p > 0$. The most important result in this direction is the Dress Induction Theorem (see [5], or also [4, Prop. 9.3, 9.4, 9.5] for a more general form). In Section 3 of this paper, we will prove a strong form of this theorem. One of the main tools in our proof will be the theory of multiplicity modules (cf. [10]). In Sections 1 and 2, we will recall the relevant notions and results, together with related correspondence and reduction theorems.

We will then go on to derive a canonical induction formula for the Green ring, at least in an important special case. Our formula here generalizes the canonical induction formula for projective FG-modules obtained in [1]. It is integral, and it applies to all indecomposable modules whose vertex is normal in G. There are two different ways to derive this canonical induction formula. In Section 4 we apply the general machinery of [1]. Alternatively, one could use the theory of multiplicity modules again; this last approach is outlined at the end of the paper.

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In [2], a canonical induction formula for p-permutation modules was obtained. The formula obtained in Section 4 also generalizes this formula in the case of a normal vertex.

1. The Fitting Correspondence

Let F be a field, and let A be a finite-dimensional F-algebra. In the following, all modules will be finitely generated, and we will denote by mod_A the category of all finitely generated right A-modules. Moreover, we denote by $a(A)$ the *Green group* of right A-modules. This is the group generated by all isomorphism classes $[M]$ of right A-modules M, subject to all relations

$$[M \oplus M'] = [M] + [M'],$$

where M and M' are arbitrary right A-modules. By the Krull–Schmidt Theorem, the symbols $[U]$, where U ranges over a set of representatives for the isomorphism classes of indecomposable right A-modules, form a $\mathbb{Z}$-basis of $a(A)$. We denote by $a(A, 1)$ the subgroup of $a(A)$ generated by all symbols $[P]$ where P ranges over all projective right A-modules. If $P_1, \dots, P_n$ form a set of representatives for the isomorphism classes of indecomposable projective right A-modules, then $[P_1], \dots, [P_n]$ form a $\mathbb{Z}$-basis of $a(A, 1)$.

If G is a finite group and if FG denotes the group algebra of G over F, then the tensor product $\otimes_F$ induces a ring structure on $a(FG)$. In this case $a(FG)$ is called the *Green ring* of FG, and $a(FG, 1)$ is an ideal in $a(FG)$.

Let again A be a finite-dimensional F-algebra. Every right A-module V has a finite-dimensional endomorphism algebra $E := \mathrm{End}_A(V)$, and V can be considered as an E-A-bimodule. There exists a well-known multiplicity preserving bijection between

- isomorphism classes of indecomposable direct summands U of V_A, and
- isomorphism classes of indecomposable projective right E-modules P.

The bijection is given in the following way: If $V = U \oplus U'$ with an A-submodule U' and if $e \in E$ denotes the corresponding projection map onto U, then $P := eE \cong \mathrm{Hom}_A(V, U)$ is the corresponding indecomposable projective right E-module. Conversely, if $P \cong eE$ with a primitive idempotent $e \in E$, then $e(V) \cong P \otimes_E V$ is the corresponding indecomposable summand of V_A. We will refer to this bijection as the *Fitting correspondence* for V. It can be used to reduce questions about *arbitrary* indecomposable modules to questions about *projective* indecomposable modules. For example, it can be used in order to reduce the Krull–Schmidt Theorem to the Jordan–Hölder Theorem.

The above considerations can also be made on the category level for the category proj_E of projective right E-modules and the full subcategory $\mathsf{add}_A(V)$ of

mod_A consisting of all A-modules which are isomorphic to direct sums of direct summands of V.

THEOREM 1.1. *The functors*

$$\mathcal{H}\colon \mathsf{add}_A(V) \to \mathsf{proj}_E, \quad U \mapsto \mathrm{Hom}_A(V, U),$$

and

$$\mathcal{T}\colon \mathsf{proj}_E \to \mathsf{add}_A(V), \quad P \mapsto P \otimes_E V,$$

are inverse category equivalences.

2. Vertices, Sources, Multiplicity Modules

In this section we will present an approach that is well known to specialists but we will recall it for the reader's convenience. For similar results in more general situations and more details see for instance [3].

Let F be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. For right FG-modules U and U', we write $U' \mid U$ if U' is isomorphic to a direct summand of U. If H is a subgroup of G then $U_H := \mathrm{Res}_H^G(U)$ denotes the corresponding restricted right FH-module. Conversely, for a right FH-module V we denote by $V^G := \mathrm{Ind}_H^G(V) = V \otimes_{FH} FG$ the corresponding induced FG-module.

2.1. Suppose that U is indecomposable. We recall that a subgroup D of G is called a *vertex* of U if D is minimal subject to the condition $U \mid (U_D)^G$. It is well known that the vertices of U form a conjugacy class of p-subgroups of G.

We fix a vertex D of U and denote by Z a *source* of U. Thus, Z is an indecomposable right FD-module with vertex D such that $Z \mid U_D$ and $U \mid Z^G$. Moreover, Z is unique up to isomorphism and conjugation with elements in $H := N_G(D)$. Here, for $x \in H$, $Z^x := Z \otimes x \subseteq Z \otimes_{FD} FH$ denotes the corresponding *conjugate* right FD-module.

In the following, we denote by V the *Green correspondent* of U in H. Thus, V is an indecomposable right FH-module with vertex D and source Z. Moreover, $V \mid U_H$ with multiplicity 1, $U \mid V^G$ with multiplicity 1, and V is uniquely determined by U, up to isomorphism. Conversely, U is uniquely determined by V, up to isomorphism.

Now let

$$I := N_G(D, Z) := \{x \in N_G(D) \mid Z^x \cong Z\}$$

denote the inertia group of Z in $N_G(D)$. By Clifford theory, there is an indecomposable right FI-module W with vertex D and source Z such that $V \cong W^H$. Moreover, W is uniquely determined, up to isomorphism, by V and Z. We will refer to W as the *Clifford correspondent* of V with respect to Z.

2.2. The endomorphism algebra $E := \mathrm{End}_{FI}(Z^I)$ is a finite-dimensional F-algebra *graded* by $\bar{I} := I/D$. This means that E has a decomposition into F-subspaces,

$$E = \bigoplus_{\bar{x} \in \bar{I}} E_{\bar{x}},$$

such that $E_{\bar{x}} E_{\bar{y}} \subseteq E_{\overline{xy}}$ for $\bar{x}, \bar{y} \in \bar{I}$. In our situation, we have

$$E_{\bar{x}} = \{f \in E \mid f(Z \otimes 1) \subseteq Z \otimes x\} \cong \mathrm{Hom}_{FD}(Z, Z \otimes x)$$

for $x \in I$ and $\bar{x} = xD \in I/D = \bar{I}$. The *identity component* E_1 of E is a unitary local subalgebra of E isomorphic to $\mathrm{End}_{FD}(Z)$. Since $Z \cong Z \otimes x$ for $x \in I$, the $\bar{I}$-graded F-algebra E is in fact a crossed product. This means that every component $E_{\bar{x}}$ contains a unit of E. Such a unit is constructed in the following way: For $x \in I$, let

$$u_x \colon Z \otimes 1 \to Z \otimes x$$

be an FD-isomorphism. Then u_x extends uniquely to an FI-automorphism $Z^I \to Z^I$ which is contained in $E_{\bar{x}}$ and which we denote by u_x again.

The induced left E-module ${}^E Z := E \otimes_{E_1} Z$ becomes an E-FI-bimodule in the following way: For $z \in Z$, we write $u_x(z \otimes 1) = v_x(z) \otimes x$ with $v_x(z) \in Z$. This defines a map $v_x \colon Z \to Z$ satisfying

$$v_x(zd) = v_x(z) x d x^{-1}$$

for all $z \in Z$ and $d \in D$. The right FI-module structure of ${}^E Z$ is then defined by

$$(f \otimes z)x := f u_x \otimes v_x^{-1}(z)$$

for $f \in E$, $z \in Z$ and $x \in I$. In this way, ${}^E Z$ becomes an E-FI-bimodule. In fact, ${}^E Z$ is an $\bar{I}$-graded E-FI-bimodule. (Recall that an A-B-bimodule M, where A and B are G-graded F-algebras is called G-*graded*, if M has a decomposition into F-subspaces, $M = \bigoplus_{y \in G} M_y$, such that $A_x M_y B_z \subseteq M_{xyz}$ for all $x, y, z \in G$.) Note that, in our situation, both ${}^E Z$ and FI are naturally $\bar{I}$-graded. In a similar way, $Z^I = Z \otimes_{FD} FI$ is an $\bar{I}$-graded E-FI-bimodule.

We now have the following well known result.

THEOREM 2.3. *The map*

$$\Phi \colon {}^E Z \to Z^I, \quad f \otimes z \mapsto f(z \otimes 1),$$

is an isomorphism of $\bar{I}$*-graded* E-FI*-bimodules.*

This shows also that the FI-module structure on ${}^E Z$ which was defined above is independent of the choices of u_x. In the following, we will often identify Z^I and ${}^E Z$ via Φ.

2.4. Similarly, we have for each intermediate group $D \leqslant J \leqslant I$ an isomorphism

$$\Phi_J \colon {}^{E_{\bar{J}}}Z \to Z^J, \quad f \otimes z \mapsto f(z \otimes 1),$$

where

$$E_{\bar{J}} := \bigoplus_{\bar{x} \in \bar{J}} E_{\bar{x}}$$

is an F-subalgebra of E which is naturally isomorphic to $\mathrm{End}_{FJ}(Z^J)$, so that Z^J is an $E_{\bar{J}}$-FJ-bimodule. Moreover, with the obvious embeddings ${}^{E_{\bar{K}}}Z \subseteq {}^{E_{\bar{J}}}Z$ and $Z^K \subseteq Z^J$ for $D \leqslant K \leqslant J \leqslant I$, the restriction of $\Phi_{\bar{J}}$ to ${}^{E_{\bar{K}}}Z$ is equal to $\Phi_{\bar{K}}$.

2.5. Now recall that W is an indecomposable direct summand of Z^I (as a right FI-module). So W has a Fitting correspondent $P = \mathrm{Hom}_{FI}(Z^I, W)$, which is an indecomposable projective right E-module such that

$$W \cong P \otimes_E Z^I \cong P \otimes_E {}^E Z \cong P \otimes_{E_1} Z.$$

More generally, for each intermediate group $D \leqslant J \leqslant I$, let $\mathsf{mod}_{FJ}^{(D,Z)}$ denote the full subcategory of mod_{FJ} consisting of all direct sums of indecomposable FJ-modules with vertex D and source Z. Note that for all $X \in \mathsf{mod}_{FJ}^{(D,Z)}$ one has $\mathrm{Res}_D^J(X) \cong Z \oplus \cdots \oplus Z$. We have functors

$$\mathcal{H}_{\bar{J}} \colon \mathsf{mod}_{FJ}^{(D,Z)} \to \mathsf{proj}_{E_{\bar{J}}}, \quad X \mapsto \mathrm{Hom}_{FJ}(Z^J, X),$$
$$\mathcal{T}_{\bar{J}} \colon \mathsf{proj}_{E_{\bar{J}}} \to \mathsf{mod}_{FJ}^{(D,Z)}, \quad Q \mapsto Q \otimes_{E_{\bar{J}}} {}^{E_{\bar{J}}}Z,$$

which are inverse equivalences. Moreover, one has obvious conjugation, restriction and induction functors

$$C_J^x \colon \mathsf{mod}_{FJ}^{(D,Z)} \to \mathsf{mod}_{FJ^x}^{(D,Z)}, \quad C_J^x \colon \mathsf{proj}_{E_{\bar{J}}} \to \mathsf{proj}_{E_{\overline{J^x}}},$$
$$\mathrm{Res}_K^J \colon \mathsf{mod}_{FJ}^{(D,Z)} \to \mathsf{mod}_{FK}^{(D,Z)}, \quad \mathrm{Res}_K^J \colon \mathsf{proj}_{E_{\bar{J}}} \to \mathsf{proj}_{E_{\bar{K}}},$$
$$\mathrm{Ind}_K^J \colon \mathsf{mod}_{FK}^{(D,Z)} \to \mathsf{mod}_{FJ}^{(D,Z)}, \quad \mathrm{Ind}_K^J \colon \mathsf{proj}_{E_{\bar{K}}} \to \mathsf{proj}_{E_{\bar{J}}},$$

for $x \in I$ and $D \leqslant K \leqslant J \leqslant I$.

The following theorem is now an easy verification.

THEOREM 2.6. *The inverse equivalences $\mathcal{H}_{\bar{J}}$ and $\mathcal{T}_{\bar{J}}$ between $\mathsf{mod}_{FJ}^{(D,Z)}$ and $\mathsf{proj}_{E_{\bar{J}}}$, $D \leqslant J \leqslant I$, commute with the conjugation, restriction and induction functors up to natural isomorphisms.*

2.7. The Jacobson radical $\mathrm{Rad}(E_1)$ is a nilpotent ideal of E_1. Hence, $\mathrm{Rad}(E_1)E = E\mathrm{Rad}(E_1)$ is a nilpotent graded ideal of E. Thus, $\widetilde{E} := E/E\mathrm{Rad}(E_1)$ is an $\bar{I}$-graded F-algebra, with $\bar{x}$-component

$$\widetilde{E}_{\bar{x}} = (E_{\bar{x}} + E\mathrm{Rad}(E_1))/E\mathrm{Rad}(E_1) \cong E_{\bar{x}}/E_{\bar{x}}\mathrm{Rad}(E_1),$$

for $\bar{x} \in \bar{I}$. Since E is a crossed product, so is $\widetilde{E}$. Moreover, since E_1 is a local F-algebra, we have $\widetilde{E}_1 \cong F$. This means that $\widetilde{E}$ is in fact a twisted group algebra of $\bar{I}$ over F. We choose a fixed nonzero element $b_{\bar{x}}$ in each $\widetilde{E}_{\bar{x}}$, denote the corresponding 2-cocycle $\bar{I} \times \bar{I} \to F^\times$ by γ, and have $\widetilde{E} = F_\gamma \bar{I}$.

Recall that $P \in \mathsf{proj}_E$ is the Fitting correspondent of the Clifford correspondent $W \in \mathsf{mod}_{FI}^{(D,Z)}$ of U. Now, $P\mathrm{Rad}(E_1)$ is an E-submodule of P, and $\widetilde{P} := P/P\mathrm{Rad}(E_1)$ becomes an indecomposable projective right $\widetilde{E}$-module which is called the *multiplicity module* of U. Conversely, P is the projective cover of $\widetilde{P}$, considered as an E-module via inflation.

More generally, for each $D \leqslant J \leqslant I$, also

$$\widetilde{E}_{\bar{J}} := \bigoplus_{\bar{x} \in \bar{J}} \widetilde{E}_{\bar{x}} \cong E_{\bar{J}}/E_{\bar{J}}\mathrm{Rad}(E_1)$$

is a twisted group algebra $F_\gamma \bar{J}$ with the 2-cocycle γ restricted to $\bar{J}$ if we use the basis elements $b_{\bar{x}}$, $\bar{x} \in \bar{J}$. The resulting F-algebra epimorphisms $\pi_J \colon E_{\bar{J}} \to F_\gamma \bar{J}$, $D \leqslant J \leqslant I$, have the property that $\pi_J|_{E_{\bar{K}}} = \pi_K$.

The following theorem is again well known to specialists.

THEOREM 2.8. *For each intermediate subgroup $D \leqslant J \leqslant I$, one has a functor*

$$\mathsf{proj}_{E_{\bar{J}}} \to \mathsf{proj}_{F_\gamma \bar{J}}, \quad Q \mapsto \widetilde{Q} := Q/Q\mathrm{Rad}(E_1)$$

which induces a bijection between the sets of isomorphism classes of the respective projective modules and which preserves indecomposability. Moreover, these functors commute up to natural isomorphisms with the respective conjugation, restriction and induction functors between the categories $\mathsf{proj}_{E_{\bar{J}}}$, $D \leqslant J \leqslant I$, and the categories $\mathsf{proj}_{F_\gamma \bar{J}}$, $D \leqslant J \leqslant I$.

2.9. Let us summarize: Every indecomposable right FG-module U determines a triple $(D, Z, \widetilde{P})$ consisting of its vertex D (a p-subgroup of G), its source Z (an indecomposable FD-module with vertex D), and its multiplicity module $\widetilde{P}$ (an indecomposable projective module over a twisted group algebra of $N_G(D, Z)/D$). Moreover, the triple $(D, Z, \widetilde{P})$ is uniquely determined by U, up to conjugation in G. For more details see [10].

2.10. We will make one further step. We may and will assume from now on that the cocycle $\gamma \in Z^2(\bar{I}, F^\times)$ was chosen such that it takes values in a finite cyclic subgroup of $F^\times$. It is well known that one can find a central extension

$$1 \longrightarrow C \longrightarrow \widehat{I} \stackrel{\nu}{\longrightarrow} \bar{I} \longrightarrow 1$$

of $\bar{I}$, given by a 2-cocycle $\delta \in Z^2(\bar{I}, C)$ with a cyclic p'-group C, and a homomorphism $\vartheta \colon C \to F^\times$ such that $\gamma = \vartheta \circ \delta$. We may assume that C is a subgroup of

$\widehat{I}$. In this situation, it is well known that one has an isomorphism of F-algebras,

$$F\widehat{I}e_\vartheta \to F_\gamma \bar{I}, \quad \widehat{x} \cdot e_\vartheta \mapsto b_{\nu(\widehat{x})},$$

where $e_\vartheta := |C|^{-1} \sum_{c \in C} \vartheta(c^{-1})c \in FC$ is the idempotent associated to ϑ. This isomorphism restricts to isomorphisms $F\widehat{J}e_\vartheta \to F_\gamma \bar{J}$, $D \leqslant J \leqslant I$, where $\widehat{J} := \nu^{-1}(\bar{J})$. This implies the following theorem in which $\mathsf{proj}_{F\widehat{J}}^{(C,\vartheta)}$ denotes the full subcategory of $\mathsf{proj}_{F\widehat{J}}$ consisting of all projective $F\widehat{J}$-modules M which satisfy $\mathrm{Res}_C^{\widehat{J}}(M) \cong F_\vartheta \oplus \cdots \oplus F_\vartheta$, where F_ϑ denotes the one-dimensional FC-module F on which C acts via ϑ. This condition is equivalent to $M = Me_\vartheta$.

THEOREM 2.11. *The isomorphisms in* (2.10) *induce category equivalences* $\mathsf{proj}_{F_\gamma \bar{J}} \simeq \mathsf{proj}_{F\widehat{J}}^{(C,\vartheta)}$, $D \leqslant J \leqslant I$, *which commute up to natural isomorphism with the conjugation, restriction and induction functors on the categories* $\mathsf{proj}_{F_\gamma \bar{J}}$, $D \leqslant J \leqslant I$, *and on the categories* $\mathsf{proj}_{F\widehat{J}}^{(C,\vartheta)}$, $D \leqslant J \leqslant I$.

Altogether, the Theorems 2.6, 2.8, and 2.11 imply the following result. Note that all the functors established in these theorems are F-linear on the morphism spaces and respect direct sums.

COROLLARY 2.12. *There exist functors*

$$\mathsf{mod}_{FJ}^{(D,Z)} \to \mathsf{proj}_{F\widehat{J}}^{(C,\vartheta)}, \quad D \leqslant J \leqslant I,$$

which commute up to natural isomorphisms with the conjugation, restriction and induction functors. For every $D \leqslant J \leqslant I$, *the corresponding functor induces a bijection between the set of isomorphism classes of (indecomposable) right* FJ*-modules* X *whose indecomposable direct summands have vertex* D *and source* Z *and the set of isomorphism classes of (indecomposable) projective right* $F\widehat{J}$*-modules* $\widehat{Q}$ *with* $\mathrm{Res}_C^{\widehat{J}}(\widehat{Q}) \cong F_\vartheta \oplus \cdots \oplus F_\vartheta$. *Moreover, the induced isomorphisms between the Green groups of* $\mathsf{mod}_{FJ}^{(D,Z)}$ *and* $\mathsf{proj}_{F\widehat{J}}^{(C,\vartheta)}$ *form an isomorphism of Mackey functors.*

2.13. We illustrate the situation by the following diagram:

G, U			
$H = N_G(D)$, V			
$I = N_G(D,Z)$, W	E, P	$\widetilde{E} = F_\gamma \bar{I}$, $\widetilde{P}$	$F\widehat{I}$, $\widehat{P}$
J, X	$E_{\bar{J}}$, Q	$F_\gamma \bar{J}$, $\widetilde{Q}$	$F\widehat{J}$, $\widehat{Q}$
K, Y	$E_{\bar{K}}$, R	$F_\gamma \bar{K}$, $\widetilde{R}$	$F\widehat{K}$, $\widehat{R}$
D, Z	E_1, E_1	$\widetilde{E}_1 = F$, F	FC, F_ϑ

If $X \in \mathsf{mod}_{FJ}^{(D,Z)}$ corresponds to $\widehat{Q} \in \mathsf{proj}_{F\widehat{J}}^{(C,\vartheta)}$ and $Y \in \mathsf{mod}_{FK}^{(D,Z)}$ corresponds to $\widehat{R} \in \mathsf{proj}_{F\widehat{K}}^{(C,\vartheta)}$, then one has for instance

$$\begin{aligned}
&X \text{ indecomposable} \iff \widehat{Q} \text{ indecomposable},\\
&X \cong \mathrm{Ind}_K^J(Y) \iff \widehat{Q} \cong \mathrm{Ind}_{\widehat{K}}^{\widehat{J}}(\widehat{R}),\\
&Y \cong \mathrm{Res}_K^J(X) \iff \widehat{R} \cong \mathrm{Res}_{\widehat{K}}^{\widehat{J}}(\widehat{Q}),\\
&Y \mid \mathrm{Res}_K^J(X) \iff \widehat{R} \mid \mathrm{Res}_{\widehat{K}}^{\widehat{J}}(\widehat{Q}),
\end{aligned}$$

etc.

The approach presented in this section can be used in order to reduce questions about indecomposable modules U over group algebras to questions about indecomposable projective modules $\widehat{P}$ or $\widetilde{P}$ over (twisted) group algebras. In the following, we will give an example of this method.

3. Explicit Dress Induction

Let F and G be as above. Our aim in this section is to prove a strong form of the Dress induction theorem, using the correspondences given in Section 2. Other proofs of the Dress induction theorem can be found in [5], [4, Section 9], [9], [8], [6], and [7], for example. We start with a special case of the Dress induction theorem which follows easily from Brauer's induction theorem.

PROPOSITION 3.1. *Let P be a projective right FG-module Then there are elementary p'-subgroups $H_1, \dots, H_n$ of G, one-dimensional modules $V_1, \dots, V_n$ over $FH_1, \dots, FH_n$, respectively, and integers $a_1, \dots, a_n$ such that*

$$[P] = \sum_{i=1}^{n} a_i [V_i^G]$$

in $a(FG)$.

Proof. We choose a complete discrete valuation ring R of characteristic 0 such that $R/\mathrm{Rad}(R) = F$. We may assume that R contains a primitive $|G|$th root of unity. By Brauer's induction theorem, there exist elementary subgroups $L_1, \dots, L_n$ of G, characters λ_1 of $L_1, \dots, \lambda_n$ of L_n, and integers $a_1, \dots, a_n$ such that

$$1_G = \sum_{i=1}^{n} a_i \lambda_i^G.$$

We denote by η the (Brauer) character of P. Multiplication by η yields

$$\eta = \sum_{i=1}^{n} a_i (\eta\lambda_i)^G.$$

Each $\eta\lambda_i$ is a projective character of L_i. Since L_i is nilpotent, $\eta\lambda_i$ is induced from the Hall p'-subgroup K_i of L_i: $\eta\lambda_i = \varphi_i^{L_i}$, where φ_i is a character of K_i. Since K_i is nilpotent, every irreducible character of K_i is induced from a linear character of a subgroup H_i of K_i. The result follows, since every projective FG-module is determined, up to isomorphism, by its (Brauer) character. □

Next we extend the above proposition to twisted group algebras. In the following, we denote by $\gamma\colon G \times G \to F^\times$ a 2-cocycle and by $F_\gamma G$ the corresponding twisted group algebra.

PROPOSITION 3.2. *Let P be a projective right $F_\gamma G$-module. Then there exist elementary p'-subgroups $H_1, \dots, H_n$ of G, one-dimensional modules $V_1, \dots, V_n$ over $F_\gamma H_1, \dots, F_\gamma H_n$, respectively, and integers $a_1, \dots, a_n$ such that*

$$[P] = \sum_{i=1}^{n} a_i [V_i^G]$$

in $a(F_\gamma G)$.

Proof. There exists a central extension

$$1 \longrightarrow Z \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

where Z is a cyclic p'-group, and there exists a linear character ϑ of Z such that $F_\gamma G \cong e_\vartheta F\widehat{G}$ where $e_\vartheta = |Z|^{-1} \sum_{z\in Z} \vartheta(z^{-1})z$ is a primitive idempotent in FZ. Thus, P can be viewed as a (projective) right $F\widehat{G}$-module via the map $F\widehat{G} \to e_\vartheta F\widehat{G} \cong F_\gamma G$. By the previous proposition, there exist elementary p'-subgroups $\widehat{H}_1, \dots, \widehat{H}_n$ of $\widehat{G}$, one-dimensional modules $\widehat{V}_1, \dots, \widehat{V}_n$ over $F\widehat{H}_1, \dots, F\widehat{H}_n$, respectively, and integers $a_1, \dots, a_n$ such that

$$[P] = \sum_{i=1}^{n} a_i [\widehat{V}_i^{\widehat{G}}]$$

in $a(F\widehat{G})$. For $i = 1, \dots, n$, Z is a central subgroup of $\widetilde{H}_i := \widehat{H}_i Z$, and $H_i := \widetilde{H}_i/Z$ is an elementary p'-subgroup. The kernel K_i of $\widehat{V}_i$ is a normal subgroup of $\widetilde{H}_i$, and $\widetilde{H}_i/K_i$ is Abelian. So $\widehat{V}_i$ extends to $F\widetilde{H}_i$, and $(\widehat{V}_i)^{\widetilde{H}_i}$ is a direct sum of one-dimensional modules. This means that

$$[P] = \sum_{j=1}^{m} b_j [\widetilde{W}_j^{\widehat{G}}],$$

where each b_j is an integer and each $\widetilde{W}_j$ is a one-dimensional module over $F\widetilde{H}_j$. Moreover, $\widetilde{H}_j$ contains Z and $\widetilde{H}_j/Z$ is an elementary p'-group. Since $P = Pe_\vartheta$, we conclude that

$$[P] = \sum_{j=1}^{m} b_j [\widetilde{W}_j^{\widehat{G}} e_\vartheta] = \sum_{j=1}^{m} b_j [(\widetilde{W}_j e_\vartheta)^{\widehat{G}}],$$

where each $\widetilde{W}_j e_\vartheta$ is either one-dimensional or zero. We may assume that each $\widetilde{W}_j e_\vartheta$ is one-dimensional. Then $V_j := \widetilde{W}_j e_\vartheta$ can be viewed as a module over the subalgebra $F_\gamma H_j$ of $F_\gamma G$. We obtain

$$[P] = \sum_{j=1}^{m} b_j [V_j^G]$$

and the proof is complete. □

Now we come to a strong version of the Dress induction theorem.

THEOREM 3.3 (Dress). *Let U be an indecomposable right FG-module with vertex D and source Z. Then, in the Green ring of FG, we have*

$$[U] = \sum_{i=1}^{n} a_i [V_i^G]$$

where, for $i = 1, \ldots, n$, a_i is an integer and V_i is an indecomposable right FH_i-module for a subgroup H_i of G. Moreover, V_i has vertex $D_i := O_p(H_i) \leqslant D$ and source $(V_i)_{D_i} \mid Z_{D_i}$, and H_i/D_i is an elementary p'-group.

Proof. We argue by induction on $|D|$. If $|D| = 1$, the result follows from Proposition 3.1. So we assume that $|D| > 1$. Let V be the Green correspondent of U in $H := N_G(D)$. Then

$$V^G \cong U \oplus U_1 \oplus \cdots \oplus U_m,$$

where, for $j = 1, \ldots, m$, U_j is an indecomposable right FG-module having a proper subgroup E_j of D as a vertex. (The case $m = 0$ is possible.) Moreover, U_j has a source Z_j such that $Z_j \mid Z_{E_j}$. By induction, there is a formula for U_j in the Green ring of FG,

$$[U_j] = \sum_{k=1}^{n_j} a_{jk} [V_{jk}^G],$$

of the type we want. This means that it suffices to prove the existence of a similar formula for $[V] \in a(FH)$.

Let $I := N_G(D, Z)$, and let the right FI-module W be the Clifford correspondent of V with respect to Z. Then W has vertex D and source Z, and $V \cong W^H$. This means that it suffices to prove a formula for $[W] \in a(FI)$.

Let $\bar{I} := I/D$. Then the endomorphism algebra $E := \mathrm{End}_{FI}(Z^I)$ is an $\bar{I}$-graded F-algebra and a crossed product. Let P denote the Fitting correspondent of W, so that P is an indecomposable projective right E-module.

Now $E\mathrm{Rad}(E_1) = \mathrm{Rad}(E_1)E$ is a nilpotent ideal of E, and $\widetilde{E} := E/E\mathrm{Rad}(E_1)$ is a twisted group algebra of $\bar{I}$ over F. We write $\widetilde{E} = F_\gamma \bar{I}$ where $\gamma\colon \bar{I} \times \bar{I} \to F^\times$ is a 2-cocycle. Moreover, $\widetilde{P} := P/P\mathrm{Rad}(E_1)$ is an indecomposable projective right $F_\gamma \bar{I}$-module. By the previous proposition, there is an induction formula

$$[\widetilde{P}] = \sum_{i=1}^{n} a_i [\widetilde{P}_i^{\bar{I}}]$$

in $a(F_\gamma \bar{I})$. Here, for $i = 1, \ldots, n$, a_i is an integer, and $\widetilde{P}_i$ is a one-dimensional module over $F_\gamma \bar{H}_i =: \widetilde{E}_{\bar{H}_i}$, where $\bar{H}_i$ is an elementary p'-subgroup of $\bar{I}$. Let $E_{\bar{H}_i} := \bigoplus_{\bar{x} \in \bar{H}_i} E_{\bar{x}}$, so that $E_{\bar{H}_i}$ is a subalgebra of E isomorphic to $\mathrm{End}_{FH_i}(Z^{H_i})$, where H_i is defined by $H_i/D = \bar{H}_i$.

Moreover, let the $E_{\bar{H}_i}$-module P_i be the projective cover of the inflation of $\widetilde{P}_i$ to $E_{\bar{H}_i}$. Then $\mathrm{Res}_D^{H_i}(P_i) \cong E_1$, since $\mathrm{Res}_D^{H_i}(\widetilde{P}_i) \cong \widetilde{E}_1 = F$, and P_i^E is the projective cover of the inflation of $\widetilde{P}_i^{\widetilde{E}} \cong \widetilde{P}_i^{\bar{I}}$ to E. Since P is the projective cover of the inflation of $\widetilde{P}$ to E, we obtain, by Theorem 2.8,

$$[P] = \sum_{i=1}^{n} a_i [P_i^E]$$

in $a(E)$. Thus,

$$[W] = [P \otimes_E Z^I] = \sum_{i=1}^{n} a_i [P_i^E \otimes_E Z^I]$$

in $a(FI)$. For $i = 1, \ldots, n$, we have

$$\begin{aligned} P_i^E \otimes_E Z^I &\cong P_i \otimes_{E_{\bar{H}_i}} E \otimes_E Z \otimes_{FD} FI \\ &\cong P_i \otimes_{E_{\bar{H}_i}} Z \otimes_{FD} FH_i \otimes_{FH_i} FI \cong (P_i \otimes_{E_{\bar{H}_i}} Z^{H_i})^I = V_i^I, \end{aligned}$$

where $V_i := P_i \otimes_{E_{\bar{H}_i}} Z^{H_i}$ is the Fitting correspondent of P_i, an indecomposable direct summand of Z^{H_i}. Since $H_i/D = \bar{H}_i$ is a p'-group, it follows easily that V_i has vertex D and source Z. Moreover, $\mathrm{Res}_D^{H_i}(P_i) \cong E_1$ implies $\mathrm{Res}_D^{H_i}(V_i) \cong Z$. So the result is proved. □

4. A Canonical Induction Formula for the Green Ring

Using the Green correspondence we reduced the proof of Theorem 3.3 to the situation where V is an indecomposable FG-module with vertex D which is normal in G. The remaining constructions in the proof of the theorem can be done in the framework of canonical induction formulas as introduced in [1]. However, to obtain a *canonical* induction formula one has to pay the price of allowing more general subgroups H than those with H/D elementary and of p'-order as it is already the case for the canonical Brauer induction formula. In fact, the version presented in the following will use subgroups H with H/D solvable and of p'-order. For the details of the constructions and arguments in the following subsection see [1].

4.1. Throughout this section we fix the following notation: Let G be a finite group and $D \trianglelefteq G$ a normal p-subgroup. For any intermediate group $D \leqslant H \leqslant G$, we denote by $M(H)$ the $\mathbb{Z}$-span of all elements $[V] \in a(FH)$, where $V \in \mathsf{mod}^D_{FH}$, the category of direct sums of indecomposable right FH-modules with vertex D. As one can easily verify using the Mackey decomposition formula, conjugation, restriction and induction induce well-defined maps

$$c^g_H\colon M(H) \to M(H^g), \quad \mathrm{res}^H_K\colon M(H) \to M(K),$$
$$\mathrm{ind}^H_K\colon M(K) \to M(H),$$

for all $g \in G$ and all $D \leqslant K \leqslant H \leqslant G$. This way, M becomes a Mackey functor on the group $\bar{G} := G/D$ over the integers. Note here that if $g \in D$, then $c^g_H = \mathrm{id}_{M(H)}$ for all $D \leqslant H \leqslant G$. For each intermediate subgroup $D \leqslant H \leqslant G$ such that H/D is a solvable p'-subgroup, let $\mathcal{B}(H) \subseteq M(H)$ denote the set of all elements $[V]$, where $V \in \mathsf{mod}^D_{FH}$ and $\mathrm{Res}^H_D(V)$ is indecomposable. If $D \leqslant H \leqslant G$ is an intermediate subgroup such that H/D is not a solvable p'-group, we set $\mathcal{B}(H) := \emptyset$. For every $D \leqslant H \leqslant G$, the set $\mathcal{B}(H)$ is $\mathbb{Z}$-linearly independent in $a(FH)$ and we write $A(H)$ for its $\mathbb{Z}$-span. We have $A(H) \subseteq M(H)$ for all $D \leqslant H \leqslant G$ and it is again easy to see that the conjugation maps c^g_H (resp. restriction maps res^H_K) from the above map $A(H)$ to $A(H^g)$ (resp. to $A(K)$). Thus, in the terminology of [1], A is a restriction subfunctor of the Mackey functor M. For every intermediate subgroup $D \leqslant H \leqslant G$, let $p_H\colon M(H) \to A(H)$ denote the projection map which, for an indecomposable module $V \in \mathsf{mod}^D_{FH}$, maps $[V]$ to itself, if $[V] \in \mathcal{B}(H)$ and maps $[V]$ to 0 otherwise. Then, p_H commutes with the conjugation maps on $M(H)$ and $A(H)$ and we are in the situation of Hypothesis 8.1 in [1].

For every intermediate subgroup $D \leqslant H \leqslant G$, let $\mathcal{M}(H)$ denote the set of pairs $(K, [V])$, where $D \leqslant K \leqslant H \leqslant G$ and $[V] \in \mathcal{B}(K)$. Then, $\mathcal{M}(H)$ is an H-set via conjugation: $(K, [V])^h := (K^h, [V^h])$ for $h \in H$ and $(K, [V]) \in \mathcal{M}(H)$. Moreover, $\mathcal{M}(H)$ is a poset by defining $(L, [W]) \leqslant (K, [V])$ if $L \leqslant K$ and $[W] = \mathrm{res}^K_L([V])$. Note that the action of H on $\mathcal{M}(H)$ respects this partial order. We set

$$A_+(H) := \Bigl(\bigoplus_{K \leqslant H} A(K)\Bigr)_H,$$

where the index H denotes taking coinvariants with respect to the conjugation action of H on the direct sum. If we denote the class in $A_+(H)$ of an element $a \in A(K)$ by $[K, a]_H$, then the elements $[K, [W]]_H$, where $(K, [W])$ ranges over a set of representatives of the H-orbits of $\mathcal{M}(H)$, form a $\mathbb{Z}$-basis of $A_+(H)$. The groups $A_+(H)$, $D \leqslant H \leqslant G$, form again a Mackey functor on $\bar{G}$ with conjugation, restriction, and induction maps given by

$$c^H_g\colon A_+(H) \to A_+(H^g), \quad [K, [W]]_H \mapsto [K^g, [W^g]]_{H^g},$$

$$\mathrm{res}_I^H\colon\ A_+(H) \to A_+(I), \quad [K,[W]]_H \mapsto \sum_{h\in K\backslash H/I} [I\cap K^h, \mathrm{res}^{K^h}_{I\cap K^h}([W^h])]_I,$$

$$\mathrm{ind}_I^H\colon\ A_+(I) \to A_+(H), \quad [L,[X]]_I \mapsto [L,[X]]_H,$$

for $g \in G$, intermediate subgroups $D \leqslant I \leqslant H \leqslant G$, and elements $(K,[W]) \in \mathcal{M}(H)$, $(L,[X]) \in \mathcal{M}(I)$.

For each intermediate group $D \leqslant H \leqslant G$ we also have a map

$$b_H\colon\ A_+(H) \to M(H), \quad [K,[W]]_H \mapsto [\mathrm{Ind}_H^G(W)].$$

The collection b_H, $D \leqslant H \leqslant G$, is a morphism b: $A_+ \to M$ of Mackey functors, i.e., the maps b_H commute with the respective conjugation, restriction, and induction maps.

For each intermediate subgroup $D \leqslant H \leqslant G$, we define the map

$$a_H\colon\ M(H) \to \mathbb{Q}\otimes A_+(H),$$
$$[V] \mapsto \frac{1}{|H|} \sum_{(H_0,[W_0])<\cdots<(H_n,[W_n])} (-1)^n |H_0| m_{W_n}(\mathrm{res}^H_{H_n}(V))[H_0,[W_0]]_H,$$

associated to the data (M, A, p) as in [1, 6.1]. Here the sum is running over all chains in $\mathcal{M}(H)$ and $m_W(V)$ denotes the multiplicity of an indecomposable module W as a direct summand of a module V. The collection a of maps a_H, $D \leqslant H \leqslant G$, is a morphism of restriction functors, i.e., the maps a_H commute with the respective conjugation and restriction maps.

The following lemma is well known. We leave the proof to the reader.

LEMMA 4.2. *Let $D \leqslant H \leqslant G$ such that H/D is a cyclic p'-group and let $V \in \mathsf{mod}_{FH}$ be indecomposable with vertex D and H-stable source $Z \in \mathsf{mod}_{FD}$. Then $\mathrm{Res}^H_D(V)$ is indecomposable. In particular, $A(H) = M(H)$.*

Now we are ready to prove the following theorem about the $\mathbb{Q}$-tensored versions $a_H\colon \mathbb{Q}\otimes M(H) \to \mathbb{Q}\otimes A_+(H)$ and $b_H\colon\ \mathbb{Q}\otimes A_+(H) \to \mathbb{Q}\otimes M(H)$ of the above maps a_H and b_H.

THEOREM 4.3. *The morphism a: $\mathbb{Q}\otimes M \to Q\otimes A_+$ of restriction functors is a canonical induction formula, i.e., $b_H \circ a_H = \mathrm{id}_{\mathbb{Q}\otimes M(H)}$ for all $D \leqslant H \leqslant G$.*

Proof. By [1, Proposition 6.4], it suffices to show that for every $D \leqslant H \leqslant G$ and every indecomposable $V \in \mathsf{mod}^D_{FH}$ one has

$$[V] - p_H([V]) \in \sum_{D\leqslant K<H} \mathrm{ind}^H_K(\mathbb{Q}\otimes M(K)).$$

So let H and V be as above, and let $Z \in \mathsf{mod}_{FD}$ be a source of V. If Z is not H-stable, then V is properly induced by its Clifford correspondent and the above equation holds. So we may assume that Z is H-stable. If H/D is not a cyclic p'-group, then by Corollary 2.12, the above equation holds again, since the class of a projective module is a $\mathbb{Q}$-linear combination of elements induced from cyclic p'-subgroups. So, we may assume that H/D is a cyclic p'-subgroup. But in this case the above lemma implies that $p([V]) = [V]$ and the theorem is proved. □

Next we show that the canonical induction formula a is integral.

THEOREM 4.4. *For every intermediate group $D \leqslant H \leqslant G$ and every $V \in \mathsf{mod}^D_{FH}$ one has*

$$a_H([V]) = \sum_{\sigma=((H_0,[W_0])<\cdots<(H_n,[W_n]))} (-1)^n \frac{|(N_H(\sigma)/H_0)_{p'}|}{|N_H(\sigma)/H_0|} m_{W_n}(\mathrm{Res}^H_{H_n}(V))[H_0,[W_0]]_H,$$

where the sum runs over a set of representatives σ of the H-orbits of strictly increasing chains in $\mathcal{M}(H)$, and $N_H(\sigma)$ denotes the stabilizer of σ in H under the conjugation action. Moreover, the rational number

$$\frac{|(N_H(\sigma)/H_0)_{p'}|}{|N_H(\sigma)/H_0|} m_{W_n}(\mathrm{Res}^H_{H_n}(V))$$

is an integer for all $\sigma \in \mathcal{M}(H)$. In particular, a_H is integral, i.e., $a_H(M(H)) \subseteq A_+(H)$.

Proof. First we prove the formula for $a_H([V]) \in \mathbb{Q} \otimes A_+(H)$. By [1, Theorem 9.3] it suffices to show that the following condition is satisfied: If $D \leqslant J \trianglelefteq I \leqslant H$ are intermediate groups such that I/J is a cyclic p'-group, if $X \in \mathsf{mod}^D_{FJ}$ is indecomposable and I-stable such that $\mathrm{Res}^J_D(X)$ also is indecomposable, and if $W \in \mathsf{mod}^D_{FI}$ is indecomposable, then

$$m_X(\mathrm{Res}^I_J(W)) = \sum_{\substack{[W']\in\mathcal{B}(I)\\ \mathrm{Res}^I_J(W')\cong X}} m_{W'}(W).$$

But if $X \nmid \mathrm{Res}^I_J(W)$, then obviously both sides of the above equation vanish. And if $X \mid \mathrm{Res}^I_J(W)$, then $\mathrm{Res}^I_J(W) \cong X$ by Clifford theory, and since $X \in \mathcal{B}(J)$, also $W \in \mathcal{B}(I)$. This implies that both sides of the above equation are equal to 1, and the formula for $a_H([V])$ is proved.

In order to show the integrality statement one can proceed with exactly the same arguments as in [1, Example 9.9]. □

The canonical induction formula a respects Clifford theory in the following sense.

PROPOSITION 4.5. *Let $D \leqslant H \leqslant G$ be an intermediate subgroup, let $V \in \mathsf{mod}^D_{FH}$ be indecomposable with source $Z \in \mathsf{mod}_{FD}$, let $I := N_H(D, Z)$ be the inertial group of Z, and let $W \in \mathsf{mod}^{(D,Z)}_{FI}$ be such that $\mathrm{Ind}^H_I(W) = V$. Then*

$$a_H([V]) = \mathrm{ind}^H_I(a_I([W])).$$

Proof. We consider the formula for $a_H([V])$ from Theorem 4.4. To every chain $\sigma = (H_0, [W_0]) < \cdots < (H_n, [W_n])$ in $\mathcal{M}(H)$ we can associate an isomorphism class $[Z']$ of an indecomposable FD-module Z' with vertex D, namely $[Z'] = \mathrm{res}_D^{H_0}([W_0])$. In this case we say that σ lies over $[Z']$. If Z' is not isomorphic to one of the modules Z^h for $h \in H$, then $m_{W_n}(\mathrm{res}_{H_n}^H(V)) = 0$ and the chain does not contribute to the formula. One has an obvious bijection between the set of I-orbits of chains in $\mathcal{M}(H)$ that lie over $[Z]$ and the set of H-orbits of chains in $\mathcal{M}(H)$ that lie over $[{}^hZ]$ for some $h \in H$. Moreover, if σ as above lies over $[Z]$, then $\mathrm{Res}_D^{H_n}(W_n) \cong Z$ and therefore $H_n \leqslant I$. In this case we also have $N_H(\sigma) = N_I(\sigma)$. Moreover, $m_{W_n}(\mathrm{Res}_{H_n}^H(V)) = m_{W_n}(\mathrm{Res}_{H_n}^I(W))$. In fact,

$$\mathrm{Res}_I^H(V) \cong \bigoplus_{h \in I\backslash H/I} \mathrm{Ind}_{I\cap I^h}^I(\mathrm{Res}_{I\cap I^h}^{I^h}(W^h)),$$

and for $h \in H$ one has

$$\begin{aligned}\mathrm{Res}_D^I(\mathrm{Ind}_{I\cap I^h}^I(\mathrm{Res}_{I\cap I^h}^{I^h}(W^h))) &\cong \bigoplus_{x \in I/I\cap I^h} \mathrm{Res}_D^{I^{hx}}(W^{hx}) \\ &= \bigoplus_{x \in I/I\cap I^h} (\mathrm{Res}_D^I(W))^{hx}.\end{aligned}$$

Since $\mathrm{Res}_D^I(W)$ is isomorphic to a direct sum of copies of Z, the summands with $h \notin I$ do not contribute and the above claim follows. Therefore we can rewrite the explicit formula for $a_H([V])$ from Theorem 4.4 by summing over representatives of the I-orbits of chains $\sigma \in \mathcal{M}(I)$, replace $N_H(\sigma)$ by $N_I(\sigma)$ and $m_{W_n}(\mathrm{Res}_{H_n}^H(V))$ by $m_{W_n}(\mathrm{Res}_{H_n}^I(W))$. The resulting expression is then equal to $\mathrm{ind}_I^H(a_I([W]))$. □

Remark 4.6. Let $D \leqslant H \leqslant G$. The last proposition showed that the computation of $a_H([V])$ for $V \in \mathsf{mod}_{FH}^D$ can be reduced to the case where V has an H-stable source $Z \in \mathsf{mod}_{FD}$. We fix an indecomposable FD-module Z with vertex D and an intermediate group $D \leqslant I \leqslant N_G(D, Z)$. As already explained in the proof of Theorem 3.3, there exists a cyclic p'-group C, a 2-cocycle $\delta \in Z^2(\bar{I}, C)$ with $\bar{I} := I/D$, and a homomorphism $\vartheta\colon C \to F^\times$ such that for the central extension

$$1 \longrightarrow C \longrightarrow \widehat{I} \xrightarrow{\nu} \bar{I} \longrightarrow 1$$

associated to δ one has an F-algebra isomorphism

$$F\widehat{I}e_\vartheta \cong F_\gamma \bar{I}$$

with $\gamma := \vartheta \circ \delta$, and such that for every intermediate group $D \leqslant J \leqslant I$ one has a bijection between isomorphism classes of indecomposable FJ-modules X with vertex D and source Z and indecomposable projective $F\widehat{J}$-modules Q with $\mathrm{Res}_C^{\widehat{I}}(Q) \cong F_\vartheta \oplus \cdots \oplus F_\vartheta$. This bijection commutes with induction, restriction, conjugation, and direct sums. Thus, under the bijection, the elements in $\mathcal{B}(J)$ correspond to extensions of ϑ to $\widehat{J}$. This shows that the canonical induction formula a defined in 4.1 could alternatively have been obtained by using the canonical induction formula for projective $F\widehat{I}$-modules from [1] and the above bijection.

Remark 4.7. If V is a trivial source FG-module with normal vertex D, then V is the inflation of a projective $F[G/D]$-module $\widetilde{V}$. In [2] a canonical induction formula for trivial source modules was constructed that commutes with inflation. Therefore, the canonical induction formula for V from [2] is the inflation of the canonical induction formula from [2] for $\widetilde{V}$ which coincides with the one for projective modules from [1] for $\widetilde{V}$. On the other hand, the one constructed here for V is also equal to the inflation of the one constructed in [1] (by comparing the explicit formulas). Thus, the one constructed here for V coincides with the one constructed in [2] for V. Strictly speaking, the foregoing statement is only true if one either changes the constructions in [1] and [2] to the effect that the formula only induces from solvable subgroups, or if one relaxes the formula constructed in Section 4, by allowing all p'-subgroups of G/D. The proofs remain the same for either of these changes.

However, if one takes in the general case with nonnormal vertex the recursive approach using the Green correspondence combined with the normal vertex formula developed here, the result is (in general) different from the canonical induction formula for trivial source modules from [2].

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