Group Algebra Extensions of Depth One*

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Abstract

A ring extension $A \subseteq B$ is said to have depth one if B is isomorphic to a direct summand of A^n as an (A, A)-bimodule, for some positive integer n. We prove group-theoretic characterizations of this property in the case $kH \subseteq kG$ where H is a subgroup of a finite group G and k is a field. Moreover, we determine when the source algebra of a block of kG with defect group P is a depth one extension of kP.

Introduction

A *ring extension* is a unitary ring homomorphism $f: A \to B$ between two rings A and B. In this situation, the ring B can be viewed as an (A, A)-bimodule using the map f. A ring extension $f: A \to B$ is said to be of *depth one* (or *centrally projective*, cf. [Ka1]) if B is isomorphic, as (A, A)-bimodule, to a direct summand of A^n for some positive integer n. We will write $B \mid A^n$

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for this condition. Whenever A is a unitary subring of B and $f\colon A \to B$ is the inclusion map we denote the corresponding ring extension by $A \subseteq B$.

To the best of our knowledge, centrally projective ring extensions were first considered by Hirata, cf. [Hi]. The identification of centrally projective ring extensions with ring extensions of depth one appeared in [K-S]. Ring extensions of higher depth were studied in [Ka2], for example.

In this paper we try to answer the question when a ring extension $kH \subseteq kG$ of group rings has depth one. Here and throughout this paper we will denote by k a commutative ring, by G a group and by H a subgroup of finite index in G. In [B-K] we considered the question when the ring extension $kH \subseteq kG$ has depth two (i.e., when $kG \otimes_{kH} kG \mid (kG)^n$ as (kG, kH)-bimodules, or equivalently as (kH, kG)-bimodules, for some positive integer n). It turns out that this is equivalent to H being normal in G, independently of k. If the ring extension $kH \subseteq kG$ has depth one it also has depth two, since one can apply the functor $kG \otimes_{kH} -$ to the relation $kG \mid (kH)^n$. In particular, H has to be normal in G. The converse is not true in general, as the main results, Theorems 1.8 and 1.11, of this paper show. In these theorems we have to assume that k is a field (or a complete discrete valuation ring of characteristic 0 and positive residual characteristic p). In both cases the depth two condition is equivalent to a purely group theoretic condition on the inclusion $H \leqslant G$, namely that $G = HC_G(X)$, for every cyclic subgroup X of H in the characteristic 0 case, and for every p-hypoelementary subgroup X of H in the modular case (see Remark 1.9 for a definition of p-hypoelementary groups). Therefore, the group theoretic depth one condition does depend on the base ring k. We do not have a group theoretic reformulation in the case $k = \mathbb{Z}$.

At the end we study the depth one condition for the ring extension $FP \to A$, where P is the defect group of a block of a group algebra over a field F of positive characteristic p, and A is a source algebra of the block.

1 Depth One for Group Algebra Extensions

1.1 Assume that $g \in N_G(H)$, the normalizer of H in G. Then

$$c_{g^{-1}}\colon kH \to kH\,, \quad a \mapsto g^{-1}ag\,,$$

is a k-algebra automorphism of kH. Restriction along this automorphism defines a functor

$$C_g\colon {}_{kH}\mathsf{Mod} \to {}_{kH}\mathsf{Mod}\,, \quad M \mapsto {}^gM\,,$$

on the category of left kH-modules. More explicitly, for $M \in {}_{kH}\mathsf{Mod}$, the left kH-module ${}^gM$ is defined to be equal to M as an abelian group and it is endowed with the module structure $a * m := (g^{-1}ag) \cdot m$, where $\cdot$ denotes the original kH-module structure of M. The functor C_g maps a homomorphism $f\colon M_1 \to M_2$ in ${}_{kH}\mathsf{Mod}$ to ${}^gf := f\colon {}^gM_1 \to {}^gM_2$.

If C_g is naturally equivalent to the identity functor on ${}_{kH}\mathsf{Mod}$ we say that g *acts trivially on* ${}_{kH}\mathsf{Mod}$. For the purpose of this paper we say that G *acts trivially on* ${}_{kH}\mathsf{Mod}$ if H is normal in G and g acts trivially on ${}_{kH}\mathsf{Mod}$ for every $g \in G$. Note that H acts trivially on ${}_{kH}\mathsf{Mod}$. This is also an immediate consequence of the next proposition.

The subset $kgH = gkH = kHg$ of kG is a (kH, kH)-subbimodule of kG. It is isomorphic to ${}^{(g,1)}kH$ if we view (kH, kH)-bimodules M as left $k[H \times H]$-modules via $(h_1, h_2) \cdot m := h_1 m h_2^{-1}$ for $h_1, h_2 \in H$ and $m \in M$.

1.2 Proposition *For $g \in N_G(H)$, the following are equivalent:*
(i) *$kgH \cong kH$ as (kH, kH)-bimodules.*
(ii) *There exists a unit u of kH such that $gag^{-1} = uau^{-1}$ for all $a \in kH$.*
(iii) *g acts trivially on ${}_{kH}\mathsf{Mod}$.*

Proof (i) $\Rightarrow$ (ii): Let $\varphi\colon kgH \to kH$ be an isomorphism of (kH, kH)-bimodules and set $u := \varphi(g)$. Since φ is surjective, there exists $v \in kH$ such that $1 = \varphi(gv) = \varphi(g)v = uv$. Since $kgH = kHg$, there also exists $w \in kH$ such that $1 = \varphi(wg) = wu$. Thus, u is a unit of kH. For every $a \in kH$ we have $\varphi(u^{-1}ga) = u^{-1}\varphi(g)a = a = au^{-1}\varphi(g) = \varphi(au^{-1}g)$. The injectivity of φ implies that $u^{-1}ga = au^{-1}g$ or equivalently that $gag^{-1} = uau^{-1}$.

(ii) $\Rightarrow$ (iii): For $M \in {}_{kH}\mathsf{Mod}$ we define $\alpha_M\colon {}^gM \to M$ by $\alpha_M(m) := um$ for $m \in {}^gM$. Then, for $a \in kH$, we have $\alpha_M(a*m) = ug^{-1}agm = uu^{-1}aum = a\alpha_M(m)$. Thus, α_M is a kH-module homomorphism. It is bijective, since u is a unit in kH. Finally, if also $M' \in {}_{kH}\mathsf{Mod}$ and $f \in \mathrm{Hom}_{kH}(M, M')$ then $(\alpha_{M'} \circ {}^gf)(m) = \alpha_{M'}(f(m)) = uf(m) = f(um) = (f \circ \alpha_M)(m)$ for all $m \in M$.

(iii) $\Rightarrow$ (i): Let α be a natural isomorphism from C_g to the identity functor on ${}_{kH}\mathsf{Mod}$. We will show that $\varphi\colon gkH \to kH$, $ga \mapsto \alpha_{kH}(a)$, is a (kH, kH)-bimodule isomorphism. Since α_{kH} is bijective, also φ is bijective. Moreover, for $a, b \in kH$, one has

$$\varphi(bga) = \varphi(gg^{-1}bga) = \alpha_{kH}(g^{-1}bga) = \alpha_{kH}(b * a) = b\alpha_{kH}(a) = b\varphi(ga).$$

Finally, since $kH \to kH$, $a \mapsto ab$, is a left kH-module homomorphism, the naturality of α yields $\varphi(gab) = \alpha_{kH}(ab) = \alpha_{kH}(a)b = \varphi(ga)b$. □

For every subset X of H we denote by $C_G(X)$ the centralizer of X in G.

1.3 Corollary *If $G = HC_G(H)$ then the ring extension $kH \subseteq kG$ has depth one. Conversely, if the ring extension $kH \subseteq kG$ has depth one then H is normal in G.*

Proof Suppose first that $G = HC_G(H)$. Then every $g \in G$ satisfies Condition (ii) in Proposition 1.2. Using Condition (i) in Proposition 1.2 together with the decomposition $kG = \bigoplus_{gH \in G/H} kgH$ into (kH, kH)-subbimodules the first assertion follows. The second assertion was already observed in the introduction. □

1.4 If Λ is a k-order (i.e., a k-algebra which is finitely generated and projective as k-module) we say that the *Krull-Schmidt theorem holds* for Λ-lattices if the following two properties hold for every Λ-module M which is finitely generated and projective as k-module:

(i) M has a decomposition $M = U_1 \oplus \cdots \oplus U_r$ into indecomposable Λ-submodules, and

(ii) if $M = U_1 \oplus \cdots \oplus U_r = V_1 \oplus \cdots \oplus V_s$ are two decompositions into indecomposable Λ-submodules then $r = s$ and there exists a permutation σ of $\{1, \ldots, r\}$ such that $U_i \cong V_{\sigma(i)}$ for all $i \in \{1, \ldots, r\}$.

If k is a field or a complete discrete valuation ring then the Krull-Schmidt theorem holds for every k-order Λ, cf. [C-R, Theorem 6.12].

1.5 Proposition *Assume that G is finite and that the Krull-Schmidt theorem holds for $k[H \times H]$-lattices. The following are equivalent:*

(i) *$kH \subseteq kG$ is a ring extension of depth one.*

(ii) *H is normal in G and $kgH \cong kH$ as (kH, kH)-bimodules for every $g \in G$.*

(iii) *H is normal in G and $kG \cong (kH)^{[G:H]}$ as (kH, kH)-bimodules.*

(iv) *$kG \cong (kH)^{[G:H]}$ as (kH, kH)-bimodules.*

Proof (i) $\Rightarrow$ (ii): Since the ring extension $kH \subseteq kG$ has depth one it also has depth two. By Corollary 1.3, this implies that H is normal in G. Therefore, for every $g \in G$, one has $kgH \mid kG \mid (kH)^n$ as (kH, kH)-bimodules for some positive integer n. Since the Krull-Schmidt theorem holds for $k[H \times H]$-lattices, every indecomposable direct summand of the $k[H \times H]$-module kgH is isomorphic to an indecomposable direct summand of the $k[H \times H]$-module kH. But, the indecomposable direct summands of kH are the blocks of kH, and they are pairwise non-isomorphic. Since kgH is isomorphic to ${}^{(g,1)}kH$ and since $C_{(g,1)}\colon {}_{k[H\times H]}\mathsf{Mod} \to {}_{k[H\times H]}\mathsf{Mod}$ is a category equivalence, also the indecomposable direct summands of kgH are pairwise non-isomorphic. Thus, we can decompose kgH and kH multiplicity-free into a direct sum of indecomposable $k[H \times H]$-submodules. The number of these summands coincides, since $kgH \cong {}^{(g,1)}kH$. Since every summand of kgH occurs as a summand of kH, we can conclude that $kgH \cong kH$ as (kH, kH)-bimodules.

(ii) $\Rightarrow$ (iii): The decomposition $kG = \bigoplus_{gH \in G/H} kgH$ into (kH, kH)-subbimodules shows that (ii) implies (iii).

(iii) $\Rightarrow$ (iv): This is trivial.

(iv) $\Rightarrow$ (i): This is immediate from the definition of depth one. □

1.6 Remark In the proof of Proposition 1.5 the Krull-Schmidt property is only used for permutation $k[H \times H]$-modules. By [B-G, Theorem 1.6], the four conditions in Proposition 1.5 are still equivalent when k is a local domain containing a root of unity of order e, where e is defined as the exponent $\exp(H)$ of H in the case that k has characteristic 0, and as the p'-part of $\exp(H)$ in the case that k has positive characteristic p.

Next we will study the case where k is a field of characteristic 0 and H is a finite group. In this case we will denote by $\overline{k}$ an algebraic closure of k and by $\mathrm{Irr}(H)$ the set of irreducible characters of $\overline{k}H$. Recall that $N_G(H)$ acts from the left on $\mathrm{Irr}(H)$ via $\chi \mapsto {}^g\chi$ for $g \in N_G(H)$, where ${}^g\chi(h) = \chi(g^{-1}hg)$, for $h \in H$. Recall also that $N_G(H)$ acts on the set of conjugacy classes of H via $\mathcal{K} \mapsto {}^g\mathcal{K}$, where ${}^g\mathcal{K} = \{ghg^{-1} \mid h \in \mathcal{K}\}$ for a conjugacy class $\mathcal{K}$ of H.

1.7 Lemma *Assume that k is a field of characteristic 0 and that H is finite. For each $g \in N_G(H)$ the following are equivalent:*

(i) *$kgH \cong kH$ as (kH, kH)-bimodules.*

(ii) *One has ${}^g\chi = \chi$ for all $\chi \in \mathrm{Irr}(H)$.*

(iii) *One has ${}^g\mathcal{K} = \mathcal{K}$ for every conjugacy class $\mathcal{K}$ of H.*

(iv) *One has* $g \in HC_G(x)$ *for every* $x \in H$.

Proof (i) $\iff$ (ii): First note that, by the Deuring-Noether theorem (cf. [N-T, Theorem 3.1]), $kgH \cong kH$ as (kH, kH)-bimodules if and only if $\overline{k}gH \cong \overline{k}H$ as $(\overline{k}H, \overline{k}H)$-bimodules. By Proposition 1.2, the latter is equivalent to g acting trivially on ${}_{\overline{k}H}\mathsf{Mod}$, which implies that g acts trivially on $\mathrm{Irr}(H)$. Conversely, if ${}^g\chi = \chi$ for every $\chi \in \mathrm{Irr}(H)$ then $\overline{k}gH \cong \overline{k}H$ as $\overline{k}[H \times H]$-modules, since the character of $\overline{k}H$ is equal to $\sum_{\chi \in \mathrm{Irr}(H)} \chi \times \overline{\chi}$ and the character of $\overline{k}gH$ is equal to

$$ {}^{(g,1)}\Big(\sum_{\chi \in \mathrm{Irr}(H)} \chi \times \overline{\chi}\Big) = \sum_{\chi \in \mathrm{Irr}(H)} {}^g\chi \times \overline{\chi} = \sum_{\chi \in \mathrm{Irr}(H)} \chi \times \overline{\chi}\,. $$

(ii) $\Rightarrow$ (iii): This follows immediately from Brauer's permutation lemma, cf. [N-T, Lemma 2.19].

(iii) $\Rightarrow$ (iv): Let $x \in H$. Since gxg^{-1} lies in the same conjugacy class as x, there exists $h \in H$ such that $gxg^{-1} = hxh^{-1}$. This implies $h^{-1}g \in C_G(x)$ and $g \in hC_G(x) \subseteq HC_G(x)$.

(iv) $\Rightarrow$ (ii): Let $\chi \in \mathrm{Irr}(H)$ and $x \in H$. Then there exists $h \in H$ and $c \in C_G(x)$ such that $g = hc$. Hence $\chi(gxg^{-1}) = \chi(hcxc^{-1}h^{-1}) = \chi(hxh^{-1}) = \chi(x)$. Thus ${}^{g^{-1}}\chi = \chi$ and $\chi = {}^g\chi$. ∎

1.8 Theorem *Assume that* k *is a field of characteristic* 0 *and that* G *is finite. The following are equivalent:*

(i) $kH \subseteq kG$ *is a ring extension of depth one.*

(ii) H *is normal in* G *and* $kgH \cong kH$ *as* (kH, kH)*-bimodules for every* $g \in G$.

(iii) $kG \cong (kH)^{[G:H]}$ *as* (kH, kH)*-bimodules.*

(iv) H *is normal in* G *and* G *acts trivially on* ${}_{kH}\mathsf{Mod}$.

(v) H *is normal in* G *and* G *acts trivially on* $\mathrm{Irr}(H)$.

(vi) H *is normal in* G *and* G *acts trivially on the set of conjugacy classes of* H.

(vii) $G = HC_G(X)$ *for every cyclic subgroup* X *of* H.

Proof Assertions (i), (ii) and (iii) are equivalent by Proposition 1.5. Assertions (ii) and (iv) are equivalent by Proposition 1.2. Finally, (ii), (v), (vi) and (vii) are equivalent by Lemma 1.7, noting that the condition in (vii) implies immediately that H is normal in G. ∎

1.9 Remark Next we study the depth one condition for the ring extension $kH \subseteq kG$ in the case where k is a field of positive characteristic p, or where k is a complete discrete valuation ring of characteristic 0 and positive residual characteristic p. We will need the theory of species developed by Benson and Parker, cf. [B, Section 5.5]. For this remark assume that G is finite and that k contains a root of unity whose order is equal to $\exp(G)$ if k has characteristic 0, and to the p'-part of $\exp(G)$ if k has characteristic p. Let S and T be finite left G-sets. We denote the corresponding permutation kG-modules by kS and kT. The goal of this remark is to derive a criterion for kS being isomorphic to kT. For this purpose, denote by $\mathcal{H}_p(G)$ the set of p-hypoelementary subgroups E of G, i.e., subgroups E which have a normal (Sylow) p-subgroup P such that E/P is cyclic. We claim that

$$kS \cong kT \text{ as } kG\text{-modules} \iff |S^E| = |T^E| \text{ for all } E \in \mathcal{H}_p(G), \quad (1.9.a)$$

where $|S^E|$ denotes the cardinality of the set S^E of E-fixed points on S. In order to see the above equivalence, it suffices to show that

$$s_{E,g}(kS) = |S^E| \quad (1.9.b)$$

for all $E \in \mathcal{H}_p(G)$ and all p'-elements $g \in E$ such that gP generates E/P, where P denotes the Sylow p-subgroup of E. For a definition of $s_{E,g}$ see [B, Section 5.5]. Since $s_{E,g}(kS) = s_{E,g}(\mathrm{Res}^G_E(kS))$, we may assume that $G = E$. Moreover, since $s_{E,g}$ is additive, we may assume that S is a transitive E-set, i.e., $S = E/D$ for some subgroup D of E. Then $kS \cong \mathrm{Ind}_D^E(k)$. If P is not contained in D then no indecomposable direct summand of kS has vertex P, and both sides of Equation (1.9.b) are equal to 0. If $P \leqslant D < E$ then the Brauer species of $\mathrm{Ind}_D^E(k)$ at g is equal to 0, since $g \notin D$, and again both sides in Equation (1.9.b) are equal to 0. Finally, if $D = E$ it is immediate that both sides of the equation are equal to 1.

In the next lemma we will apply the criterion in (1.9.a) to the $H \times H$-sets gH and H for $g \in N_G(H)$.

For a subgroup X of H we set $\Delta X := \{(x, x) \mid x \in X\}$. This is a subgroup of $H \times H$.

1.10 Lemma *Assume that H is finite and that k is a field of characteristic $p > 0$ or a complete discrete valuation ring of characteristic 0 with residual characteristic $p > 0$. For $g \in N_G(H)$, the following are equivalent:*

(i) $kgH \cong kH$ *as* (kH,kH)*-bimodules.*
(ii) $|(gH)^E| = |H^E|$ *for all* $E \in \mathcal{H}_p(H \times H)$.
(iii) $|(gH)^{\Delta X}| = |H^{\Delta X}|$ *for all* $X \in \mathcal{H}_p(H)$.
(iv) $g \in C_G(X)H$ *for all* $X \in \mathcal{H}_p(H)$.

Proof (i) $\iff$ (ii): This follows immediately from Remark 1.9. In fact, by the Deuring-Noether theorem (cf. [N-T, Theorem 3.1]) and by Corollary 3.11.4(i) in [B], we have $kgH \cong kH$ as $k[H \times H]$-modules if and only if $k'gH \cong k'H$ as $k'[H \times H]$-modules, where k' is obtained from k by adjoining a root of unity whose order is equal to $\exp(H)$ if k has characteristic 0, and to the p'-part of $\exp(H)$ if k has characteristic p.

(ii) $\Rightarrow$ (iii): This is trivial.

(iii) $\Rightarrow$ (iv): Since $1 \in H^{\Delta X}$, the set $(gH)^{\Delta X}$ is non-empty. Let $h \in H$ such that $gh \in (gH)^{\Delta X}$. Then $gh \in C_G(X)$ and $g \in C_G(X)H$.

(iv) $\Rightarrow$ (ii): Note that H and gH are transitive $H \times H$ sets with stabilizers ΔH of $1 \in H$ and ${}^{(g,1)}\Delta H$ of $g \in gH$, respectively. Let $E \in \mathcal{H}_p(H \times H)$. One has $|H^E| = 0 = |(gH)^E|$ unless E is $H \times H$-conjugate to a subgroup of ΔH or ${}^{(g,1)}\Delta H$. Since the number of fixed points does not change if we replace E by an $H \times H$-conjugate of E, we may assume that $E \leqslant \Delta H$ or $E \leqslant {}^{(g,1)}\Delta H$. We first assume that $E \leqslant \Delta H$. Then $E = \Delta X$ for some $X \in \mathcal{H}_p(H)$. Since $g \in C_G(X)H$ we can write $g = ch$ with $c \in C_G(X)$ and $h \in H$. Then $gH = cH$ and for $h' \in H$ we have:

$$h' \in H^{\Delta X} \iff h' \in C_G(X) \iff ch' \in C_G(X) \iff ch' \in (cH)^{\Delta X}.$$

It follows that $|H^{\Delta X}| = |(cH)^{\Delta X}| = |(gH)^{\Delta X}|$. Finally, if $E \leqslant {}^{(g,1)}\Delta H$ then ${}^{(g^{-1},1)}E = \Delta X$ for some $X \in \mathcal{H}_p(H)$. Again we can write $g = ch$ with $c \in C_G(X)$ and $h \in H$. Then $g = h'c$ with $h' = ghg^{-1} \in H$ and $E = {}^{(g,1)}(\Delta X) = {}^{(h'c,1)}(\Delta X) = {}^{(h',1)}(\Delta X)$ is $H \times H$-conjugate to ΔX. This implies $|(gH)^E| = |(gH)^{\Delta X}| = |H^{\Delta X}| = |H^E|$ by the first case. □

1.11 Theorem *Assume that G is finite and that k is a field of characteristic $p > 0$ or a complete discrete valuation ring of characteristic 0 with residual characteristic $p > 0$. The following are equivalent:*

(i) $kH \subseteq kG$ *is a ring extension of depth one.*

(ii) H *is normal in* G *and* $kgH \cong kH$ *as* (kH,kH)*-bimodules for every* $g \in G$.

(iii) $kG \cong (kH)^{[G:H]}$ *as* (kH,kH)*-bimodules.*

(iv) *H is normal in G and G acts trivially on ${}_{kH}\mathsf{Mod}$.*
(v) *$G = C_G(X)H$ for every $X \in \mathcal{H}_p(H)$.*

Proof Assertions (i), (ii) and (iii) are equivalent by Proposition 1.5, and the assertions in (ii) and (iv) are equivalent by Proposition 1.2. Finally, (ii) and (v) are equivalent by Lemma 1.10, noting that (v) immediately implies the normality of H in G. ∎

1.12 Remark In the case $k = \mathbb{Z}$ we do not know if there is a similar equivalence (i) $\iff$ (v) as in Theorem 1.11 with $\mathcal{H}_p(H)$ replaced by some other set $\mathcal{S}(H)$ of subgroups of H. Even if there existed such a set $\mathcal{S}(H)$, we don't have a good guess what it should be.

If $\mathbb{Z}H \subseteq \mathbb{Z}G$ has depth one then $kH \subseteq kG$ has depth one for every commutative ring k (by scalar extension). In particular this implies that $G = HC_G(X)$ for every p-hypoelementary subgroup X of H for all primes p. We do not know if the converse holds. On the other hand, if $G = HC_G(H)$ then $\mathbb{Z}H \subseteq \mathbb{Z}G$ has depth one by Corollary 1.3. However, the converse is not true. In fact, by Theorem A in [He], there exists a finite group H (metabelian of order $2^{25} \cdot 97^2$), a non-inner automorphism g of H, and a unit u of $\mathbb{Z}H$ with $g(a) = uau^{-1}$. We set $G := H \rtimes \langle g \rangle$. By Proposition 1.2, we obtain $\mathbb{Z}g^iH \cong \mathbb{Z}H$ as $(\mathbb{Z}H, \mathbb{Z}H)$-bimodules for every integer i. This implies that $\mathbb{Z}G = \bigoplus_{xH \in G/H} \mathbb{Z}xH \cong (\mathbb{Z}H)^{[G:H]}$ and that $\mathbb{Z}H \subseteq \mathbb{Z}G$ has depth one. But $g \notin C_G(H)H$, since g is not an inner automorphism of H. This shows that if, for each finite group H, there exists a set of subgroups $\mathcal{S}(H)$ of H which replaces $\mathcal{H}_p(H)$ in Theorem 1.11(v) in the case $k = \mathbb{Z}$ then $H \notin \mathcal{S}(H)$ for Hertweck's group H.

2 Depth One for Source Algebras of Blocks

2.1 Let G be a finite group, let p be a prime, and let (K, R, F) be a p-modular system. Thus, R is a complete discrete valuation ring of characteristic zero, K is the field of fractions of R, and F, the residue field of R, has characteristic p. We assume that R contains a root of unity of order $\exp(G)$ and that F is algebraically closed. Then K and F are splitting fields for KG and FG, respectively. For an R-order A we denote by $\overline{A}$ the finite-dimensional F-algebra $F \otimes_R A$. In the following, let $k \in \{R, F\}$.

In this section, we will consider the depth one condition for blocks and source algebras. For general background, we refer to the books [T] and [Ku]. For the convenience of the reader, we recall some of the basic concepts.

An *interior G-algebra* over k consists of a k-order A and a group homomorphism $i\colon G \to A^\times$, where $A^\times$ denotes the group of units of A. In this case, we will consider the k-linear extension $kG \to A$ of i as a ring extension. Two interior G-algebras A_1 and A_2 are called *isomorphic* if there exists an isomorphism $f\colon A_1 \to A_2$ commuting with the structural maps $i_1\colon G \to A_1^\times$ and $i_2\colon G \to A_2^\times$.

If A is an interior G-algebra then a *point* of a subgroup H of G on A is an $(A^H)^\times$-conjugacy class β of primitive idempotents in the subalgebra $A^H := \{a \in A \mid ha = ah \text{ for all } h \in H\}$ of A. In this case the pair $(H, \beta) =: H_\beta$ is called a *pointed group* on A.

The point β of H on A is called *local* if $\beta \not\subseteq \mathrm{Tr}_L^H(A^L)$ for every proper subgroup L of H; here $\mathrm{Tr}_L^H\colon A^L \to A^H$, $a \mapsto \sum_{hL\in H/L} hah^{-1}$, is the *relative trace map*. If β is a local point of H on A then H_β is called a *local pointed group* on A. One can show that in this case H has to be a p-group.

Let H_β and L_γ be pointed groups on A. We write $L_\gamma \leq H_\beta$ if $L \leqslant H$ and $jAj \subseteq iAi$ for suitable idempotents $i \in \beta$, $j \in \gamma$. This defines a partial order on the set of pointed groups on A. The group G acts by conjugation on the set of all pointed groups H_β on A, and this action is compatible with the partial order relation. We denote by $N_G(H_\beta)$ the stabilizer of H_β in G. Thus, $N_G(H_\beta)$ is a subgroup of $N_G(H)$.

A *block* of kG is an indecomposable direct summand B of kG, considered as a (kG, kG)-bimodule. In this case, B is a k-order in its own right. We will consider B as an interior G-algebra via the group homomorphism $G \to B^\times$, $g \mapsto g1_B = 1_Bg$. Then $\alpha := \{1_B\}$ is a point of G on B, and we will consider G_α as a pointed group on B.

The maximal local pointed groups $P_\gamma \leqslant G_\alpha$ are called *defect pointed groups* of G_α (and of B). They are unique up to conjugation in G. If P_γ is a defect pointed group on B then P is also called a *defect group* of B. For $i \in \gamma$, the k-order $B_\gamma = iBi = ikGi$ is called a *source algebra* of B. One can show that $BiB = B$, so that B and iBi are Morita equivalent k-orders via multiplication with i. The source algebra iBi will always be considered as an interior P-algebra via the map $P \to (iBi)^\times$, $x \mapsto ix = xi$.

The block B is called *nilpotent* if $N_G(Q_\delta)/C_G(Q)$ is a p-group for every local pointed group $Q_\delta \leq G_\alpha$ on B. (Note that indeed $C_G(Q) \subseteq N_G(Q_\delta)$ here.) Puig [P] determined the structure of the source algebra of a nilpotent block. It is a consequence of his results that every nilpotent block has a unique simple module in characteristic p, up to isomorphism. We will make use of Puig's results in the following theorem.

2.2 Theorem *Let B be a block of RG with defect pointed group P_γ, and let B_γ be a corresponding source algebra. Then the following assertions are equivalent:*

(i) *The ring extension $FP \to \overline{B_\gamma}$ defined by the canonical map $P \to \overline{B_\gamma}^\times$ has depth one.*

(ii) *B_γ and RP are isomorphic as interior P-algebras.*

(iii) *B is a nilpotent block, and the unique simple $\overline{B}$-module M has a trivial source.*

Proof (i) $\Rightarrow$ (ii): Suppose that the ring extension $FP \to \overline{B_\gamma}$ has depth one. Then $\overline{B_\gamma} \mid (FP)^n$ as an (FP, FP)-bimodule, for some positive integer n. Thus every indecomposable direct summand of the (FP, FP)-bimodule $\overline{B_\gamma}$ is isomorphic to FP. Hence [T, Theorem 44.3] implies that $N_G(P_\gamma) = PC_G(P)$ and $\overline{B_\gamma} \cong FP$, as an (FP, FP)-bimodule; in particular, we have $\mathrm{rk}_R(B_\gamma) = \dim_F \overline{B_\gamma} = |P|$. [T, Theorem 44.3] now implies that $B_\gamma \cong RP$ as interior P-algebras.

(ii) $\Rightarrow$ (iii): Suppose that B_γ and RP are isomorphic interior P-algebras. Then a result by Puig [P, Theorem 1.6] implies that the block B is nilpotent (cf. [T, Remark 50.10]). We write $\overline{B_\gamma} = iFGi$ where i is a primitive idempotent in $(FG)^P$. Since every block has at least one simple module whose vertices are defect groups of the block, P is a vertex of the unique simple $\overline{B}$-module M. By [T, Proposition 38.3], M has an FP-source V such that $V \mid iM$, as an FP-module. Since $\overline{B}$ and $\overline{B_\gamma}$ are Morita equivalent via multiplication with i, the $\overline{B_\gamma}$-module iM is simple. Since $\overline{B_\gamma} \cong FP$, iM is trivial as an FP-module, and so is V.

(iii) $\Rightarrow$ (i): Suppose that B is nilpotent and that the unique simple $\overline{B}$-module M has a trivial source. Then M has vertex P, as above, and a result by Puig [T, Theorem 50.6] implies that $\overline{B_\gamma} \cong S \otimes_F FP$ as interior P-algebras where S is an interior P-algebra which is simple as an F-algebra. (Note that the tensor product of two interior P-algebras is again an interior P-algebra via the diagonal map.) As above, we write $\overline{B_\gamma} = iFGi$ where i is a primitive idempotent in $(FG)^P$. Since $\overline{B}$ and $\overline{B_\gamma}$ are Morita equivalent via multiplication with i, the module iM is the unique simple $\overline{B_\gamma}$-module, up to isomorphism. Thus, S and $\mathrm{End}_F(iM)$ are isomorphic interior P-algebras; in particular, $S^P \cong \mathrm{End}_{FP}(iM)$ as F-algebras. But S^P is a local ring (since $\overline{B_\gamma}^P$ is), so iM is indecomposable as an FP-module. On the other hand, [T, Proposition 38.3] implies that iM has a direct summand, as an FP-module, which is a source of M. Thus $\dim_F iM = 1$. Hence $\dim_F S = 1$, so $S \cong F$ and $\overline{B_\gamma} \cong FP$. In particular, the ring extension $FP \to \overline{B_\gamma}$ has depth one. □

2.3 It would be interesting to have a similar description of the depth two condition for source algebras of blocks. The goal of this subsection is to show that $RP \to B_\gamma$ (and also $FP \to \overline{B_\gamma}$) is a *symmetric Frobenius extension* so that the left and right depth two conditions are equivalent (cf. [K-S, Proposition 6.4]).

Recall from [Ka1, Theorem I.1.2] that a ring extension $f\colon \Gamma \to \Delta$ is called a *Frobenius extension* if there exists a (Γ,Γ)-bimodule homomorphism $E\colon \Delta \to \Gamma$ and elements $x_j, y_j \in \Delta$, $j = 1, \ldots, n$, such that

$$\sum_{j=1}^{n} x_j E(y_j a) = a = \sum_{j=1}^{n} E(a x_j) y_j \qquad (2.3.a)$$

for all $a \in \Delta$. If in addition

$$E(ca) = E(ac) \qquad (2.3.b)$$

holds for all $a \in \Delta$ and $c \in C_\Delta(\Gamma)$ then one calls the extension $f\colon \Gamma \to \Delta$ a *symmetric Frobenius extension*.

If $\Gamma \subseteq \Delta$ is a symmetric Frobenius extension and e is an idempotent in $C_\Delta(\Gamma)$ then also $e\Gamma e \subseteq e\Delta e$ is a symmetric Frobenius extension. In fact, if $E\colon \Delta \to \Gamma$ satisfies Equations (2.3.a) and (2.3.b) then it is easy to verify that $\widetilde{E}\colon e\Delta e \to e\Gamma e$, $a \mapsto eE(a)e$, satisfies

$$\sum_{j=1}^{n} e x_j e \widetilde{E}(e y_j e a) = a = \sum_{j=1}^{n} \widetilde{E}(a e x_j e) e y_j e$$

for all $a \in e\Delta e$. Moreover, Equation (2.3.b) implies $\widetilde{E}(ca) = \widetilde{E}(ac)$ for all $a \in e\Delta e$ and $c \in C_{e\Delta e}(e\Gamma e) = eC_\Delta(\Gamma)e$.

Note that if H is a subgroup of G then $kH \subseteq kG$ is a symmetric Frobenius algebra. In fact, one can choose for $E\colon kG \to kH$ the canonical projection and for x_j and y_j coset representatives of G/H and their inverses. Thus, if e

is an idempotent in $(kG)^H$ then also $ekHe \to ekGe$ is a symmetric Frobenius extension. This holds even over arbitrary commutative rings k.

Now our claim follows by specializing to $H = P$ and $e = 1_{B_\gamma}$ (or $e = 1_{\overline{B_\gamma}}$), and noting that $kP \mapsto ekPe$, $a \mapsto eae = ea = ae$, is an isomorphism of k-algebras.

By Subsection 2.3, we do not need to distinguish between the left and right depth two condition in the following proposition.

2.4 Proposition *Let B be the principal block of RG, let P_γ be a maximal local pointed group on B (so that P is a Sylow p-subgroup of G), and set $E := N_G(P_\gamma)/PC_G(P)$. Moreover, let B_γ be a source algebra of B. Then the following assertions are equivalent:*

(i) *The ring extension $FP \to \overline{B_\gamma}$ defined by the structural map $P \to \overline{B_\gamma}^\times$ has depth two.*

(ii) *B_γ is isomorphic to a twisted group algebra $R_\sharp[P \rtimes E]$ of the semidirect product $P \rtimes E$, as an interior P-algebra.*

Proof (i) $\Rightarrow$ (ii): Suppose that the ring extension $FP \to \overline{B_\gamma}$ has depth two, and write $A := \overline{B_\gamma} = iFGi$ where i is a primitive idempotent in $(FG)^P$. Then there exists a positive integer n such that

$$\mathrm{Res}^A_{FP}\mathrm{Ind}^A_{FP}\mathrm{Res}^A_{FP}(iM) \mid \mathrm{Res}^A_{FP}(iM)^n$$

for every B-module M. Taking for M the trivial FG-module F we obtain $A \otimes_{FP} iF \mid (iF)^n$ in ${}_{FP}\mathsf{Mod}$. Thus, P acts trivially on $A \otimes_{FP} iF$. On the other hand, A is a direct sum of (FP, FP)-bimodules of the form $F[PgP]$, for suitable $g \in G$. It is easy to see that $F[PgP] \otimes_{FP} iF \cong \mathrm{Ind}^P_{P\cap gPg^{-1}}(F)$ in ${}_{FP}\mathsf{Mod}$. And if P acts trivially on $\mathrm{Ind}^P_{P\cap gPg^{-1}}(F)$ then $g \in N_G(P)$. Thus A is in fact a direct sum of (FP, FP)-bimodules of the form $F[PgP]$, for suitable $g \in N_G(P)$. Hence [T, Theorem 44.3], a result by Puig, implies that $\mathrm{rk}_R B_\gamma = \dim_F \overline{B_\gamma} = |P| \cdot |E|$. Thus [T, Theorem 45.11], another result by Puig, implies (ii).

(ii) $\Rightarrow$ (i): Suppose that (ii) holds. Since $R_\sharp[P \rtimes E]$ is a strongly E-graded ring with 1-component $R_\sharp P \cong RP$, [B-K, Proposition 1.5] shows that the ring extension $RP \to R_\sharp[P \rtimes E]$ has depth two. Tensoring with F, we obtain (i). ∎

2.5 Remark We note that the implication (ii) $\Rightarrow$ (i) is valid for arbitrary blocks B of RG. Also, if (ii) holds one can show that every simple $\overline{B}$-module M has trivial source by noting that P acts trivially on iM.

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