CANONICAL BRAUER INDUCTION AND SYMMETRIC GROUPS

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Abstract. Imitating the approach of canonical induction formulas we derive a formula that expresses every character of the symmetric group as an integer linear combination of Young characters. It is different from the well-known formula that uses the determinantal form.
Imitando l'approccio della formula canonica dell'induzione, otteniamo una formula che esprime ogni carattere del gruppo simmetrico come combinazione lineare intera di caratteri di Young. E' diversa dalla formula ben nota che usa la forma del determinante.

Let n be a positive integer. We denote by $\mathrm{Irr}(S_n)$ the set of irreducible characters of the symmetric group S_n of degree n. It is well-known that every $\chi \in \mathrm{Irr}(S_n)$ can be written as an integral linear combination of Young characters, i.e., permutation characters on cosets of Young subgroups. An explicit formula is given by the determinantal form (cf. Theorem 2.3.15 in [JK]).

In this short note we will present a somewhat differently looking formula. In order to explain this in more detail, let us introduce the following notation. By $\mathcal{P}(n)$ we denote the set of partitions of the set $\{1, \ldots, n\}$. Then $\mathcal{P}(n)$ is a partially ordered set (poset) with respect to the refinement relation $\leq$. Moreover, S_n acts on $\mathcal{P}(n)$ by

$$g\Lambda := \{g(\Lambda_1), \ldots, g(\Lambda_k)\} \qquad (\Lambda = \{\Lambda_1, \ldots, \Lambda_k\} \in \mathcal{P}(n),\ g \in S_n)$$

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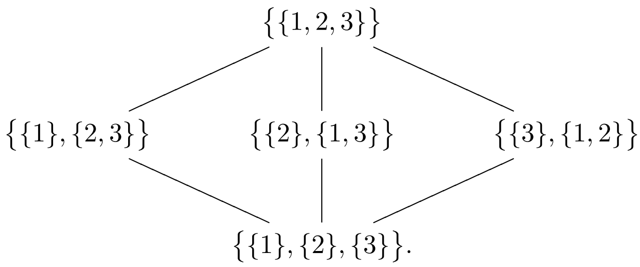
This action is compatible with the partial order $\leq$ on $\mathcal{P}(n)$, so that $\mathcal{P}(n)$ becomes an S_n–poset in this way. For example, the poset $\mathcal{P}(3)$ can be illustrated by the diagram



We denote the Möbius function of $\mathcal{P}(n)$ by μ. Recall that μ is defined recursively by

$$\sum_{\substack{\Delta \\ \Gamma\leq\Delta\leq\Lambda}} \mu(\Delta,\Lambda) = \begin{cases} 1 & \text{if } \Gamma=\Lambda \\ 0 & \text{otherwise,} \end{cases}$$

for $\Gamma, \Lambda \in \mathcal{P}(n)$. Suppose that $\Gamma = \{\Gamma_1, \ldots, \Gamma_k\} \leq \Delta = \{\Delta_1, \ldots, \Delta_l\}$. Furthermore, suppose that each block Δ_j of Δ contains precisely m_j blocks Γ_i of Γ. Then, as is well-known (cf. Example 2.2.23 in [SO]), we have

$$\mu(\Gamma,\Delta) = (-1)^{k-l} \prod_{j=1}^{l} (m_j-1)!\ .$$

Every $\Lambda = \{\Lambda_1, \ldots \Lambda_k\} \in \mathcal{P}(n)$ defines a Young subgroup

$$S_\Lambda := S_{\Lambda_1} \times \ldots \times S_{\Lambda_k}$$

of S_n. Then $gS_\Lambda g^{-1} = S_{g\Lambda}$ for $g \in S_n$. If $\Lambda_i = \{\alpha_{i1}, \ldots, \alpha_{il_i}\}$ with $\alpha_{i1} < \ldots < \alpha_{il_i}$ for $i = 1, \ldots, k$ then let

$$x_\Lambda := (\alpha_{11}, \ldots, \alpha_{1l_1}) \ldots (\alpha_{k1}, \ldots, \alpha_{kl_k}) \in S_\Lambda \leq S_n\,.$$

It is easy to see that

$$x_\Delta \in S_\Lambda \Leftrightarrow \Delta \leq \Lambda \Leftrightarrow S_\Delta \leq S_\Lambda\,.$$

We can now state our main result.

Theorem 1. *For $\chi \in \mathrm{Irr}(S_n)$, we have*

$$\chi = \frac{1}{n!} \sum_{\Gamma\leq\Delta} |S_\Gamma|\, \mu(\Gamma,\Delta)\, \chi(x_\Delta)\ 1_{S_\Gamma}^{S_n}\,,$$

where the sum runs over all pairs $(\Gamma,\Delta) \in \mathcal{P}(n)^2$ with $\Gamma \leq \Delta$.

Proof. Let $\Theta \in \mathcal{P}(n)$. Then

$$\begin{aligned} y &:= \sum_{\Gamma\leq\Delta} |S_\Gamma|\, \mu(\Gamma,\Delta)\, \chi(x_\Delta)\, 1_{S_\Gamma}^{S_n}(x_\Theta) \\ &= \sum_{\Gamma\leq\Delta} \mu(\Gamma,\Delta)\, \chi(x_\Delta) \sum_{\substack{g\in S_n \\ gx_\Theta g^{-1}\in S_\Gamma}} 1\,. \end{aligned}$$

Note that $x_\Theta \in g^{-1}\, S_\Gamma\, g = S_{g^{-1}\Gamma}$ if and only if $\Theta \leq g^{-1}\Gamma$. Hence

$$\begin{aligned} y &= \sum_{g\in S_n} \sum_{g\Theta\leq\Gamma\leq\Delta} \mu(\Gamma,\Delta)\, \chi(x_\Delta) \\ &= \sum_{g\in S_n} \sum_{g\Theta\leq\Delta} \chi(x_\Delta) \sum_{g\Theta\leq\Gamma\leq\Delta} \mu(\Gamma,\Delta) \end{aligned}$$

The inner sum vanishes unless $g\Theta = \Delta$. Thus

$$y = \sum_{g\in S_n} \chi(x_{g\Theta}) = n!\,\chi(x_\Theta)\,,$$

and the result is proved. □

Let us illustrate Theorem 1 with two specific examples. In case $n = 3$, the formula yields

$$\begin{aligned}\chi &= \chi((1,2,3))\,\varphi_{(3)} + \big[\chi((1,2)) - \chi((1,2,3))\big]\varphi_{(2,1)} \\ &+ \frac{1}{6}\big[\chi(1) - 3\chi((1,2)) + 2\chi((1,2,3))\big]\ \varphi_{(1,1,1)}\,;\end{aligned}$$

here we have used partitions λ of n in order to label the Young characters $\varphi_\lambda = 1_{S_\lambda}^{S_n}$ of S_n. For $n = 4$, the formula reads as follows:

$$\begin{aligned}\chi &= \chi((1,2,3,4))\,\varphi_{(4)} + \big[\chi((1,2,3)) - \chi((1,2,3,4))\big]\,\varphi_{(3,1)} \\ &+ \frac{1}{2}\big[\chi((1,2)(3,4)) - \chi((1,2,3,4))\big]\,\varphi_{(2,2)} \\ &+ \frac{1}{2}\big[\chi((1,2)) - 2\chi((1,2,3)) - \chi((1,2)(3,4)) + 2\chi((1,2,3,4))\big]\,\varphi_{(2,1,1)} \\ &+ \frac{1}{24}\big[\chi(1) - 6\chi((1,2)) + 8\chi((1,2,3)) + 3\chi((1,2)(3,4)) - 6\chi((1,2,3,4))\big]\,\varphi_{(1,1,1,1)}\,.\end{aligned}$$

Note the difference between Theorem 1 and the determinantal form (cf. Theorem 2.3.15 in [JK]). Theorem 1 gives a generic formula which works for *all* $\chi \in \mathrm{Irr}(S_n)$ and, more generally, for every generalized character χ of S_n.
On the other hand, the determinantal form looks different for irreducible characters χ_λ labeled by different partitions λ of n. For example, the number $k = l(\lambda)$ of parts of $\lambda = (\lambda_1,\ldots,\lambda_k)$ is responsible for the size of the determinant. Of course, since the Young characters φ_λ form a $\mathbb{Z}$–basis for the group of generalized characters of S_n, both Theorem 1 and the determinantal form yield the same expression when applied to $\chi_\lambda \in \mathrm{Irr}(S_n)$.

Although the proof of Theorem 1 is quite short and elementary it does not tell how the induction formula was found. This resulted from applying the general machinery of 'canonical induction formulae', cf. [B], to the special case of the 'Mackey functor' for S_n given by the representation rings of S_n and its Young subgroups and to the constant 'restriction functor' for S_n given by assigning $\mathbb{Z}$ to each Young subgroup of S_n. Strictly speaking, the setting in [B] is not general enough to cover the above situation, since there only Mackey functors and restriction functors on *all* subgroups were considered. But it would be straightforward to adapt all the results to the more general situation where one considers only subgroups satisfying certain axioms (cf. [BB], where such a notion was introduced as a *Mackey system*). The symmetric group S_n together with its Young subgroups forms a Mackey system. Since we have the above direct proof of Theorem 1, there was no need to introduce all these notions.

We now illustrate how Theorem 1 can be used in order to prove Solomon's formula for the alternating character of S_n (cf. Theorem (66.29) in [CR]). In the following, we denote by $\wp(n)$ the set of partitions λ of n, and

$$\binom{n}{a_1,\ldots,a_k} = \frac{n!}{a_1!\,\ldots\,a_k!}$$

denotes a multinomial coefficient (where $a_1 + \ldots + a_k = n$).

Corollary 2. *Let ε denote the alternating character of S_n. Then*

$$\varepsilon = \sum_{\lambda\in\wp(n)} \binom{l(\lambda)}{a_1(\lambda), a_2(\lambda),\ldots} (-1)^{n-l(\lambda)}\varphi_\lambda\,.$$

Here we have written a partition λ of n in the form $\lambda = (1^{a_1(\lambda)}, 2^{a_2(\lambda)}, \ldots)$.

Proof. By Theorem 1, we have

$$\varepsilon = \sum_{\Gamma\leq\Delta} \frac{|S_\Gamma|}{n!}\mu(\Gamma,\Delta)\ \varepsilon(x_\Delta)\ 1_{S_\Gamma}^{S_n}\,.$$

Let us fix $\Gamma=\{\Gamma_1,\ldots,\Gamma_k\}\in\mathcal{P}(n)$. Then the coefficient c_Γ of $1_{S_\Gamma}^{S_n}$ is

$$c_\Gamma = \frac{|S_\Gamma|}{n!}\sum_{\Gamma\leq\Delta}\mu(\Gamma,\Delta)\varepsilon(x_\Delta)\,.$$

Let $\Gamma\leq\Delta=\{\Delta_1,\ldots,\Delta_l\}$, and suppose that each block Δ_j of Δ contains exactly $m_j(\Delta)$ blocks Γ_i of Γ. Then, as above, we have

$$\mu(\Gamma,\Delta) = (-1)^{k-l}\prod_{j=1}^{l}\,(m_j(\Delta)-1)\,!\ .$$

Moreover we have $\varepsilon(x_\Delta)=(-1)^{n-l}$. Thus

$$\begin{aligned} c_\Gamma &= \frac{|S_\Gamma|}{n!}\sum_{\Gamma\leq\Delta}(-1)^{n-k}\prod_{j=1}^{|\Delta|}(m_j(\Delta)-1)\,! \\ &= \frac{|S_\Gamma|}{n!}\,(-1)^{n-k}\sum_{\Lambda\in\mathcal{P}(k)}\prod_{j=1}^{|\Lambda|}(|\Lambda_j|-1)\,!\ . \end{aligned}$$

Now we would like to replace the summation over $\mathcal{P}(k)$ by a summation over $\wp(k)$. Each $\lambda=(1^{a_1},2^{a_2},\ldots)\in\wp(k)$ corresponds to exactly

$$\frac{k!}{(1!)^{a_1}\ a_1!\ (2!)^{a_2}\ a_2!\ldots}$$

elements $\Lambda\in\mathcal{P}(k)$. Thus

$$c_\Gamma = \frac{|S_\Gamma|}{n!}\,(-1)^{n-k}\sum_{\lambda\in\wp(k)}\frac{k!}{1^{a_1(\lambda)}\,a_1(\lambda)!\ 2^{a_2(\lambda)}\,a_2(\lambda)!\ \ldots}\ .$$

The fraction on the right hand side equals the length of the conjugacy class of S_k parametrized by λ. Hence

$$c_\Gamma = \frac{|S_\Gamma|}{n!}\,(-1)^{n-k}\ k!\ ,$$

so that we have

$$\varepsilon = \sum_{\Gamma\in\mathcal{P}(n)}\frac{|S_\Gamma|}{n!}\,(-1)^{n-|\Gamma|}\ (|\Gamma|!)\ 1_{S_\Gamma}^{S_n}\ .$$

Next we replace the summation over $\mathcal{P}(n)$ by a summation over $\wp(n)$ and obtain

$$\begin{aligned} \varepsilon &= \sum_{\gamma\in\wp(n)}|S_n:N_{S_n}(S_\gamma)|\ \frac{|S_\gamma|}{n!}\ (-1)^{n-l(\gamma)}\ (l(\gamma)!)\ 1_{S_\gamma}^{S_n} \\ &= \sum_{\gamma\in\wp(n)}|N_{S_n}(S_\gamma):S_\gamma|^{-1}\ (-1)^{n-l(\gamma)}\ (l(\gamma)!)\ \varphi_\gamma\ . \end{aligned}$$

Let us fix $\gamma=(1^{a_1},\ 2^{a_2},\ldots)\in\wp(n)$. Then

$$\begin{aligned} |S_\gamma| &= (1!)^{a_1}\ (2!)^{a_2}\ \ldots\ , \\ |N_{S_n}(S_\gamma)| &= (1!)^{a_1}a_1!\ (2!)^{a_2}a_2!\ \ldots\ , \\ |N_{S_n}(S_\gamma):S_\gamma| &= a_1!\ a_2!\ \ldots\ , \\ l(\gamma) &= a_1+a_2+\ldots\ . \end{aligned}$$

We conclude that

$$\varepsilon = \sum_{\gamma \in \wp(n)} \binom{l(\gamma)}{a_1(\gamma),\ a_2(\gamma),\ \dots} (-1)^{n-l(\gamma)} \varphi_\gamma \ ,$$

and the proof is complete. □

It is not immediate from Theorem 1 that every $\chi \in \mathrm{Irr}(S_n)$ is an *integral* linear combination of Young characters. For completeness, we will now give an easy proof of this well-known fact.

In the following, we denote by $\mathcal{X}(S_n)$ the group of virtual characters of S_n, by $\mathcal{Y}(S_n)$ the group generated by all Young characters φ_λ ($\lambda \in \wp(n)$), and by $\mathcal{Z}(S_n)$ the group of all class functions $S_n \to \mathbb{Z}$. Then we have

$$\mathcal{Y}(S_n) \subseteq \mathcal{X}(S_n) \subseteq \mathcal{Z}(S_n).$$

Proposition 3. *We have* $\mathcal{Y}(S_n) = \mathcal{X}(S_n)$.

Proof. It suffices to show that

$$|\mathcal{Z}(S_n) : \mathcal{Y}(S_n)| = |\mathcal{Z}(S_n) : \mathcal{X}(S_n)| < \infty.$$

Let us first compute $|\mathcal{Z}(S_n) : \mathcal{Y}(S_n)|$. This index coincides with the determinant of the matrix

$$Y_n := (\varphi_\lambda(x_\mu) : \lambda,\ \mu \in \wp(n))$$

provided that $\det Y_n \neq 0$. Note that

$$\begin{aligned} \varphi_\lambda(x_\mu) &= 1_{S_\lambda}^{S_n}(x_\mu) = |\{g\, S_\lambda \in S_n/S_\lambda : x_\mu \in g\, S_\lambda\, g^{-1}\}| \\ &= |\{g\, S_\lambda \in S_n/S_\lambda : S_\mu \leq g\, S_\lambda\, g^{-1}\}|\,, \end{aligned}$$

for $\lambda, \mu \in \wp(n)$. Hence, with an appropriate ordering of $\wp(n)$, Y_n is an upper triangular matrix with diagonal entries $|N_{S_n}(S_\lambda) : S_\lambda|$, $\lambda \in \wp(n)$. We conclude that

$$\det Y_n = \prod_{\lambda \in \wp(n)} |N_{S_n}(S_\lambda) : S_\lambda| = \prod_{\lambda \in \wp(n)} a_1(\lambda)!\ a_2(\lambda)!\ \dots\ .$$

Next we compute $|\mathcal{Z}(S_n) : \mathcal{X}(S_n)|$. This index coincides with the determinant of the character table

$$X_n = (\chi_\lambda(x_\mu) : \lambda,\ \mu \in \wp(n))$$

of S_n provided that $\det X_n \neq 0$. The orthogonality relations imply that

$$X_n^T X_n = \mathrm{diag}(|C_{S_n}(x_\lambda)| : \lambda \in \wp(n))\,.$$

Hence

$$\begin{aligned} (\det X_n)^2 &= \prod_{\lambda \in \wp(n)} |C_{S_n}(x_\lambda)| \\ &= \prod_{\lambda \in \wp(n)} 1^{a_1(\lambda)} a_1(\lambda)!\ 2^{a_2(\lambda)} a_2(\lambda)!\ \dots\ . \end{aligned}$$

We now use the following well-known fact:

$$(*) \qquad \prod_{\lambda \in \wp(n)} 1^{a_1(\lambda)}\ 2^{a_2(\lambda)}\ \dots\ = \prod_{\lambda \in \wp(n)} a_1(\lambda)!\ a_2(\lambda)!\ \dots\ .$$

Let us indicate a short proof of $(*)$. We set

$$T := \{(\lambda, i, j) \in \wp(n) \times \mathbb{N} \times \mathbb{N} : j \leq a_i(\lambda)\}$$

and define a map $\tau\colon T \to T$ by $\tau(\lambda, i, j) := (\mu, j, i)$ where $\mu \in \wp(n)$ is obtained from λ by replacing j parts equal to i by i parts equal to j. It is immediate that $\tau(\lambda, i, j) \in T$ and that $\tau^2 = \mathrm{id}_T$; in particular, τ is a bijection. Hence

$$\prod_{\lambda\in\wp(n)} 1^{a_1(\lambda)}\ 2^{a_2(\lambda)}\ \ldots = \prod_{(\lambda,i,j)\in T} i = \prod_{(\lambda,i,j)\in T} j = \prod_{\lambda\in\wp(n)} a_1(\lambda)!\ a_2(\lambda)!\ \ldots\ ,$$

and $(*)$ is proved. It follows that

$$\det X_n = \prod_{\lambda\in\wp(n)} a_1(\lambda)!\ a_2(\lambda)!\ \ldots = \det Y_n \neq 0$$

which finishes the proof. □

We can use the result above in order to characterize the group of virtual characters $\mathcal{X}(S_n)$ as a subgroup of the group of class functions $\mathcal{Z}(S_n)$ by a system of congruences.

Corollary 4. *A class function $\alpha : S_n \to \mathbb{Z}$ is a virtual character if and only if*

$$\sum_{\Gamma\leq\Delta} \mu(\Gamma,\Delta)\ \alpha(x_\Delta) \equiv 0 \pmod{|N_{S_n}(S_\Gamma) : S_\Gamma|},$$

for $\Gamma \in \mathcal{P}(n)$.

Proof. Let A be the subgroup of $\mathcal{Z}(S_n)$ consisting of all $\alpha \in \mathcal{Z}(S_n)$ satisfying the congruences above. Theorem 1 implies that A contains $\mathcal{X}(S_n)$. On the other hand, A has index

$$\prod_{\gamma\in\wp(n)} |N_{S_n}(S_\gamma) : S_\gamma| = \prod_{\gamma\in\wp(n)} a_1(\gamma)!\ a_2(\gamma)!\ \ldots$$

in $\mathcal{Z}(S_n)$. In the proof of Proposition 3, we have seen that this is also the index of $\mathcal{X}(S_n)$ in $\mathcal{Z}(S_n)$. So the result follows. □

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