

Monomial Resolutions of Trivial Source Modules*

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Abstract

For trivial source modules admitting a filtration related to their generalized Brauer constructions, a resolution in terms of monomial modules is given which provides a categorical interpretation for the canonical induction formula applied to these modules.

Introduction

This article should be considered as a first attempt to define the notion of a monomial resolution of a trivial source module for a finite group G over a field F of characteristic $p > 0$ such that it generalizes the definition of a monomial resolution of a complex representation as defined in [Bo97].

There are several motivations for doing this. One is the connection to canonical induction formulas. The monomial resolution as defined in [Bo97] has the property that it lifts the canonical Brauer induction formula for complex characters, a map between Grothendieck groups, to a functor between suitable categories. Since there is also a canonical induction formula for trivial source modules, it would be nice to have a similar categorical interpretation of it. Another motivation is the occurrence of trivial source modules in so-called Rickard complexes which conjecturally exist and provide equivalences between the homotopy categories of blocks of group algebras in specific situations. These equivalences have been conjectured by Broué, cf. [Br90], [Br95]. One might hope that replacing trivial source modules via their resolutions by objects of a finer structure could help to construct such chain complexes; the more so if the morphisms between the finer objects can be constructed inductively layer by layer going down a filtration, as it is the case with finite G -equivariant line bundles, a category introduced and studied in [Bo97].

Our original goal of generalizing the results in [Bo97] in all aspects is not achieved by the construction we present in this article. The reason could be that the categorical framework we chose is not yet the ‘right’ one. There are still other possibilities to be examined in the future, but they will build on results established here. One aspect that is not satisfactory is the fact that not every trivial source module has a monomial resolution. We are able to characterize those which admit a monomial resolution as the ones which admit a certain filtration which we call a Brauer filtration. A similar (maybe equivalent) type of filtration was considered in Bouc’s work on resolutions of Mackey functors. The second flaw of our construction is the missing functoriality. Not every homomorphism between trivial source modules can be extended to a

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chain map between given monomial resolutions. But if it can be extended, then the extension is unique up to homotopy. Also, we cannot exclude the possibility that two different Brauer filtrations of one trivial source module can give rise to two not homotopy equivalent monomial resolutions. Thus under a categorical point of view it is more natural to work with objects consisting of a trivial source module together with a Brauer filtration on it. In fact, on such a category we can consider our construction as a functor into the homotopy category of finite G -equivariant line bundles or into the homotopy category of Brauer sheaves (see Section 2 for a definition).

The article is arranged in five sections as follows. In Section 1 we recall the generalized Brauer construction $\overline{V}(H, \varphi)$ for an FG -module V , a subgroup $H \leq G$, and a homomorphism $\varphi: H \rightarrow F^\times$ from [BK00]. This construction generalizes the well known Brauer construction in the case where H is a p -subgroup. The dimensions of the F -spaces $\overline{V}(H, \varphi)$ yield via a Möbius inversion formula the canonical induction formula of a trivial source FG -module V . These spaces (for fixed V and varying (H, φ)) form a rigid algebraic object by the existence of conjugation, restriction, corestriction, and transitivity maps between them. We investigate the properties of these maps and their interplay. In Section 2, by axiomatizing the situation studied in Section 1, we define the notion of a *Brauer sheaf*, whose objects are families of F -vector spaces, indexed by pairs (H, φ) together with conjugation, restriction and transitivity maps satisfying a set of natural compatibilities. One important feature of the restriction maps is that they are only required between spaces indexed by pairs $(I, \psi) \leq (H, \varphi)$ if p does not divide $[H : I]$. In fact, there is no natural map $\overline{V}(H, \varphi) \rightarrow \overline{V}(I, \psi)$ if p divides $[H : I]$. Besides the generalized Brauer constructions which give rise to Brauer sheaves, there exists another natural source of examples, namely finite G -equivariant line bundles. The Grothendieck group of the category ${}_{FG}\text{mon}$ of these line bundles is precisely the range of the canonical induction formula of trivial source modules. We define a monomial resolution of a trivial source module V in Section 3 as a chain complex M_* in ${}_{FG}\text{mon}$ using the embeddings of V and M_* into the category of Brauer sheaves in such a way that the Lefschetz element of M_* in the Grothendieck group of ${}_{FG}\text{mon}$ has to coincide with the canonical induction formula of V . It turns out that not every trivial source module can have such a monomial resolution, the obstruction being the (im)possibility of extending the restriction maps between the generalized Brauer construction to *all* pairs $(I, \psi) \leq (H, \varphi)$. This is equivalent to admitting a filtration which we call *Brauer filtration*. If V has a Brauer filtration we can consider it as a sheaf in yet another category $\text{Sh}_F(G)$ of sheaves which we already used in [Bo97]. In Section 4 we study more generally monomial resolutions of sheaves in the category $\text{Sh}_F(G)$. If the sheaf structure comes from a Brauer filtration of an FG -module V , such a resolution automatically yields a monomial resolution in the original sense, and every monomial resolution in the original sense has to come this way. This allows us to characterize the modules V which have (finite,) (locally split) monomial resolutions. Section 5 concludes the article with studying under which conditions FG -homomorphisms can be extended to chain maps between monomial resolutions, with giving examples, and with stating a list of questions which remain open.

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Notation Throughout this article F denotes a field of characteristic $p > 0$ and F^\times the group of units of F . By G we will denote a finite group and by \widehat{G} the multiplicative group $\text{Hom}(G, F^\times)$. We fix an embedding of the group of roots of unity in F into the group of roots of unity in \mathbb{C} . Brauer characters of FG -modules will always be defined using this embedding. If H is a subgroup of G and $g \in G$ we set ${}^gH := gHg^{-1}$. For any partially ordered set we use the symbol ‘ $<$ ’ for ‘strictly smaller’ and ‘ \leq ’ for ‘smaller or equal’. For a ring R , ${}_R\text{mod}$ denotes the category of finitely generated left R -modules, and for $X, Y \in {}_R\text{mod}$ we use the notation $X \mid Y$ to express that X is isomorphic to a direct summand of Y .

1 The generalized Brauer construction and some natural maps

1.1 We recall from [BK00] the definition and some properties of the generalized Brauer construction.

For a finite group G , let $\mathcal{M} = \mathcal{M}_F(G)$ denote the set of pairs (H, φ) , where $H \leq G$ and $\varphi \in \widehat{H}$. For any $(H, \varphi) \in \mathcal{M}$ we denote by F_φ the FH -module with underlying additive group F and H -action $h\alpha := \varphi(h)\alpha$, for $h \in H$, $\alpha \in F$. Note that $\varphi \mapsto F_\varphi$ defines a bijection between \widehat{G} and the set of isomorphism classes of one-dimensional FG -modules. If $\varphi = 1$, the trivial homomorphism, we simply write F instead of F_1 . The set \mathcal{M} is a poset if we set $(I, \psi) \leq (H, \varphi)$ if $I \leq H$ and $\psi = \varphi|_I$. Moreover, G acts on \mathcal{M} by conjugation. More precisely, for $g \in G$ and $(H, \varphi) \in \mathcal{M}$, we set ${}^g(H, \varphi) := ({}^gH, {}^g\varphi)$ with ${}^g\varphi(ghg^{-1}) = \varphi(h)$ for all $h \in H$. Note that this action respects the partial order on \mathcal{M} .

For $V \in {}_{FG}\text{mod}$ and $(H, \varphi) \in \mathcal{M}$, the (H, φ) -fixed points $V^{(H, \varphi)}$ are defined by

$$V^{(H, \varphi)} := \{v \in V \mid hv = \varphi(h)v \text{ for all } h \in H\}.$$

This is an $FN_G(H, \varphi)$ -submodule of V , where $N_G(H, \varphi)$ denotes the stabilizer of (H, φ) in G . If $f: V \rightarrow W$ is an FG -module homomorphism, then we denote by $f^{(H, \varphi)}: V^{(H, \varphi)} \rightarrow W^{(H, \varphi)}$ the restriction of f to $V^{(H, \varphi)}$. This way we obtain a functor

$$?^{(H, \varphi)}: {}_{FG}\text{mod} \rightarrow {}_{FN_G(H, \varphi)}\text{mod}.$$

For $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} , there is a relative trace map

$$\text{tr}_{(I, \psi)}^{(H, \varphi)}: V^{(I, \psi)} \longrightarrow V^{(H, \varphi)}, \quad v \mapsto \sum_{h \in H/I} \varphi(h^{-1})hv.$$

The above summation runs over a set of representatives for the cosets H/I and is independent of the choice of such a set. The *generalized Brauer construction* $\overline{V}(H, \varphi)$ of V with respect to $(H, \varphi) \in \mathcal{M}$ is defined as

$$\overline{V}(H, \varphi) := V^{(H, \varphi)} / \sum_{\substack{(I, \psi) < (H, \varphi) \\ p \mid [H:I]}} \text{tr}_{(I, \psi)}^{(H, \varphi)}(V^{(I, \psi)})$$

and the canonical surjection

$$\text{Br}_{(H, \varphi)}^V: V^{(H, \varphi)} \longrightarrow \overline{V}(H, \varphi)$$

is called the *Brauer map* with respect to (H, φ) . If there is no risk of confusion, we will also write $\text{Br}_{(H, \varphi)}$ instead of $\text{Br}_{(H, \varphi)}^V$ and, for $v \in V^{(H, \varphi)}$ we will just write \overline{v} instead of $\text{Br}_{(H, \varphi)}(v)$. The F -space $\overline{V}(H, \varphi)$ is again an $FN_G(H, \varphi)$ -module, since $\ker(\text{Br}_{(H, \varphi)})$ is an $FN_G(H, \varphi)$ -submodule of $V^{(H, \varphi)}$. For an FG -module homomorphism $f: V \rightarrow W$, there exists a unique $FN_G(H, \varphi)$ -module homomorphism $\overline{f}(H, \varphi): \overline{V}(H, \varphi) \rightarrow \overline{W}(H, \varphi)$ such that the diagram

$$\begin{array}{ccc} V^{(H, \varphi)} & \xrightarrow{f^{(H, \varphi)}} & W^{(H, \varphi)} \\ \text{Br}_{(H, \varphi)}^V \downarrow & & \downarrow \text{Br}_{(H, \varphi)}^W \\ \overline{V}(H, \varphi) & \xrightarrow{\overline{f}(H, \varphi)} & \overline{W}(H, \varphi) \end{array}$$

commutes, and the generalized Brauer construction becomes an F -functor, i.e., a functor which is F -linear on morphisms,

$$\overline{?}^{(H, \varphi)}: {}_{FG}\text{mod} \rightarrow {}_{FN_G(H, \varphi)}\text{mod}.$$

Note that if $(H, \varphi) \in \mathcal{M}$ with $\varphi = 1$, then $V^{(H, \varphi)}$ is the usual fixed point set V^H and $\text{tr}_{(I, \psi)}^{(H, \varphi)}$ is the usual trace map tr_I^H . Moreover, if P is a p -subgroup of G , then $\overline{V}(P, 1)$ coincides with the usual Brauer construction and we will write $\overline{V}(P)$ for it and Br_P^V instead of $\text{Br}_{(P, 1)}^V$. We will often refer to [BK00] for facts about the generalized Brauer construction.

In the following, we consider a trivial source FG -module V . This is an object $V \in {}_{FG}\text{mod}$ which is isomorphic to a direct summand of a permutation FG -module. We refer the reader to [Br85] and [Bo98a] for general facts on trivial source modules that will be used in this article. The category of trivial source FG -modules will be denoted by ${}_{FG}\text{triv}$. For $V \in {}_{FG}\text{mod}$ and $(H, \varphi) \in \mathcal{M}$ we denote by $m_V(H, \varphi)$ the multiplicity with which the FH -module F_φ occurs as a direct summand in $\text{Res}_H^G(V)$.

1.2 Lemma *Let $V \in {}_{FG}\text{triv}$, $(H, \varphi) \in \mathcal{M}$, and $N := N_G(H, \varphi)$.*

(a) *There exist FN -submodules X, Y of V such that*

$$\text{Res}_N^G(V) = X \oplus Y, \quad \text{Res}_H^N(X) \cong F_\varphi \oplus \cdots \oplus F_\varphi, \quad F_\varphi \nmid \text{Res}_H^N(Y).$$

(b) *One has $\dim_F \overline{V}(H, \varphi) = m_V(H, \varphi)$.*

(c) *If X and Y are as in (a), then $V^{(H, \varphi)} = X \oplus Y^{(H, \varphi)}$ as FN -modules and $Y^{(H, \varphi)} = \ker(\text{Br}_{(H, \varphi)}^V)$. In particular, $\text{Br}_{(H, \varphi)}^V$ is split surjective and induces an isomorphism $X \rightarrow \overline{V}(H, \varphi)$ of FN -modules, and $\overline{V}(H, \varphi) \in {}_{FN}\text{triv}$.*

Proof (a) This follows immediately from Theorem 3.3(iii) in [BK00].

(b) This is proved in Theorem 3.5 in [BK00].

(c) This follows from part (b) and the facts that the generalized Brauer construction $\overline{V}(H, \varphi)$ only depends on $\text{Res}_N^G(V)$ and is compatible with direct sums. \square

In the situation of the above lemma, a warning seems to be appropriate: Although $Y^{(H, \varphi)}$ is uniquely determined as the kernel of $\text{Br}_{(H, \varphi)}^V$, neither X nor Y are unique in general.

For the rest of this section, let G be a finite group and let $V \in {}_{FG}\text{mod}$. We will construct some natural maps between the generalized Brauer constructions of V .

1.3 Lemma *Let $g \in G$ and $(H, \varphi) \in \mathcal{M}$.*

(i) *There exists a unique F -linear map*

$$\text{con}_{(H, \varphi)}^g = {}_V\text{con}_{(H, \varphi)}^g : \overline{V}(H, \varphi) \longrightarrow \overline{V}^g(H, \varphi)$$

such that the diagram

$$\begin{array}{ccc} V^{(H, \varphi)} & \xrightarrow{g} & V^g(H, \varphi) \\ \text{Br}_{(H, \varphi)} \downarrow & & \downarrow \text{Br}_{g(H, \varphi)} \\ \overline{V}(H, \varphi) & \xrightarrow{\text{con}_{(H, \varphi)}^g} & \overline{V}^g(H, \varphi) \end{array}$$

commutes. Moreover, $\text{con}_{(H, \varphi)}^g$ is a bijection.

(ii) *Triviality: For $h \in H$ and $v \in \overline{V}(H, \varphi)$ one has $\text{con}_{(H, \varphi)}^h(v) = \varphi(h)v$.*

(iii) *Transitivity: For $g' \in G$ one has $\text{con}_{(H, \varphi)}^{g'g} = \text{con}_{g'(H, \varphi)}^{g'} \circ \text{con}_{(H, \varphi)}^g$.*

(iv) *Functoriality: For each $W \in {}_{FG}\text{mod}$ and each $f \in \text{Hom}_{FG}(V, W)$, the diagram*

$$\begin{array}{ccc}
\overline{V}(H, \varphi) & \xrightarrow{\overline{f}(H, \varphi)} & \overline{W}(H, \varphi) \\
\downarrow v \text{con}_{(H, \varphi)}^g & & \downarrow w \text{con}_{(H, \varphi)}^g \\
\overline{V}^g(H, \varphi) & \xrightarrow{\overline{f}^g(H, \varphi)} & \overline{W}^g(H, \varphi)
\end{array}$$

commutes.

Proof Straight forward. □

We call the maps $\text{con}_{(H, \varphi)}^g$ the *conjugation maps* of V .

1.4 Remark By (ii) and (iii) in Lemma 1.3, $\overline{V}(H, \varphi)$ becomes an $FN_G(H, \varphi)$ -module with $g\overline{v} = \text{con}_{(H, \varphi)}^g(\overline{v})$ for $g \in N_G(H, \varphi)$, $v \in V^{(H, \varphi)}$. This is the same $FN_G(H, \varphi)$ -module structure as the one introduced in 1.1. Note that $\ker(\varphi)$ and in particular H' , the commutator subgroup of H , acts trivially on $\overline{V}(H, \varphi)$.

1.5 Lemma Let $(I, \psi) \leq (H, \varphi)$ be in \mathcal{M} such that $p \nmid [H : I]$.

(i) There exists a unique F -linear map

$$\text{res}_{(I, \psi)}^{(H, \varphi)} = v \text{res}_{(I, \psi)}^{(H, \varphi)} : \overline{V}(H, \varphi) \longrightarrow \overline{V}(I, \psi)$$

such that the diagram

$$\begin{array}{ccc}
V^{(H, \varphi)} & \xrightarrow{\quad} & V^{(I, \psi)} \\
\downarrow \text{Br}_{(H, \varphi)} & & \downarrow \text{Br}_{(I, \psi)} \\
\overline{V}(H, \varphi) & \xrightarrow{\text{res}_{(I, \psi)}^{(H, \varphi)}} & \overline{V}(I, \psi)
\end{array}$$

commutes.

(ii) Triviality: $\text{res}_{(H, \varphi)}^{(H, \varphi)} = \text{id}_{\overline{V}(H, \varphi)}$.

(iii) G -equivariance: For $g \in G$ one has $\text{con}_{(I, \psi)}^g \circ \text{res}_{(I, \psi)}^{(H, \varphi)} = \text{res}_{(I, \psi)}^g \circ \text{con}_{(H, \varphi)}^g$. In particular, $\text{res}_{(I, \psi)}^{(H, \varphi)}$ is $F[N_G(H, \varphi) \cap N_G(I, \psi)]$ -linear.

(iv) Transitivity: For $(J, \lambda) \leq (I, \psi)$ in \mathcal{M} with $p \nmid [I : J]$ one has $\text{res}_{(J, \lambda)}^{(I, \psi)} \circ \text{res}_{(I, \psi)}^{(H, \varphi)} = \text{res}_{(J, \lambda)}^{(H, \varphi)}$.

(v) Functoriality: For each $W \in {}_{FG}\text{mod}$ and each $f \in \text{Hom}_{FG}(V, W)$, the diagram

$$\begin{array}{ccc}
\overline{V}(H, \varphi) & \xrightarrow{\overline{f}(H, \varphi)} & \overline{W}(H, \varphi) \\
\downarrow v \text{res}_{(I, \psi)}^{(H, \varphi)} & & \downarrow w \text{res}_{(I, \psi)}^{(H, \varphi)} \\
\overline{V}(I, \psi) & \xrightarrow{\overline{f}(I, \psi)} & \overline{W}(I, \psi)
\end{array}$$

commutes.

Proof (i) It suffices to show that $\ker(\mathrm{Br}_{(H,\varphi)}) \subseteq \ker(\mathrm{Br}_{(I,\psi)})$. Recall that

$$\ker(\mathrm{Br}_{(H,\varphi)}) = \sum_{\substack{(U,\mu) \leq (H,\varphi) \\ p \nmid [H:U]}} \mathrm{tr}_{(U,\mu)}^{(H,\varphi)}(V^{(U,\mu)}).$$

So let $(U,\mu) < (H,\varphi)$ in \mathcal{M} with $p \mid [H:U]$ and let $v \in V^{(U,\mu)}$. Then, by the Mackey formula [BK00, Lemma 1.3(e)], one has

$$\mathrm{tr}_{(U,\mu)}^{(H,\varphi)}(v) = \sum_{h \in I \backslash H/U} \varphi(h^{-1}) \mathrm{tr}_{(I \cap {}^h U, \psi)}^{(I,\psi)}(hv).$$

For $h \in H$ one has $p \mid [H:U] = [H: {}^h U] \mid [H: I \cap {}^h U]$. Since $p \nmid [H:I]$, this implies $p \mid [I: I \cap {}^h U]$ and $\mathrm{tr}_{(U,\mu)}^{(H,\varphi)}(v) \in \ker(\mathrm{Br}_{(I,\psi)})$.

(ii)–(v) follow immediately from the definition of $\mathrm{res}_{(I,\psi)}^{(H,\varphi)}$. \square

We call the maps ${}_V \mathrm{res}_{(I,\psi)}^{(H,\varphi)}$ the *restriction maps* of V .

Similar to properties of cohomology groups one can reconstruct $\overline{V}(H,\varphi)$ as the set of stable elements in $\overline{V}(P,1)$, where P is a Sylow p -subgroup of H . For the proof of this result we will need the following lemma.

1.6 Lemma *Let $(I,\psi) \leq (H,\varphi)$ be in \mathcal{M} such that $p \nmid [H:I]$.*

(i) *There exists a unique F -linear map*

$${}_V \mathrm{cor}_{(I,\psi)}^{(H,\varphi)} = \mathrm{cor}_{(I,\psi)}^{(H,\varphi)} : \overline{V}(I,\psi) \longrightarrow \overline{V}(H,\varphi)$$

such that the diagram

$$\begin{array}{ccc} \overline{V}(I,\psi) & \xrightarrow{\mathrm{tr}_{(I,\psi)}^{(H,\varphi)}} & \overline{V}(H,\varphi) \\ \mathrm{Br}_{(I,\psi)} \downarrow & & \downarrow \mathrm{Br}_{(H,\varphi)} \\ \overline{V}(I,\psi) & \xrightarrow{\mathrm{cor}_{(I,\psi)}^{(H,\varphi)}} & \overline{V}(H,\varphi) \end{array}$$

commutes.

(ii) Triviality: $\mathrm{cor}_{(H,\varphi)}^{(H,\varphi)} = \mathrm{id}_{\overline{V}(H,\varphi)}$.

(iii) G -equivariance: For $g \in G$ one has $\mathrm{con}_{(H,\varphi)}^g \circ \mathrm{cor}_{(I,\psi)}^{(H,\varphi)} = \mathrm{cor}_{g(I,\psi)}^{g(H,\varphi)} \circ \mathrm{con}_{(I,\psi)}^g$. In particular, $\mathrm{cor}_{(I,\psi)}^{(H,\varphi)}$ is $F[N_G(I,\psi) \cap N_G(H,\varphi)]$ -linear.

(iv) Transitivity: For $(J,\lambda) \leq (I,\psi)$ in \mathcal{M} with $p \nmid [I:J]$, one has $\mathrm{cor}_{(I,\psi)}^{(H,\varphi)} \circ \mathrm{cor}_{(J,\lambda)}^{(I,\psi)} = \mathrm{cor}_{(J,\lambda)}^{(H,\varphi)}$.

(v) Cohomological property: $\mathrm{cor}_{(I,\psi)}^{(H,\varphi)} \circ \mathrm{res}_{(I,\psi)}^{(H,\varphi)} = [H:I] \cdot \mathrm{id}_{\overline{V}(H,\varphi)}$. In particular, $\mathrm{res}_{(I,\psi)}^{(H,\varphi)}$ is injective and $\mathrm{cor}_{(I,\psi)}^{(H,\varphi)}$ is surjective.

(vi) Mackey formula: For $(J,\lambda) \leq (H,\varphi)$ in \mathcal{M} with $p \nmid [H:J]$, one has

$$\mathrm{res}_{(J,\lambda)}^{(H,\varphi)} \circ \mathrm{cor}_{(I,\psi)}^{(H,\varphi)} = \sum_{\substack{h \in J \backslash H/I \\ p \nmid [J:J \cap {}^h I]}} \varphi(h^{-1}) \mathrm{cor}_{(J \cap {}^h I, \lambda)}^{(J,\lambda)} \circ \mathrm{res}_{(J \cap {}^h I, {}^h \psi)}^{({}^h I, {}^h \psi)} \circ \mathrm{con}_{(I,\psi)}^h.$$

(Note that $\lambda|_{J \cap {}^h I} = {}^h \psi|_{J \cap {}^h I}$, since both homomorphisms are restrictions of φ , and that $p \nmid [J : J \cap {}^h I]$ implies $p \nmid [{}^h I : J \cap {}^h I]$, since $p \nmid [H : J]$.)

(vii) Functoriality: For each $W \in {}_{FG}\text{Mod}$ and each $f \in \text{Hom}_{FG}(V, W)$, the diagram

$$\begin{array}{ccc} \overline{V}(I, \psi) & \xrightarrow{\overline{f}(I, \psi)} & \overline{W}(I, \psi) \\ \downarrow \text{v cor}_{(I, \psi)}^{(H, \varphi)} & & \downarrow \text{w cor}_{(I, \psi)}^{(H, \varphi)} \\ \overline{V}(H, \varphi) & \xrightarrow{\overline{f}(H, \varphi)} & \overline{W}(H, \varphi) \end{array}$$

commutes.

Proof (i) It is easy to see that $\text{tr}_{(I, \psi)}^{(H, \varphi)}(\ker(\text{Br}_{(I, \psi)})) \subseteq \ker(\text{Br}_{(H, \varphi)})$, since the trace maps are transitive (cf. [BK00, Lemma 1.3(b)]), and the assertion in (i) follows.

(ii), (iii), (iv), (v) and (vii) follow from the respective properties of the trace maps (cf. [BK00, Lemma 1.3(a), (b), (d)]).

(vi) Let $v \in V^{(I, \psi)}$ and set $\overline{v} := \text{Br}_{(I, \psi)}(v) \in \overline{V}(I, \psi)$. Then

$$\begin{aligned} \text{res}_{(J, \lambda)}^{(H, \varphi)}\left(\text{cor}_{(I, \psi)}^{(H, \varphi)}(\overline{v})\right) &= \text{res}_{(J, \lambda)}^{(H, \varphi)}\left(\text{Br}_{(H, \varphi)}\left(\text{tr}_{(I, \psi)}^{(H, \varphi)}(v)\right)\right) \\ &= \text{Br}_{(J, \lambda)}\left(\text{tr}_{(I, \psi)}^{(H, \varphi)}(v)\right) = \sum_{h \in J \backslash H/I} \varphi(h^{-1}) \text{Br}_{(J, \lambda)}\left(\text{tr}_{(J \cap {}^h I, \lambda)}^{(J, \lambda)}(hv)\right) \\ &= \sum_{\substack{h \in J \backslash H/I \\ p \nmid [J : J \cap {}^h I]}} \varphi(h^{-1}) \text{cor}_{(J \cap {}^h I, \lambda)}^{(J, \lambda)}\left(\text{Br}_{(J \cap {}^h I, \lambda)}(hv)\right) \\ &= \sum_{\substack{h \in J \backslash H/I \\ p \nmid [J : J \cap {}^h I]}} \varphi(h^{-1}) \text{cor}_{(J \cap {}^h I, \lambda)}^{(J, \lambda)}\left(\text{res}_{(J \cap {}^h I, {}^h \psi)}^{({}^h I, {}^h \psi)}\left(\text{Br}_{({}^h I, {}^h \psi)}(hv)\right)\right) \\ &= \sum_{\substack{h \in J \backslash H/I \\ p \nmid [J : J \cap {}^h I]}} \varphi(h^{-1}) \text{cor}_{(J \cap {}^h I, \lambda)}^{(J, \lambda)}\left(\text{res}_{(J \cap {}^h I, {}^h \psi)}^{({}^h I, {}^h \psi)}\left(\text{con}_{(I, \psi)}^h(\overline{v})\right)\right), \end{aligned}$$

where we obtained the fourth term from the third by using the Mackey formula for traces, cf. [BK00, Lemma 1.3(e)]. \square

We call the maps $\text{v cor}_{(I, \psi)}^{(H, \varphi)}$ the *corestriction maps* of V . They exist as defined in part (i) of Lemma 1.6 even without the requirement that $p \nmid [H : I]$. But if $p \mid [H : I]$, then clearly $\text{cor}_{(I, \psi)}^{(H, \varphi)} = 0$.

1.7 Definition Let $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} be such that $p \nmid [H : I]$. An element $\overline{v} \in \overline{V}(I, \psi)$ is called (H, φ) -stable if

$$\text{res}_{(I \cap {}^h I, {}^h \psi)}^{({}^h I, {}^h \psi)}\left(\text{con}_{(I, \psi)}^h(\overline{v})\right) = \varphi(h) \text{res}_{(I \cap {}^h I, \psi)}^{(I, \psi)}(\overline{v}),$$

for all $h \in H$ with $p \nmid [I : I \cap {}^h I]$ ($= [{}^h I : I \cap {}^h I]$). Note that ${}^h \psi|_{I \cap {}^h I} = {}^h \varphi|_{I \cap {}^h I} = \varphi|_{I \cap {}^h I} = \psi|_{I \cap {}^h I}$ for all $h \in H$.

1.8 Remark If $(H, \varphi) \in \mathcal{M}$ and P is a Sylow p -subgroup of H , then an element $\overline{v} \in \overline{V}(P, 1)$ is (H, φ) -stable if and only if $\overline{v} \in \overline{V}(P, 1)^{(N_H(P), \varphi)}$.

1.9 Proposition Let $(I, \psi) \leq (H, \varphi)$ be in \mathcal{M} such that $p \nmid [H : I]$. Then

$$\text{res}_{(I, \psi)}^{(H, \varphi)}(\overline{V}(H, \varphi)) = \{\overline{v} \in \overline{V}(I, \psi) \mid \overline{v} \text{ is } (H, \varphi)\text{-stable}\}.$$

In particular, if P is a Sylow p -subgroup of H , then

$$\text{res}_{(P, 1)}^{(H, \varphi)}(\overline{V}(H, \varphi)) = \overline{V}(P, 1)^{(N_H(P), \varphi)}.$$

Proof Suppose first that $v \in V^{(H, \varphi)}$, set $\overline{v} := \text{Br}_{(H, \varphi)}(v) \in \overline{V}(H, \varphi)$, and assume that $h \in H$ with $p \nmid [I : I \cap {}^h I]$. Then

$$\begin{aligned} & \text{res}_{(I \cap {}^h I, {}^h \psi)}^{(hI, {}^h \psi)}(\text{con}_{(I, \psi)}^h(\text{res}_{(I, \psi)}^{(H, \varphi)}(\overline{v}))) \\ &= \text{res}_{(I \cap {}^h I, {}^h \psi)}^{(hI, {}^h \psi)}(\text{res}_{(hI, {}^h \psi)}^{(H, \varphi)}(\text{con}_{(H, \varphi)}^h(\overline{v}))) \\ &= \text{res}_{(I \cap {}^h I, {}^h \psi)}^{(H, \varphi)}(\varphi(h)\overline{v}) \\ &= \varphi(h)\text{res}_{(I \cap {}^h I, \psi)}^{(I, \psi)}(\text{res}_{(I, \psi)}^{(H, \varphi)}(\overline{v})), \end{aligned}$$

and $\text{res}_{(I, \psi)}^{(H, \varphi)}(\overline{v})$ is (H, φ) -stable. Conversely, assume that $v \in V^{(I, \psi)}$ and that $\overline{v} := \text{Br}_{(I, \psi)}(v)$ is (H, φ) -stable. Then

$$\begin{aligned} \text{res}_{(I, \psi)}^{(H, \varphi)}(\text{cor}_{(I, \psi)}^{(H, \varphi)}(\overline{v})) &= \sum_{\substack{h \in I \backslash H/I \\ p \nmid [I : I \cap {}^h I]}} \varphi(h^{-1})\text{cor}_{(I \cap {}^h I, \psi)}^{(I, \psi)}(\text{res}_{(I \cap {}^h I, {}^h \psi)}^{(hI, {}^h \psi)}(\text{con}_{(I, \psi)}^h(\overline{v}))) \\ &= \sum_{\substack{h \in I \backslash H/I \\ p \nmid [I : I \cap {}^h I]}} \varphi(h^{-1})\text{cor}_{(I \cap {}^h I, \psi)}^{(I, \psi)}(\varphi(h)\text{res}_{(I \cap {}^h I, \psi)}^{(I, \psi)}(\overline{v})) \\ &= \sum_{\substack{h \in I \backslash H/I \\ p \nmid [I : I \cap {}^h I]}} [I : I \cap {}^h I] \cdot \overline{v} = \sum_{h \in I \backslash H/I} [I : I \cap {}^h I] \cdot \overline{v} = [H : I] \cdot \overline{v}, \end{aligned}$$

and \overline{v} is in the image of $\text{res}_{(I, \psi)}^{(H, \varphi)}$. □

For $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} with $H \leq N_G(I, \psi)$ we will write $(I, \psi) \trianglelefteq (H, \varphi)$. The following generalizes a well-known construction in the case that I and H are p -subgroups of G .

1.10 Lemma Let $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} .

(i) There exists a unique F -linear map

$$\text{ts}_{(I, \psi)}^{(H, \varphi)} = \text{vts}_{(I, \psi)}^{(H, \varphi)} : \overline{V}(H, \varphi) \longrightarrow \overline{\overline{V}(I, \psi)}(H, \varphi)$$

such that the diagram

$$\begin{array}{ccccc} V^{(I, \psi)} & \longleftarrow & (V^{(I, \psi)})^{(H, \varphi)} = V^{(H, \varphi)} & \xrightarrow{\text{Br}_{(H, \varphi)}^V} & \overline{V}(H, \varphi) \\ \text{Br}_{(I, \psi)}^V \downarrow & & (\text{Br}_{(I, \psi)}^V)^{(H, \varphi)} \downarrow & & \downarrow \text{ts}_{(I, \psi)}^{(H, \varphi)} \\ \overline{V}(I, \psi) & \longleftarrow & \overline{V}(I, \psi)^{(H, \varphi)} & \xrightarrow{\text{Br}_{(H, \varphi)}^{\overline{V}(I, \psi)}} & \overline{\overline{V}(I, \psi)}(H, \varphi) \end{array}$$

commutes.

(ii) G -equivariance: For $g \in G$ the left hand square in the diagram

$$\begin{array}{ccccc} \overline{V}(H, \varphi) & \xrightarrow{\text{ts}_{(I, \psi)}^{(H, \varphi)}} & \overline{\overline{V}(I, \psi)}(H, \varphi) & \xleftarrow{\text{Br}_{(H, \varphi)}^{\overline{V}(I, \psi)}} & \overline{V}(I, \psi)^{(H, \varphi)} \\ \downarrow \text{con}_{(H, \varphi)}^g & & \downarrow & & \downarrow \\ \overline{V}^g(H, \varphi) & \xrightarrow{\text{ts}_{(I, \psi)}^g} & \overline{\overline{V}^g(I, \psi)}^g(H, \varphi) & \xleftarrow{\text{Br}_{(H, \varphi)}^{\overline{V}^g(I, \psi)}} & \overline{V}^g(I, \psi)^g(H, \varphi) \end{array}$$

commutes, where the middle vertical map is induced by the commutativity of the right hand square and the right vertical map is the restriction of the conjugation map $\text{con}_{(I, \psi)}^g : \overline{V}(I, \psi) \rightarrow \overline{V}^g(I, \psi)$.

(iii) Transitivity: If also $(J, \lambda) \trianglelefteq (I, \psi)$ in \mathcal{M} with $(J, \lambda) \trianglelefteq (H, \varphi)$, then the diagram

$$\begin{array}{ccc} \overline{V}(H, \varphi) & \xrightarrow{\text{vts}_{(I, \psi)}^{(H, \varphi)}} & \overline{\overline{V}(I, \psi)}(H, \varphi) \\ \text{vts}_{(J, \lambda)}^{(H, \varphi)} \downarrow & & \downarrow \text{vts}_{(J, \lambda)}^{\overline{V}(I, \psi)} \\ \overline{\overline{V}(J, \lambda)}(H, \varphi) & \xrightarrow{\overline{V}(J, \lambda) \text{ts}_{(I, \psi)}^{(H, \varphi)}} & \overline{\overline{\overline{V}(J, \lambda)}(I, \psi)}(H, \varphi) \end{array}$$

commutes.

(iv) Compatibility with restrictions: If

$$\begin{array}{ccc} (H, \varphi) & \geq & (J, \lambda) \\ \nabla \downarrow & & \nabla \downarrow \\ (I, \psi) & \geq & (K, \mu) \end{array}$$

are elements in \mathcal{M} such that $p \nmid [H : J]$ and $p \nmid [I : K]$, then the diagram

$$\begin{array}{ccc} \overline{V}(H, \varphi) & \xrightarrow{\text{vres}_{(J, \lambda)}^{(H, \varphi)}} & \overline{V}(J, \lambda) \\ \text{vts}_{(I, \psi)}^{(H, \varphi)} \downarrow & & \downarrow \text{vts}_{(K, \mu)}^{(J, \lambda)} \\ \overline{\overline{V}(I, \psi)}(H, \varphi) & \xrightarrow{\overline{V}(I, \psi) \text{res}_{(J, \lambda)}^{(H, \varphi)}} \overline{\overline{V}(I, \psi)}(J, \lambda) & \xrightarrow{\overline{\text{vres}_{(K, \mu)}^{(I, \psi)}}(J, \lambda)} \overline{\overline{V}(K, \mu)}(J, \lambda) \end{array}$$

commutes.

(v) Triviality: The map

$$\text{ts}_{(H, \varphi)}^{(H, \varphi)} : \overline{V}(H, \varphi) \longrightarrow \overline{\overline{V}(H, \varphi)}(H, \varphi) = \overline{V}(H, \varphi) / \{0\}$$

is the canonical isomorphism.

(vi) If $V \in {}_{FG}\text{triv}$, then $\text{ts}_{(I, \psi)}^{(H, \varphi)}$ is a bijection.

(vii) If H/I is a p' -group, then $\overline{\overline{V}(I, \psi)}(H, \varphi) \cong \overline{V}(I, \psi)^{(H, \varphi)}$. If additionally $V \in {}_{FG}\text{triv}$, then $\overline{V}(H, \varphi) \cong \overline{V}(I, \psi)^{(H, \varphi)}$.

(viii) Functoriality: For each $W \in {}_{FG}\text{mod}$ and each $f \in \text{Hom}_{FG}(V, W)$, the diagram

$$\begin{array}{ccc}
\overline{V}(H, \varphi) & \xrightarrow{\overline{f}(H, \varphi)} & \overline{W}(H, \varphi) \\
\downarrow \text{vts}_{(I, \psi)}^{(H, \varphi)} & & \downarrow \text{wts}_{(I, \psi)}^{(H, \varphi)} \\
\overline{\overline{V}(I, \psi)}(H, \varphi) & \xrightarrow{\overline{\overline{f}(I, \psi)}(H, \varphi)} & \overline{\overline{W}(I, \psi)}(H, \varphi)
\end{array}$$

commutes.

Proof (i) It suffices to show that

$$\text{Br}_{(H, \varphi)}^{\overline{V}(I, \psi)}(\text{Br}_{(I, \psi)}^V(\ker(\text{Br}_{(H, \varphi)}^V))) = 0.$$

Recall that $\ker(\text{Br}_{(H, \varphi)}^V)$ is generated by $\text{tr}_{(K, \mu)}^{(H, \varphi)}(V^{(K, \mu)})$, where $(K, \mu) < (H, \varphi)$ is such that $p \mid [H : K]$. So let (K, μ) have this property and let $v \in V^{(K, \mu)}$. Then

$$\text{tr}_{(K, \mu)}^{(H, \varphi)}(v) = \text{tr}_{(KI, \varphi)}^{(H, \varphi)}(\text{tr}_{(K, \mu)}^{(KI, \varphi)}(v)) = \text{tr}_{(KI, \varphi)}^{(H, \varphi)}(\text{tr}_{(K \cap I, \psi)}^{(I, \psi)}(v)),$$

and applying $\text{Br}_{(H, \varphi)}^{\overline{V}(I, \psi)} \circ \text{Br}_{(I, \psi)}^V$ to $w := \text{tr}_{(K, \mu)}^{(H, \varphi)}(v)$ gives

$$\text{Br}_{(H, \varphi)}^{\overline{V}(I, \psi)}(\text{Br}_{(I, \psi)}^V(w)) = \text{Br}_{(H, \varphi)}^{\overline{V}(I, \psi)}\left(\text{tr}_{(KI, \varphi)}^{(H, \varphi)}(\text{Br}_{(I, \psi)}^V(\text{tr}_{(K \cap I, \psi)}^{(I, \psi)}(v)))\right).$$

Now, $p \mid [H : K]$ implies that $p \mid [H : KI]$ or $p \mid [KI : K] = [I : K \cap I]$. In either case, the above expression is zero.

(ii) Let $v \in V^{(H, \varphi)}$ and set $u := \text{ts}_{(I, \psi)}^{(H, \varphi)}(\text{Br}_{(H, \varphi)}^V(v)) = \text{Br}_{(H, \varphi)}^{\overline{V}(I, \psi)}(\text{Br}_{(I, \psi)}^V(v))$. Then the middle vertical map sends u to

$$\text{Br}_{(H, \varphi)}^{\overline{V}^g(I, \psi)}(\text{con}_{(I, \psi)}^g(\text{Br}_{(I, \psi)}^V(v))) = \text{Br}_{(H, \varphi)}^{\overline{V}^g(I, \psi)}(\text{Br}_{(I, \psi)}^V(gv)).$$

On the other hand, $\text{con}_{(H, \varphi)}^g$ maps $\text{Br}_{(H, \varphi)}^V(v)$ to $\overline{w} := \text{Br}_{(H, \varphi)}^{\overline{V}^g(I, \psi)}(gv)$ and $\text{ts}_{(I, \psi)}^{g(H, \varphi)}$ maps \overline{w} also to $\text{Br}_{(H, \varphi)}^{\overline{V}^g(I, \psi)}(\text{Br}_{(I, \psi)}^V(gv))$.

(iii) Let $v \in V^{(H, \varphi)}$ and let u be as above. Then $\overline{\text{ts}_{(J, \lambda)}^{(I, \psi)}(H, \varphi)}$ maps u to

$$\text{Br}_{(H, \varphi)}^{\overline{\overline{V}(J, \lambda)}(I, \psi)}\left(\text{ts}_{(J, \lambda)}^{(I, \psi)}(\text{Br}_{(I, \psi)}^V(v))\right) = \text{Br}_{(H, \varphi)}^{\overline{\overline{V}(J, \lambda)}(I, \psi)}\left(\text{Br}_{(I, \psi)}^{\overline{V}(J, \lambda)}(\text{Br}_{(J, \lambda)}^V(v))\right).$$

On the other hand, $\text{ts}_{(J, \lambda)}^{(H, \varphi)}$ maps $\text{Br}_{(H, \varphi)}^V(v)$ to $x := \text{Br}_{(H, \varphi)}^{\overline{V}(J, \lambda)}(\text{Br}_{(J, \lambda)}^V(v))$ and $\overline{\text{v}_{(J, \lambda)}} \text{ts}_{(I, \psi)}^{(H, \varphi)}$ maps x also to the element on the right hand side of the above equation.

(iv) Let $v \in V^{(H, \varphi)}$ and let u be as above. Then $\overline{\text{v}_{(I, \psi)}} \text{res}_{(J, \lambda)}^{(H, \varphi)}$ maps u to $x := \text{Br}_{(J, \lambda)}^{\overline{V}(I, \psi)}(\text{Br}_{(I, \psi)}^V(v))$, and $\overline{\text{res}_{(K, \mu)}^{(I, \psi)}(J, \lambda)}$ maps x to

$$\text{Br}_{(J, \lambda)}^{\overline{V}(K, \mu)}\left(\text{res}_{(K, \mu)}^{(I, \psi)}(\text{Br}_{(I, \psi)}^V(v))\right) = \text{Br}_{(J, \lambda)}^{\overline{V}(K, \mu)}(\text{Br}_{(K, \mu)}^V(v)).$$

On the other hand, $\text{res}_{(J, \lambda)}^{(H, \varphi)}$ maps $\text{Br}_{(H, \varphi)}^V(v)$ to $y := \text{Br}_{(J, \lambda)}^V(v)$ and $\text{ts}_{(K, \mu)}^{(J, \lambda)}$ maps y also to $\text{Br}_{(J, \lambda)}^{\overline{V}(K, \mu)}(\text{Br}_{(K, \mu)}^V(v))$.

(v) By Lemma 1.2(c) $\ker(\text{Br}_{(H,\varphi)}^{\overline{V}(H,\varphi)}) = 0$, since $\overline{V}(H,\varphi)^{(H,\varphi)} = \overline{V}(H,\varphi)$. From that the assertion is obvious.

(vi) We write $\text{Res}_H^G(V) = U \oplus W$ with submodules U, W such that $U \cong F_\varphi \oplus \cdots \oplus F_\varphi$ and $F_\varphi \nmid W$. By Lemma 1.2(c), $\text{Br}_{(H,\varphi)}$ restricts to an isomorphism between U and $\overline{V}(H,\varphi)$. By the commutativity of the right hand square diagram in (i), it therefore suffices to show that $\text{Br}_{(H,\varphi)}^{\overline{V}(I,\psi)} \circ \text{Br}_{(I,\psi)}^V : U \rightarrow \overline{V}(I,\psi)(H,\varphi)$ is an isomorphism. By Lemma 1.2(i) we can write $W = X \oplus Y$ with submodules X, Y such that $\text{Res}_I^H(X) \cong F_\psi \oplus \cdots \oplus F_\psi$ and $F_\psi \nmid \text{Res}_I^H(Y)$. Then $\text{Br}_{(I,\psi)}^V$ induces an isomorphism between $U \oplus X$ and $\overline{V}(I,\psi)$, and $\text{Br}_{(H,\varphi)}^{\overline{V}(I,\psi)}$ induces an isomorphism between $\text{Br}_{(I,\psi)}^V(U)$ and $\overline{V}(I,\psi)(H,\varphi)$.

(vii) Let $(J,\lambda) < (H,\varphi)$ be such that $p \mid [H : J]$. Then

$$\text{tr}_{(J,\lambda)}^{(H,\varphi)}(\overline{v}) = \sum_{h \in I \setminus H/J} \varphi(h^{-1}) \text{tr}_{(I \cap {}^h J, \psi)}^{(I,\psi)}(h\overline{v}) = \sum_{h \in I \setminus H/J} \varphi(h^{-1}) [I : I \cap {}^h J] \cdot h\overline{v}$$

for $\overline{v} \in \overline{V}(I,\psi)^{(J,\lambda)}$ by [BK00, Lemma 1.3(e) and (d)], since $h\overline{v} \in \overline{V}(I,\psi) = \overline{V}(I,\psi)^{(I,\psi)}$. Moreover, since $p \mid [H : J]$ and $p \nmid [H : I]$, we have $p \mid [I : I \cap {}^h J]$. Thus,

$$\text{tr}_{(J,\lambda)}^{(H,\varphi)}(\overline{v}) = 0, \quad \ker(\text{Br}_{(H,\varphi)}^{\overline{V}(I,\psi)}) = 0,$$

and

$$\text{Br}_{(H,\varphi)}^{\overline{V}(I,\psi)} : \overline{V}(I,\psi)^{(H,\varphi)} \rightarrow \overline{V}(I,\psi)(H,\varphi)$$

is an isomorphism. The last statement now follows from part (vi).

(viii) Let $v \in V^{(H,\varphi)}$, $\overline{v} := \text{Br}_{(H,\varphi)}^V(v)$, and $w := f(v) \in W^{(H,\varphi)}$. Then

$$\begin{aligned} \text{wts}_{(I,\psi)}^{(H,\varphi)}(\overline{f}(H,\varphi)(\overline{v})) &= \text{wts}_{(I,\psi)}^{(H,\varphi)}(\text{Br}_{(H,\varphi)}^W(w)) \\ &= \text{Br}_{(H,\varphi)}^{\overline{W}(I,\psi)}(\text{Br}_{(I,\psi)}^W(w)) \end{aligned}$$

and also

$$\begin{aligned} \overline{f}(I,\psi)(H,\varphi)(\text{vts}_{(I,\psi)}^{(H,\varphi)}(\overline{v})) &= \overline{f}(I,\psi)(H,\varphi)\left(\text{Br}_{(H,\varphi)}^{\overline{V}(I,\psi)}(\text{Br}_{(I,\psi)}^V(v))\right) \\ &= \text{Br}_{(H,\varphi)}^{\overline{W}(I,\psi)}\left(\overline{f}(I,\psi)(\text{Br}_{(I,\psi)}^V(v))\right) \\ &= \text{Br}_{(H,\varphi)}^{\overline{W}(I,\psi)}(\text{Br}_{(I,\psi)}^W(w)). \end{aligned}$$

□

The maps $\text{vts}_{(I,\psi)}^{(H,\varphi)}$ will be called the *transitivity maps* of V . We did not state any compatibility of the corestrictions and the transitivity maps of V , because we will not need the corestrictions later. They only served to prove Proposition 1.9.

2 Brauer sheaves

In this section we axiomatize the existence of maps we have constructed between the generalized Brauer constructions with respect to the pairs $(H,\varphi) \in \mathcal{M} = \mathcal{M}_F(G)$. This results in the notion of a Brauer sheaf. Moreover, we will give another source of examples for Brauer sheaves, namely the category ${}_{FG}\text{mon}$ of finite G -equivariant line bundles over F .

2.1 Definition A *Brauer sheaf* $(\mathcal{F}, \mathcal{F}\text{con}, \mathcal{F}\text{res}, \mathcal{F}\text{ts})$ for G over F consists of the following data:

- (a) A family of F -vector spaces $\mathcal{F}_{(H,\varphi)}$, one for each $(H,\varphi) \in \mathcal{M}$.
- (b) A family of *conjugation maps*

$$\text{con}_{(H,\varphi)}^g = \mathcal{F}\text{con}_{(H,\varphi)}^g : \mathcal{F}_{(H,\varphi)} \longrightarrow \mathcal{F}_{g(H,\varphi)},$$

one for each $(H, \varphi) \in \mathcal{M}$ and each $g \in G$.

(c) A family of *restriction maps*

$$\text{res}_{(I, \psi)}^{(H, \varphi)} = \mathcal{F}\text{res}_{(I, \psi)}^{(H, \varphi)} : \mathcal{F}_{(H, \varphi)} \longrightarrow \mathcal{F}_{(I, \psi)},$$

one for each pair of elements $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} with $p \nmid [H : I]$.

(d) A family of *transitivity maps*

$$\text{ts}_{(I, \psi)}^{(H, \varphi)} = \mathcal{F}\text{ts}_{(I, \psi)}^{(H, \varphi)} : \mathcal{F}_{(H, \varphi)} \longrightarrow \overline{\mathcal{F}_{(I, \psi)}}(H, \varphi),$$

one for each pair of elements $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} .

These maps are subject to the following axioms:

(i) Transitivity of conjugations: For $g, g' \in G$ and $(H, \varphi) \in \mathcal{M}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_{(H, \varphi)} & \xrightarrow{\text{con}_{(H, \varphi)}^{g'g}} & \mathcal{F}_{g'(H, \varphi)} \\ & \searrow \text{con}_{(H, \varphi)}^g & \nearrow \text{con}_{(H, \varphi)}^{g'} \\ & \mathcal{F}_{g(H, \varphi)} & \end{array}$$

(ii) Triviality of conjugations: For $(H, \varphi) \in \mathcal{M}$, $h \in H$, and $v \in \mathcal{F}_{(H, \varphi)}$, one has $\text{con}_{(H, \varphi)}^h(v) = \varphi(h)v$. (By (i) and (ii), $\mathcal{F}_{(H, \varphi)}$ becomes an $FN_G(H, \varphi)$ -module on which $\ker(\varphi)$ acts trivially. In part (d) of this definition we view $\mathcal{F}_{(I, \psi)}$ as an $FN_G(I, \psi)$ -module so that the generalized Brauer construction $\overline{\mathcal{F}_{(I, \psi)}}(H, \varphi)$ is defined, since $H \leq N_G(I, \psi)$.)

(iii) Transitivity of restrictions: For $(J, \lambda) \leq (I, \psi) \leq (H, \varphi)$ in \mathcal{M} with $p \nmid [H : J]$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_{(H, \varphi)} & \xrightarrow{\text{res}_{(J, \lambda)}^{(H, \varphi)}} & \mathcal{F}_{(J, \lambda)} \\ & \searrow \text{res}_{(I, \psi)}^{(H, \varphi)} & \nearrow \text{res}_{(J, \lambda)}^{(I, \psi)} \\ & \mathcal{F}_{(I, \psi)} & \end{array}$$

(iv) Triviality of restrictions: For $(H, \varphi) \in \mathcal{M}$ one has $\text{res}_{(H, \varphi)}^{(H, \varphi)} = \text{id}_{\mathcal{F}_{(H, \varphi)}}$.

(v) G -equivariance of restrictions: For $g \in G$ and $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} with $p \nmid [H : I]$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_{(H, \varphi)} & \xrightarrow{\text{res}_{(I, \psi)}^{(H, \varphi)}} & \mathcal{F}_{(I, \psi)} \\ \downarrow \text{con}_{(H, \varphi)}^g & & \downarrow \text{con}_{(I, \psi)}^g \\ \mathcal{F}_{g(H, \varphi)} & \xrightarrow{\text{res}_{(I, \psi)}^g} & \mathcal{F}_{g(I, \psi)} \end{array}$$

(In particular, $\text{res}_{(I, \psi)}^{(H, \varphi)}$ is $F[N_G(H, \varphi) \cap N_G(I, \psi)]$ -linear.)

(vi) Transitivity of transitivity maps: For $(J, \lambda) \leq (I, \psi) \leq (H, \varphi)$ in \mathcal{M} with $(J, \lambda) \leq (H, \varphi)$, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}_{(H,\varphi)} & \xrightarrow{\text{ts}_{(I,\psi)}^{(H,\varphi)}} & \overline{\mathcal{F}_{(I,\psi)}}(H,\varphi) \\
\downarrow \text{ts}_{(J,\lambda)}^{(H,\varphi)} & & \downarrow \overline{\text{ts}_{(J,\lambda)}^{(I,\psi)}}(H,\varphi) \\
\overline{\mathcal{F}_{(J,\lambda)}}(H,\varphi) & \xrightarrow{\mathcal{F}_{(J,\lambda)} \text{ts}_{(I,\psi)}^{(H,\varphi)}} & \overline{\mathcal{F}_{(J,\lambda)}(I,\psi)}(H,\varphi)
\end{array}$$

(vii) Triviality of transitivity maps: For $(H, \varphi) \in \mathcal{M}$, the map

$$\text{ts}_{(H,\varphi)}^{(H,\varphi)} : \mathcal{F}_{(H,\varphi)} \longrightarrow \overline{\mathcal{F}_{(H,\varphi)}}(H,\varphi) = \mathcal{F}_{(H,\varphi)} / \{0\}$$

is the canonical isomorphism. (Note that the triviality of conjugations and Lemma 1.2(c) imply that $\overline{\mathcal{F}_{(H,\varphi)}}(H,\varphi) = \mathcal{F}_{(H,\varphi)} / \{0\}$.)

(viii) G -equivariance of transitivity maps: For $g \in G$ and $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} the left hand square in the diagram

$$\begin{array}{ccccc}
\mathcal{F}_{(H,\varphi)} & \xrightarrow{\text{ts}_{(I,\psi)}^{(H,\varphi)}} & \overline{\mathcal{F}_{(I,\psi)}}(H,\varphi) & \xleftarrow{\text{Br}_{(H,\varphi)}^{\mathcal{F}_{(I,\psi)}}} & (\mathcal{F}_{(I,\psi)})^{(H,\varphi)} \\
\downarrow \text{con}_{(H,\varphi)}^g & & \downarrow & & \downarrow \\
\mathcal{F}_{(H,\varphi)}^g & \xrightarrow{\text{ts}_{(I,\psi)}^g} & \overline{\mathcal{F}_{(I,\psi)}^g}(H,\varphi) & \xleftarrow{\text{Br}_{(H,\varphi)}^{\mathcal{F}_{(I,\psi)}^g}} & (\mathcal{F}_{(I,\psi)}^g)^{g(H,\varphi)}
\end{array}$$

is commutative, where the middle vertical map is defined by the commutativity of the right hand square and the right vertical map is the restriction of $\text{con}_{(I,\psi)}^g : \mathcal{F}_{(I,\psi)} \rightarrow \mathcal{F}_{(I,\psi)}^g$.

(ix) Compatibility of transitivity maps with restriction maps: For all elements

$$\begin{array}{ccc}
(H, \varphi) & \geq & (J, \lambda) \\
\triangledown | & & \triangledown | \\
(I, \psi) & \geq & (K, \mu)
\end{array}$$

in \mathcal{M} such that $p \nmid [H : J]$ and $p \nmid [I : K]$, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}_{(H,\varphi)} & \xrightarrow{\text{res}_{(J,\lambda)}^{(H,\varphi)}} & \mathcal{F}_{(J,\lambda)} \\
\downarrow \text{ts}_{(I,\psi)}^{(H,\varphi)} & & \downarrow \text{ts}_{(K,\mu)}^{(J,\lambda)} \\
\overline{\mathcal{F}_{(I,\psi)}}(H,\varphi) & \xrightarrow{\mathcal{F}_{(I,\psi)} \text{res}_{(J,\lambda)}^{(H,\varphi)}} & \overline{\mathcal{F}_{(I,\psi)}(J,\lambda)} \xrightarrow{\overline{\text{res}_{(K,\mu)}^{(I,\psi)}}(J,\lambda)} \overline{\mathcal{F}_{(K,\mu)}}(J,\lambda)
\end{array}$$

2.2 Definition Let $\mathcal{F} = (\mathcal{F}, \mathcal{F}\text{con}, \mathcal{F}\text{res}, \mathcal{F}\text{ts})$ and $\mathcal{G} = (\mathcal{G}, \mathcal{G}\text{con}, \mathcal{G}\text{res}, \mathcal{G}\text{ts})$ be Brauer sheaves of G over F . Then a *morphism* $f: \mathcal{F} \rightarrow \mathcal{G}$ is a family $(f_{(H,\varphi)})$ of F -linear maps $f_{(H,\varphi)}: \mathcal{F}_{(H,\varphi)} \rightarrow \mathcal{G}_{(H,\varphi)}$, $(H, \varphi) \in \mathcal{M}$, commuting with the conjugation, restriction, and transitivity maps. To be more precise, the last condition means that for all $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_{(H,\varphi)} & \xrightarrow{f_{(H,\varphi)}} & \mathcal{G}_{(H,\varphi)} \\ \mathcal{F}\text{ts}_{(I,\psi)}^{(H,\varphi)} \downarrow & & \downarrow \mathcal{G}\text{ts}_{(I,\psi)}^{(H,\varphi)} \\ \overline{\mathcal{F}_{(I,\psi)}}(H, \varphi) & \xrightarrow{\overline{f_{(I,\psi)}}(H, \varphi)} & \overline{\mathcal{G}_{(I,\psi)}}(H, \varphi) \end{array}$$

2.3 Remark (i) If $f: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of Brauer sheaves, then $f_{(H,\varphi)}$ is $FN_G(H, \varphi)$ -linear for each $(H, \varphi) \in \mathcal{M}$.

(ii) The Brauer sheaves of G over F , together with their morphisms, form an additive category, which we denote by $\mathcal{B}_F(G)$. In general the kernel and cokernel of a morphism $f: \mathcal{F} \rightarrow \mathcal{G}$ in $\mathcal{B}_F(G)$ need not exist. Where the kernel is concerned, it is easy to verify that on the F -vector spaces $\ker(f_{(H,\varphi)})$, the conjugation and restriction maps of \mathcal{F} induce again conjugation and restriction maps. But in order to have induced transitivity maps, some extra condition is needed. For example, if $\ker(f_{(I,\psi)})$ is a direct summand of $\mathcal{F}_{(I,\psi)}$ as $FN_G(I, \psi)$ -module for each $(I, \psi) \in \mathcal{M}$, then the following diagram (with $(I, \psi) \trianglelefteq (H, \varphi)$)

$$\begin{array}{ccccc} \ker(f_{(H,\varphi)}) & \xrightarrow{i_{(H,\varphi)}} & \mathcal{F}_{(H,\varphi)} & \xrightarrow{f_{(H,\varphi)}} & \mathcal{G}_{(H,\varphi)} \\ & & \downarrow \mathcal{F}\text{ts}_{(I,\psi)}^{(H,\varphi)} & & \downarrow \mathcal{G}\text{ts}_{(I,\psi)}^{(H,\varphi)} \\ \overline{\ker(f_{(I,\psi)})}(H, \varphi) & \xrightarrow{\overline{i_{(I,\psi)}}(H, \varphi)} & \overline{\mathcal{F}_{(I,\psi)}}(H, \varphi) & \xrightarrow{\overline{f_{(I,\psi)}}(H, \varphi)} & \overline{\mathcal{G}_{(I,\psi)}}(H, \varphi) \end{array}$$

with exact rows, where $i_{(H,\varphi)}$ denotes the inclusion, shows that then a transitivity map $\ker(f_{(H,\varphi)}) \rightarrow \overline{\ker(f_{(I,\psi)})}(H, \varphi)$ is induced. In fact for that it suffices that $\overline{i_{(I,\psi)}}(H, \varphi)$ is injective and the bottom row is exact.

(iii) Every $V \in {}_F G \text{mod}$ defines a Brauer sheaf \mathcal{F} of G over F by setting $\mathcal{F}_{(H,\varphi)} := \overline{V}(H, \varphi)$ for $(H, \varphi) \in \mathcal{M}$ and defining the conjugation, restriction and transitivity maps as in the Lemmas 1.3, 1.5, and 1.10.

(iv) It is immediate that (iii) yields an F -functor

$$\mathcal{I}: {}_F G \text{mod} \rightarrow \mathcal{B}_F(G).$$

2.4 Proposition The functor $\mathcal{I}: {}_F G \text{mod} \rightarrow \mathcal{B}_F(G)$ is fully faithful (i.e., bijective on morphism sets).

Proof Let $V, W \in {}_F G \text{mod}$. For $f \in \text{Hom}_{{}_F G}(V, W)$, one has the following commutative diagram in which the vertical maps are canonical isomorphisms:

$$\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\text{Br}_{(1,1)}^V \downarrow & & \downarrow \text{Br}_{(1,1)}^W \\
\overline{V}(1,1) & \xrightarrow{\overline{f}(1,1)} & \overline{W}(1,1)
\end{array}$$

Therefore, $\mathcal{I}: \text{Hom}_{FG}(V, W) \rightarrow \text{Hom}_{\mathcal{B}_F(G)}(\mathcal{I}(V), \mathcal{I}(W))$ is injective.

On the other hand, let $g: \mathcal{I}(V) \rightarrow \mathcal{I}(W)$ be a morphism in $\mathcal{B}_F(G)$. Then there exists a unique $f \in \text{Hom}_{FG}(V, W)$ such that the diagram

$$\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\text{Br}_{(1,1)}^V \downarrow & & \downarrow \text{Br}_{(1,1)}^W \\
\overline{V}(1,1) & \xrightarrow{g(1,1)} & \overline{W}(1,1)
\end{array}$$

commutes. We have to verify that $\overline{f}(H, \varphi) = g_{(H, \varphi)}$ for all $(H, \varphi) \in \mathcal{M}$. To see this we consider the two commutative diagrams

$$\begin{array}{ccc}
\overline{V}(H, \varphi) & \xrightarrow{\text{vts}_{(1,1)}^{(H, \varphi)}} & \overline{\overline{V}(1,1)}(H, \varphi) \\
g_{(H, \varphi)} \downarrow & & \downarrow \overline{g}_{(1,1)}(H, \varphi) \\
\overline{W}(H, \varphi) & \xrightarrow{\text{wts}_{(1,1)}^{(H, \varphi)}} & \overline{\overline{W}(1,1)}(H, \varphi)
\end{array}$$

and

$$\begin{array}{ccc}
\overline{V}(H, \varphi) & \xrightarrow{\overline{\text{Br}_{(1,1)}^V}(H, \varphi)} & \overline{\overline{V}(1,1)}(H, \varphi) \\
\overline{f}(H, \varphi) \downarrow & & \downarrow \overline{g}_{(1,1)}(H, \varphi) \\
\overline{W}(H, \varphi) & \xrightarrow{\overline{\text{Br}_{(1,1)}^W}(H, \varphi)} & \overline{\overline{W}(1,1)}(H, \varphi)
\end{array}$$

and observe that, for $v \in V^{(H, \varphi)}$, we have

$$(\overline{\text{Br}_{(1,1)}^V}(H, \varphi))(\text{Br}_{(H, \varphi)}^V(v)) = \text{Br}_{(H, \varphi)}^{\overline{V}(1,1)}(\text{Br}_{(1,1)}^V(v)) = \text{vts}_{(1,1)}^{(H, \varphi)}(\text{Br}_{(H, \varphi)}^V(v)).$$

Since a similar argument holds for W and since $\overline{\text{Br}_{(1,1)}^V}(H, \varphi)$ and $\text{vts}_{(1,1)}^{(H, \varphi)}$ are isomorphisms, we obtain $g_{(H, \varphi)} = \overline{f}(H, \varphi)$ and $g = \mathcal{I}(f)$. \square

2.5 Next we show that Brauer sheaves also arise from finite G -equivariant line bundles. We recall the definition of their category $FG\text{mon}$ from [Bo97]. The objects of $FG\text{mon}$ are FG -modules M together with a decomposition $M = M_1 \oplus \cdots \oplus M_r$ into one-dimensional F -subspaces M_i , $i = 1, \dots, r$, which are permuted by the G -action. In particular, M is a monomial FG -module. To each *line* M_i is attached a *stabilizing pair* $(H_i, \varphi_i) \in \mathcal{M}$, where H_i is the stabilizer of M_i under the G -action and H_i acts via φ_i on M_i . This gives rise to an \mathcal{M} -grading

$$M((H, \varphi)) := \bigoplus_{\substack{i \in \{1, \dots, r\} \\ (H, \varphi) = (H_i, \varphi_i)}} M_i, \quad (H, \varphi) \in \mathcal{M},$$

and an \mathcal{M} -filtration

$$M^{((H, \varphi))} := \bigoplus_{\substack{i \in \{1, \dots, r\} \\ (H, \varphi) \leq (H_i, \varphi_i)}} M_i, \quad (H, \varphi) \in \mathcal{M},$$

of M with the properties

$$M^{((H, \varphi))} \subseteq M^{((I, \psi))}, \quad M^{(g(H, \varphi))} = gM^{((H, \varphi))}, \quad M^{((1, 1))} = M,$$

for $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} and $g \in G$. Note that $M^{((H, \varphi))}$ is an $FN_G(H, \varphi)$ -submodule of M . Using this filtration, the set $FG\text{mon}(M, N)$ of morphisms between objects M and N from $FG\text{mon}$ is defined as the set of $f \in \text{Hom}_{FG}(M, N)$ satisfying $f(M^{((H, \varphi))}) \subseteq N^{((H, \varphi))}$, for all $(H, \varphi) \in \mathcal{M}$. The category $FG\text{mon}$ is additive but in general not abelian.

Sometimes it will be necessary to distinguish between an object $M \in FG\text{mon}$ and its underlying FG -module which is a trivial source FG -module. Therefore, we introduce the forgetful functor

$$\mathcal{V}: FG\text{mon} \longrightarrow FG\text{triv}.$$

Note that $M^{((H, \varphi))} \subseteq \mathcal{V}(M)^{(H, \varphi)}$ for all $(H, \varphi) \in \mathcal{M}$, but that in general one does not have equality (as one can see for example from Lemma 2.9).

Similar as for FG -modules V we can associate to each $M \in FG\text{mon}$ a Brauer sheaf. This is the content of the next Proposition.

2.6 Proposition (a) *Each $M \in FG\text{mon}$ defines a Brauer sheaf \mathcal{F} in the following way:*

- $\mathcal{F}_{(H, \varphi)} := M^{((H, \varphi))}$ for all $(H, \varphi) \in \mathcal{M}$.
- $\text{con}_{(H, \varphi)}^g = M\text{con}_{(H, \varphi)}^g: M^{((H, \varphi))} \rightarrow M^{(g(H, \varphi))}$, $m \mapsto gm$, for all $(H, \varphi) \in \mathcal{M}$ and $g \in G$.
- $\text{res}_{(I, \psi)}^{(H, \varphi)} = M\text{res}_{(I, \psi)}^{(H, \varphi)}: M^{((H, \varphi))} \rightarrow M^{((I, \psi))}$, $m \mapsto m$, for all $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} with $p \nmid [H : I]$.
- $\text{ts}_{(I, \psi)}^{(H, \varphi)} = M\text{ts}_{(I, \psi)}^{(H, \varphi)}: M^{((H, \varphi))} \rightarrow \overline{M^{((I, \psi))}}(H, \varphi)$, $m \mapsto \text{Br}_{(H, \varphi)}^{M^{((I, \psi))}}(m)$, for all $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} .

(b) *For each $M \in FG\text{mon}$ the restriction maps and transitivity maps in (a) are split injective $F[N_G(H, \varphi) \cap N_G(I, \psi)]$ -module homomorphisms, and $M\text{ts}_{(I, \psi)}^{(H, \varphi)}$ is bijective whenever $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} is such that H/I is a p -group. In particular, the restriction of $\text{Br}_{(P, 1)}^{\mathcal{V}(M)}$ induces an isomorphism $M^{((P, 1))} \cong \overline{M}(P)$ of $FN_G(P)$ -modules for all p -subgroups P of G .*

Proof (a) First note that for $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} , one has $M^{((H, \varphi))} \subseteq (M^{((I, \psi))})^{(H, \varphi)}$, so that $\text{ts}_{(I, \psi)}^{(H, \varphi)}$ is well-defined. Most of the properties in Definition 2.1 follow immediately from the above definitions.

To show that property (vi) holds, suppose that $(J, \lambda) \trianglelefteq (I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} with $(J, \lambda) \trianglelefteq (H, \varphi)$. Let $m \in \mathcal{F}_{(H, \varphi)} = M^{((H, \varphi))}$. Then $\text{ts}_{(I, \psi)}^{(H, \varphi)}$ maps m to $\text{Br}_{(H, \varphi)}^{M^{((I, \psi))}}(m)$, and $\text{ts}_{(J, \lambda)}^{(I, \psi)}(H, \varphi)$

maps $\text{Br}_{(H,\varphi)}^{M^{((I,\psi))}}(m)$ to $\text{Br}_{(H,\varphi)}^{\overline{M^{((J,\lambda))}}(I,\psi)}(\text{Br}_{(I,\psi)}^{M^{((J,\lambda))}}(m))$. On the other hand, $\text{ts}_{(J,\lambda)}^{(H,\varphi)}$ maps m to $\text{Br}_{(H,\varphi)}^{M^{((J,\lambda))}}(m)$, and $\text{ts}_{(I,\psi)}^{(H,\varphi)}$ maps $\text{Br}_{(H,\varphi)}^{M^{((J,\lambda))}}(m)$ also to $\text{Br}_{(H,\varphi)}^{\overline{M^{((J,\lambda))}}(I,\psi)}(\text{Br}_{(I,\psi)}^{M^{((J,\lambda))}}(m))$.

To show that property (ix) holds, suppose we have elements

$$\begin{array}{ccc} (H, \varphi) & \geq & (J, \lambda) \\ \nabla | & & \nabla | \\ (I, \psi) & \geq & (K, \mu) \end{array}$$

in \mathcal{M} with $p \nmid [I : K]$ and $p \nmid [H : J]$. Then $\text{ts}_{(I,\psi)}^{(H,\varphi)}$ maps $m \in M^{((H,\varphi))}$ to $\text{Br}_{(H,\varphi)}^{M^{((I,\psi))}}(m)$, $M^{((I,\psi))} \text{res}_{(J,\lambda)}^{(H,\varphi)}$ maps $\text{Br}_{(H,\varphi)}^{M^{((I,\psi))}}(m)$ to $\text{Br}_{(J,\lambda)}^{M^{((I,\psi))}}(m)$, and $\overline{\text{res}_{(K,\mu)}^{(I,\psi)}}(J,\lambda)$ maps $\text{Br}_{(J,\lambda)}^{M^{((I,\psi))}}(m)$ to $\text{Br}_{(J,\lambda)}^{M^{((K,\mu))}}(m)$. On the other hand, $\text{res}_{(J,\lambda)}^{(H,\varphi)}$ maps m to m , and $\text{ts}_{(K,\mu)}^{(J,\lambda)}$ maps m also to $\text{Br}_{(J,\lambda)}^{M^{((K,\mu))}}(m)$.

(b) For $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} with $p \nmid [H : I]$, the restriction map $M^{((H,\varphi))} \rightarrow M^{((I,\psi))}$ has the splitting map which vanishes on every line of $M^{((I,\psi))}$ with stabilizing pair not containing (H, φ) . For $(I, \psi) \leq (H, \varphi)$, we write $M^{((I,\psi))} = M^{((H,\varphi))} \oplus Z$ with Z the kernel of the above splitting map (which exists also if $p \mid [H : I]$). Thus the transitivity map $M^{((H,\varphi))} \rightarrow \overline{M^{((I,\psi))}}(H, \varphi)$ is split injective. Moreover, if H/I is a p -group, then the transitivity map $\text{ts}_{(I,\psi)}^{(H,\varphi)}$ is bijective, since $\text{Br}_{(H,\varphi)}^{M^{((I,\psi))}}(M^{((I,\psi))}) = \text{Br}_{(H,\varphi)}^{M^{((I,\psi))}}(M^{((H,\varphi))})$. In fact, $(M^{((I,\psi))})^{(H,\varphi)} = M^{((H,\varphi))} \oplus X$, where X is generated as F -vector space by elements of the form $\sum_{h \in H/J \cap H} hx$ where $x \in M^{((J,\lambda))}$ for some $(J, \lambda) \in \mathcal{M}$ satisfying $(J, \lambda) \geq (I, \psi)$, $(J, \lambda) \not\geq (H, \varphi)$, $\lambda|_{J \cap H} = \varphi|_{J \cap H}$. But $I \leq J \cap H < H$ implies that $\sum_{h \in H/J \cap H} hx = \text{tr}_{(J \cap H, \varphi)}^{(H, \varphi)}(x)$ lies in the kernel of $\text{Br}_{(H, \varphi)}^{M^{((I, \psi))}}$. \square

2.7 Remark It is easy to see that for each morphism $f: M \rightarrow N$ in the category $FG\text{mon}$ the restrictions

$$f^{((H,\varphi))}: M^{((H,\varphi))} \longrightarrow N^{((H,\varphi))}, \quad (H, \varphi) \in \mathcal{M},$$

of f form a morphism between the associated Brauer sheaves. From that we obtain an F -functor

$$\mathcal{J}: FG\text{mon} \longrightarrow \mathcal{B}_F(G).$$

Note that $\mathcal{J}(f)_{(1,1)} = f: M \rightarrow N$. Hence the functor $\mathcal{J}: FG\text{mon} \rightarrow \mathcal{B}_F(G)$ is faithful, i.e., injective on morphisms.

2.8 Proposition For $M, N \in FG\text{mon}$ the map

$$\begin{aligned} \text{Hom}_{\mathcal{B}_F(G)}(\mathcal{J}(M), \mathcal{J}(N)) &\longrightarrow \text{Hom}_{FG}(\mathcal{V}(M), \mathcal{V}(N)) \\ (f_{(H,\varphi)}) &\longmapsto f_{(1,1)} \end{aligned}$$

is injective with image consisting of those $f \in \text{Hom}_{FG}(\mathcal{V}(M), \mathcal{V}(N))$ satisfying $f(M^{((H,\varphi))}) \subseteq N^{((O^p(H), \varphi))}$ for all $(H, \varphi) \in \mathcal{M}$, where $O^p(H)$ denotes the smallest normal subgroup of H of p -power index.

Proof Let $M, N \in FG\text{mon}$ and consider, for $(f_{(H,\varphi)}) \in \text{Hom}_{\mathcal{B}_F(G)}(\mathcal{J}(M), \mathcal{J}(N))$, the commutative diagram

$$\begin{array}{ccc} M^{((H,\varphi))} & \xrightarrow{f_{(H,\varphi)}} & N^{((H,\varphi))} \\ \text{Mts}_{(1,1)}^{(H,\varphi)} \downarrow & & \downarrow \text{Nts}_{(1,1)}^{(H,\varphi)} \\ \overline{M}(H, \varphi) & \xrightarrow{\overline{f_{(1,1)}}(H, \varphi)} & \overline{N}(H, \varphi). \end{array}$$

Then $f_{(1,1)} \in \text{Hom}_{FG}(M, N)$ and $f_{(H,\varphi)}$ is uniquely determined by $f_{(1,1)}$, since $N\text{ts}_{(1,1)}^{(H,\varphi)}$ is injective. This proves the injectivity of our map.

Next we prove the statement about the image of our map. Note that for given $f_{(1,1)} \in \text{Hom}_{FG}(M, N)$, there exists a map $f_{(H,\varphi)}$ making the above diagram commutative if and only if

$$f_{(1,1)}(M^{((H,\varphi))}) \subseteq N^{((H,\varphi))} \oplus \ker(\text{Br}_{(H,\varphi)}^{\mathcal{V}(N)}).$$

Moreover, if the above condition holds for all $(H, \varphi) \in \mathcal{M}$, and if $f_{(H,\varphi)}$ is defined from $f_{(1,1)}$ via the above diagram, then the collection $f_{(H,\varphi)}: M^{((H,\varphi))} \rightarrow N^{((H,\varphi))}$, $(H, \varphi) \in \mathcal{M}$, is a morphism in $\text{Hom}_{\mathcal{B}_F(G)}(\mathcal{J}(M), \mathcal{J}(N))$. In fact, compatibility of $f_{(H,\varphi)}$ with conjugations is easy to verify, compatibility with restrictions follows from

$$\begin{aligned} N\text{ts}_{(1,1)}^{(I,\psi)} \circ N\text{res}_{(I,\psi)}^{(H,\varphi)} \circ f_{(H,\varphi)} &\stackrel{2.1(\text{ix})}{=} \mathcal{V}(N)\text{res}_{(I,\psi)}^{(H,\varphi)} \circ N\text{ts}_{(1,1)}^{(H,\varphi)} \circ f_{(H,\varphi)} \\ &= \mathcal{V}(N)\text{res}_{(I,\psi)}^{(H,\varphi)} \circ \overline{f_{(1,1)}}(H, \varphi) \circ M\text{ts}_{(1,1)}^{(H,\varphi)} \\ &= \overline{f_{(1,1)}}(I, \psi) \circ \mathcal{V}(M)\text{res}_{(I,\psi)}^{(H,\varphi)} \circ M\text{ts}_{(1,1)}^{(H,\varphi)} \\ &\stackrel{2.1(\text{ix})}{=} \overline{f_{(1,1)}}(I, \psi) \circ M\text{ts}_{(1,1)}^{(I,\psi)} \circ M\text{res}_{(I,\psi)}^{(H,\varphi)} = N\text{ts}_{(1,1)}^{(I,\psi)} \circ f_{(I,\psi)} \circ M\text{res}_{(I,\psi)}^{(H,\varphi)} \end{aligned}$$

and the injectivity of $N\text{ts}_{(1,1)}^{(I,\psi)}$, and compatibility with transitivity maps follows from

$$\begin{aligned} \overline{N\text{ts}_{(1,1)}^{(I,\psi)}}(H, \varphi) \circ N\text{ts}_{(I,\psi)}^{(H,\varphi)} \circ f_{(H,\varphi)} &\stackrel{2.1(\text{vi})}{=} \mathcal{V}(N)\text{ts}_{(I,\psi)}^{(H,\varphi)} \circ N\text{ts}_{(1,1)}^{(H,\varphi)} \circ f_{(H,\varphi)} \\ &= \mathcal{V}(N)\text{ts}_{(I,\psi)}^{(H,\varphi)} \circ \overline{f_{(1,1)}}(H, \varphi) \circ M\text{ts}_{(1,1)}^{(H,\varphi)} \\ &\stackrel{1.10(\text{viii})}{=} \overline{f_{(1,1)}}(I, \psi)(H, \varphi) \circ \mathcal{V}(M)\text{ts}_{(I,\psi)}^{(H,\varphi)} \circ M\text{ts}_{(1,1)}^{(H,\varphi)} \\ &\stackrel{2.1(\text{vi})}{=} \overline{f_{(1,1)}}(I, \psi)(H, \varphi) \circ \overline{M\text{ts}_{(1,1)}^{(I,\psi)}}(H, \varphi) \circ M\text{ts}_{(I,\psi)}^{(H,\varphi)} \\ &= \overline{N\text{ts}_{(1,1)}^{(I,\psi)}}(H, \varphi) \circ \overline{f_{(I,\psi)}}(H, \varphi) \circ M\text{ts}_{(I,\psi)}^{(H,\varphi)} \end{aligned}$$

and the injectivity of $\overline{N\text{ts}_{(1,1)}^{(I,\psi)}}(H, \varphi)$, which in turn follows from the split injectivity of $N\text{ts}_{(1,1)}^{(I,\psi)}: N^{((I,\psi))} \rightarrow \overline{N}(I, \psi)$ as an FH -module homomorphism, cf. Proposition 2.6(b).

Now we know that the image of the map defined in the proposition consists of exactly those $f \in \text{Hom}_{FG}(\mathcal{V}(M), \mathcal{V}(N))$ satisfying

$$f(M^{((H,\varphi))}) \subseteq N^{((H,\varphi))} \oplus \ker(\text{Br}_{(H,\varphi)}^{\mathcal{V}(N)}),$$

for all $(H, \varphi) \in \mathcal{M}$. Obviously, this is the case if and only if f satisfies

$$f(M^{((H,\varphi))}) \subseteq N^{((I,\psi))} \oplus \ker(\text{Br}_{(I,\psi)}^{\mathcal{V}(N)}),$$

for all $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} , and the following lemma concludes the proof. \square

2.9 Lemma *Let $M \in {}_{FG}\text{mon}$ and let $(H, \varphi) \in \mathcal{M}$.*

(a) *One has*

$$\begin{aligned} M^{(H,\varphi)} = M^{((H,\varphi))} \oplus &\bigoplus_{\substack{(J,\lambda) \in \mathcal{M}/H \\ (H \cap J, \lambda) < (H,\varphi) \\ p \uparrow [H: H \cap J]}} \text{tr}_{(H \cap J, \varphi)}^{(H,\varphi)}(M^{((J,\lambda))}) \\ &\oplus \bigoplus_{\substack{(J,\lambda) \in \mathcal{M}/H \\ (H \cap J, \lambda) < (H,\varphi) \\ p \uparrow [H: H \cap J]}} \text{tr}_{(H \cap J, \varphi)}^{(H,\varphi)}(M^{((J,\lambda))}), \end{aligned}$$

where the direct sums run over a set of representatives (J, λ) of H -conjugacy classes of \mathcal{M} .

(b) In the decomposition of $M^{(H, \varphi)}$ in part (a), the third direct summand is equal to $\ker(\mathrm{Br}_{(H, \varphi)}^{\mathcal{V}(M)})$.

(c) One has

$$\bigcap_{(I, \psi) \leq (H, \varphi)} \left(M^{((I, \psi))} \oplus \ker(\mathrm{Br}_{(I, \psi)}^{\mathcal{V}(M)}) \right) = M^{((O^p(H), \varphi))} \cap M^{(H, \varphi)}.$$

Proof (a) This follows from [BK00, Lemma 1.5].

(b) This follows from [BK00, Proposition 2.4].

(c) We write $M = \bigoplus_{x \in X} M_x$ with F -subspaces M_x such that $\dim M_x = 1$ and G permutes the set $\{M_x \mid x \in X\}$. The bijection $x \mapsto M_x$ induces an action of G on X . For $x \in X$, let (G_x, φ_x) denote the stabilizing pair of M_x . If $H \leq G$, then we set $H_x := G_x \cap H$ and often just write φ_x again instead of $\varphi_x|_{H_x}$.

Both sides of the equation in part (c) are contained in $M^{(H, \varphi)}$. Let $m \in M^{(H, \varphi)}$. We will show that m lies in the left hand side if and only if it lies in the right hand side of this equation. By [BK00, Lemma 1.5], we can write

$$m = \sum_{\substack{x \in X/H \\ \varphi|_{H_x} = \varphi_x}} \mathrm{tr}_{(H_x, \varphi_x)}^{(H, \varphi)}(m_x),$$

where x runs through a set of representatives of the H -orbits of X such that $(H_x, \varphi_x) \leq (H, \varphi)$, and where $m_x \in M_x$.

Now we fix $(I, \psi) \leq (H, \varphi)$. Then, by [BK00, Lemma 1.3(e)], we obtain

$$m = \sum_{\substack{x \in X/H \\ \varphi|_{H_x} = \varphi_x}} \sum_{h \in I \backslash H/H_x} \varphi(h^{-1}) \mathrm{tr}_{(I \cap {}^h H_x, \psi)}^{(I, \psi)}(hm_x). \quad (2.9.a)$$

We fix indices $x \in X/H$ and $h \in I \backslash H/H_x$. If $I \cap {}^h H_x = I$, then $\mathrm{tr}_{(I \cap {}^h H_x, \psi)}^{(I, \psi)}(hm_x) \in M^{((I, \psi))}$. If $I \cap {}^h H_x < I$ and $p \nmid [I : I \cap {}^h H_x]$, then $\mathrm{tr}_{(I \cap {}^h H_x, \psi)}^{(I, \psi)}(hm_x)$ is contained in the second direct summand of the decomposition of $M^{((I, \psi))}$ in part (a), and if $p \mid [I : I \cap {}^h H_x]$, then it is contained in the third direct summand, which equals $\ker(\mathrm{Br}_{(I, \psi)}^{\mathcal{V}(M)})$ by part (b). Note that the summands $\varphi(h^{-1}) \mathrm{tr}_{(I \cap {}^h H_x, \psi)}^{(I, \psi)}(hm_x)$ in Equation (2.9.a) are linearly independent, since they have mutually disjoint support Ihx . Thus,

$$m \in M^{((I, \psi))} \oplus \ker(\mathrm{Br}_{(I, \psi)}^{\mathcal{V}(M)}) \iff \begin{array}{l} \mathrm{tr}_{(I \cap {}^h H_x, \psi)}^{(I, \psi)}(hm_x) = 0 \text{ for all } x \in X/H \\ \text{and } h \in I \backslash H/H_x \text{ such that } I \not\leq {}^h H_x \\ \text{and } p \nmid [I : I \cap {}^h H_x]. \end{array}$$

However, $\mathrm{tr}_{(I \cap {}^h H_x, \psi)}^{(I, \psi)}(hm_x) = 0$ if and only if $hm_x = 0$, since $I \cap {}^h H_x$ is the stabilizer of hM_x in I . Moreover, $hm_x = 0$ if and only if $m_x = 0$. Therefore, the condition on the right hand side of the above equivalence is equivalent to

$$m_x = 0 \text{ for all } x \in X/H \text{ such that there exists } h \in H \\ \text{with } I \not\leq {}^h H_x \text{ and } p \nmid [I : I \cap {}^h H_x].$$

So, altogether, m is contained in the left hand side of the equation in part (c) if and only if

$$m_x = 0 \text{ for all } x \in X/H \text{ such that there exist } I \leq H \\ \text{and } h \in H \text{ with } I \not\leq {}^h H_x \text{ and } p \nmid [I : I \cap {}^h H_x]. \quad (2.9.b)$$

We claim that the condition in (2.9.b) is equivalent to

$$m_x = 0 \text{ for all } x \in X/H \text{ such that } H_x \not\leq O^p(H). \quad (2.9.c)$$

In fact, if $H_x \not\geq O^p(H)$, then there exists a p' -subgroup I of H which is not contained in H_x (since $O^p(H)$ is generated by all p' -subgroups of H). For this I we then have $I \not\leq H_x$ and $p \nmid [I : I \cap H_x]$. Conversely, assume that $H_x \geq O^p(H)$ and let $I \leq H$ and $h \in H$. If $I \not\leq {}^h H_x$, then

$$1 < [I : I \cap {}^h H_x] \mid [I : I \cap O^p(H)] = [I \cdot O^p(H) : O^p(H)] \mid [H : O^p(H)].$$

This implies $p \mid [I : I \cap {}^h H_x]$. Finally, the condition in (2.9.c) is clearly equivalent to $m \in M^{((O^p(H), \varphi))}$. \square

2.10 Remark Let $M, N \in {}_{FG}\text{mon}$ and $f \in \text{Hom}_{FG}(\mathcal{V}(M), \mathcal{V}(N))$. Note that for $H \leq U \leq G$, one has $O^p(H) \leq O^p(U)$. Therefore, $f(M^{((H, \varphi))}) \subseteq N^{((O^p(H), \varphi))}$ for all $(H, \varphi) \in \mathcal{M}$ if and only if $f(M^{((H, \varphi))}) \subseteq N^{((O^p(H), \varphi))}$ for all $(H, \varphi) \in \mathcal{M}$. Moreover, this is equivalent to $f(M^{((H, \varphi))}) \subseteq N^{((H, \varphi))}$ for all $(H, \varphi) \in \mathcal{M}$ with H a p -perfect subgroup, i.e., $O^p(H) = H$.

2.11 Proposition Let $M \in {}_{FG}\text{mon}$ and $V \in {}_{FG}\text{mod}$. Then the map

$$\begin{aligned} \text{Hom}_{FG}(\mathcal{V}(M), V) &\longrightarrow \text{Hom}_{\mathcal{B}_F(G)}(\mathcal{J}(M), \mathcal{I}(V)) \\ f &\longmapsto (f_{(H, \varphi)}) \end{aligned}$$

with

$$f_{(H, \varphi)} : M^{((H, \varphi))} \twoheadrightarrow M^{(H, \varphi)} \xrightarrow{f^{(H, \varphi)}} V^{(H, \varphi)} \xrightarrow{\text{Br}_{(H, \varphi)}^V} \overline{V}^{(H, \varphi)}$$

for $(H, \varphi) \in \mathcal{M}$, where $f^{(H, \varphi)}$ denotes the restriction of f , is an F -isomorphism. Its inverse is given by $(f_{(H, \varphi)}) \mapsto {}_V\text{can}^{-1} \circ f_{(1, 1)}$, where ${}_V\text{can} : V \rightarrow \overline{V}(1, 1) = V/\{0\}$ is the canonical isomorphism. Moreover, the above isomorphism is functorial in V and M .

Proof Let $f \in \text{Hom}_{FG}(\mathcal{V}(M), V)$. We first show that $(f_{(H, \varphi)})$ is a morphism of Brauer sheaves. For $g \in G$ and $(H, \varphi) \in \mathcal{M}$, the diagram

$$\begin{array}{ccccccc} M^{((H, \varphi))} & \twoheadrightarrow & M^{(H, \varphi)} & \xrightarrow{f^{(H, \varphi)}} & V^{(H, \varphi)} & \xrightarrow{\text{Br}_{(H, \varphi)}^V} & \overline{V}^{(H, \varphi)} \\ \downarrow M\text{con}_{(H, \varphi)}^g & & \downarrow g & & \downarrow g & & \downarrow {}_V\text{con}_{(H, \varphi)}^g \\ M^{(g(H, \varphi))} & \twoheadrightarrow & M^{g(H, \varphi)} & \xrightarrow{f^{g(H, \varphi)}} & V^{g(H, \varphi)} & \xrightarrow{\text{Br}_{g(H, \varphi)}^V} & \overline{V}^{g(H, \varphi)} \end{array}$$

commutes. Also, for $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} with $p \nmid [H : I]$, the diagram

$$\begin{array}{ccccccc} M^{((H, \varphi))} & \twoheadrightarrow & M^{(H, \varphi)} & \xrightarrow{f^{(H, \varphi)}} & V^{(H, \varphi)} & \xrightarrow{\text{Br}_{(H, \varphi)}^V} & \overline{V}^{(H, \varphi)} \\ \downarrow M\text{res}_{(I, \psi)}^{(H, \varphi)} & & \downarrow & & \downarrow & & \downarrow {}_V\text{res}_{(I, \psi)}^{(H, \varphi)} \\ M^{((I, \psi))} & \twoheadrightarrow & M^{(I, \psi)} & \xrightarrow{f^{(I, \psi)}} & V^{(I, \psi)} & \xrightarrow{\text{Br}_{(I, \psi)}^V} & \overline{V}^{(I, \psi)} \end{array}$$

commutes. Finally, whenever $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} , the diagram

$$\begin{array}{ccccc}
M^{((H,\varphi))} & \xrightarrow{\quad} & (M^{((I,\psi))})^{(H,\varphi)} & \xrightarrow{\text{Br}_{(H,\varphi)}^{M^{((I,\psi))}}} & \overline{M^{((I,\psi))}}(H,\varphi) \\
\downarrow i_{(H,\varphi)} & & \downarrow (i_{(I,\psi)})^{(H,\varphi)} & & \downarrow \overline{i_{(I,\psi)}}(H,\varphi) \\
M^{(H,\varphi)} & \xrightarrow{\text{id}} & (M^{(I,\psi)})^{(H,\varphi)} & \xrightarrow{\text{Br}_{(H,\varphi)}^{M^{(I,\psi)}}} & \overline{M^{(I,\psi)}}(H,\varphi) \\
\downarrow f_{(H,\varphi)} & & \downarrow (f_{(I,\psi)})^{(H,\varphi)} & & \downarrow \overline{f_{(I,\psi)}}(H,\varphi) \\
V^{(H,\varphi)} & \xrightarrow{\text{id}} & (V^{(I,\psi)})^{(H,\varphi)} & \xrightarrow{\text{Br}_{(H,\varphi)}^{V^{(I,\psi)}}} & \overline{V^{(I,\psi)}}(H,\varphi) \\
\downarrow \text{Br}_{(H,\varphi)}^V & & & & \downarrow \overline{\text{Br}_{(I,\psi)}^V}(H,\varphi) \\
\overline{V}(H,\varphi) & \xrightarrow{\text{vts}_{(I,\psi)}^{(H,\varphi)}} & & & \overline{V}(I,\psi)(H,\varphi)
\end{array}$$

commutes, where $i_{(H,\varphi)}$ denotes the inclusion map. Obviously, the map $f \mapsto (f_{(H,\varphi)})$ is F -linear and $\text{vcan}^{-1} \circ f_{(1,1)} = f$. Therefore, it suffices to show that for each $g \in \text{Hom}_{\mathcal{B}_F(G)}(\mathcal{J}(M), \mathcal{I}(V))$, there exists $f \in \text{Hom}_{FG}(\mathcal{V}(M), V)$ such that $f_{(H,\varphi)} = g_{(H,\varphi)}$ for all $(H,\varphi) \in \mathcal{M}$. For given g , the diagram

$$\begin{array}{ccc}
M & \xrightarrow{f} & V \\
\text{id}_M \downarrow & & \downarrow \text{Br}_{(1,1)}^V \\
M^{((1,1))} & \xrightarrow{g_{(1,1)}} & \overline{V}(1,1)
\end{array}$$

commutes for a unique $f \in \text{Hom}_{FG}(\mathcal{V}(M), V)$, since $\text{Br}_{(1,1)}^V = \text{vcan}$ is an isomorphism. Let $(H,\varphi) \in \mathcal{M}$ and define $f_{(H,\varphi)}$ as in the statement of the proposition. Then the diagram

$$\begin{array}{ccccc}
& & & & V^{(H,\varphi)} \xrightarrow{\text{Br}_{(H,\varphi)}^V} \overline{V}^{(H,\varphi)} \\
& & & \nearrow f^{(H,\varphi)} & \nearrow \overline{f}^{(H,\varphi)} \\
& & & & \overline{\text{vcan}}^{(H,\varphi)} \\
M^{((H,\varphi))} & \xrightarrow{\quad} & M^{(H,\varphi)} & \xrightarrow{\text{Br}_{(H,\varphi)}^M} & \overline{M}^{(H,\varphi)} \\
\downarrow g^{(H,\varphi)} & & \downarrow \overline{g}_{(1,1)}^{(H,\varphi)} & & \downarrow \\
\overline{V}^{(H,\varphi)} & \xrightarrow{\text{vts}_{(1,1)}^{(H,\varphi)}} & \overline{V}^{(1,1)}(H,\varphi) & &
\end{array}$$

commutes, and since $\overline{\text{vcan}}^{(H,\varphi)}$ and $\text{vts}_{(1,1)}^{(H,\varphi)}$ are inverse isomorphisms, we obtain $g^{(H,\varphi)} = \overline{f}^{(H,\varphi)}$. \square

We conclude this section with relating the category of Brauer sheaves to other similar categories one of which (the one in part (b)) we will use later.

2.12 Remark (a) Let $\mathcal{F} \in \mathcal{B}_F(G)$. Note that if $(I, \psi) \trianglelefteq (H, \varphi)$ are in \mathcal{M} with $p \nmid [H : I]$, then $\text{ts}_{(I,\psi)}^{(H,\varphi)}$ is determined by $\text{res}_{(I,\psi)}^{(H,\varphi)}$ by Axiom 2.1(ix) with $(J, \lambda) = (K, \mu) = (I, \psi)$. More generally, for arbitrary $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} , the transitivity map $\text{ts}_{(I,\psi)}^{(H,\varphi)}$ is determined by $\text{res}_{(P,\varphi)}^{(H,\varphi)}$ and $\text{ts}_{(I,\psi)}^{(P,\varphi)}$, where P/I is a Sylow p -subgroup of H/I . In fact, this is immediate from Axiom 2.1(ix) with $(J, \lambda) = (P, \varphi)$ and $(K, \mu) = (I, \psi)$.

One could define a category $\mathcal{B}_F^p(G)$ with the same data as for $\mathcal{B}_F(G)$, except that one assumes transitivity maps $\text{ts}_{(I,\psi)}^{(H,\varphi)}$ only in the case where $(I, \psi) \trianglelefteq (H, \varphi)$ are in \mathcal{M} such that H/I is a p -group, and with the same axioms (i)-(ix) as in Definition 2.1 and the same morphisms, mutatis mutandis.

Then there is an obvious forgetful functor $\mathcal{B}_F(G) \rightarrow \mathcal{B}_F^p(G)$. One can show that this functor is an isomorphism. In fact, given $\mathcal{F} \in \mathcal{B}_F^p(G)$ can one define transitivity maps $\text{ts}_{(I,\psi)}^{(H,\varphi)}$ for general $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} by completing the diagram

$$\begin{array}{ccccc}
\mathcal{F}_{(H,\varphi)} & \xrightarrow{\text{res}_{(P,\varphi)}^{(H,\varphi)}} & \mathcal{F}_{(P,\varphi)} & \xrightarrow{\text{ts}_{(I,\psi)}^{(P,\varphi)}} & \overline{\mathcal{F}}_{(I,\psi)}^{(P,\varphi)} \\
& \searrow \text{ts}_{(I,\psi)}^{(H,\varphi)} & & \uparrow \mathcal{F}_{(I,\psi)} \text{res}_{(P,\varphi)}^{(H,\varphi)} & \\
& & & & \overline{\mathcal{F}}_{(I,\psi)}^{(H,\varphi)}
\end{array}$$

where P/I is a Sylow p -subgroup of H/I . In fact, the image of the injective map $\mathcal{F}_{(I,\psi)} \text{res}_{(P,\varphi)}^{(H,\varphi)}$ is equal to $\overline{\mathcal{F}}_{(I,\psi)}^{(P,\varphi)}(N_H(P), \varphi)$ (by Proposition 1.9 applied to $V = F_{\varphi^{-1}} \otimes \mathcal{F}_{(I,\psi)} \in F[H/I]\text{mod}$) which in turn contains the image of the horizontal map in the above diagram. Using the conjugation maps, one sees that $\text{ts}_{(I,\psi)}^{(H,\varphi)}$ does not depend on the choice of P . It is a lengthy but straight forward exercise to show that with this definition of $\text{ts}_{(I,\psi)}^{(H,\varphi)}$ one obtains an element in $\mathcal{B}_F(G)$ and that the resulting functor is an inverse of the forgetful functor, so that the categories $\mathcal{B}_F(G)$ and $\mathcal{B}_F^p(G)$ are isomorphic.

(b) In [Bo97, 3.9] we introduced another sheaf like category, $\text{Sh}_F(G)$, related to the poset \mathcal{M} : The objects are families of F -vectors spaces $\mathcal{F}_{(H,\varphi)}$, one for each $(H,\varphi) \in \mathcal{M}$, together with conjugation and restriction maps as in Definition 2.1, except that one requires restriction maps for *all* $(I,\psi) \leq (H,\varphi)$ in \mathcal{M} . There are no transitivity maps required. The conjugation and restriction maps are required to satisfy all the axioms in Definition 2.1 that do not involve transitivity maps. There is a functor

$$\mathcal{B}: \text{Sh}_F(G) \rightarrow \mathcal{B}_F(G)$$

given by defining transitivity maps via

$$\text{ts}_{(I,\psi)}^{(H,\varphi)} := \mathcal{F}\text{ts}_{(I,\psi)}^{(H,\varphi)} : \mathcal{F}_{(H,\varphi)} \xrightarrow{\text{res}_{(I,\psi)}^{(H,\varphi)}} (\mathcal{F}_{(I,\psi)})^{(H,\varphi)} \xrightarrow{\text{Br}_{(H,\varphi)}^{\mathcal{F}_{(I,\psi)}}} \overline{\mathcal{F}_{(I,\psi)}}(H,\varphi)$$

for all $(I,\psi) \leq (H,\varphi)$ in \mathcal{M} . Combining 3.1 and 3.9 in [Bo97] we obtain a functor $\mathcal{J}: {}_{FG}\text{mon} \rightarrow \text{Sh}_F(G)$ that takes $M \in {}_{FG}\text{mon}$ to the collection $M^{((H,\varphi))}$, $(H,\varphi) \in \mathcal{M}$, of F -spaces, together with the obvious conjugation and restriction maps. It is now clear that the diagram

$$\begin{array}{ccc} & & \text{Sh}_F(G) \\ & \nearrow \mathcal{J} & \downarrow \mathcal{B} \\ {}_{FG}\text{mon} & & \mathcal{B}_F(G) \\ & \searrow \mathcal{J} & \end{array}$$

is commutative. Recall from [Bo97, 3.3] that the objects $\mathcal{J}(M) \in \text{Sh}_F(G)$, $M \in {}_{FG}\text{mon}$, are projective in the abelian category $\text{Sh}_F(G)$. Unfortunately, in general this is not true for $\mathcal{J}(M) \in \mathcal{B}_F(G)$. Recall also from [Bo97, 3.2] that $\mathcal{J}: {}_{FG}\text{mon} \rightarrow \text{Sh}_F(G)$ is bijective on morphism sets, and therefore is injective on isomorphism classes of objects.

3 Monomial resolutions of Brauer sheaves

3.1 Definition Let $\mathcal{F} \in \mathcal{B}_F(G)$. A *monomial resolution* of \mathcal{F} is a chain complex

$$M_*: \quad \cdots \xrightarrow{\partial_2} M_2 \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0$$

in ${}_{FG}\text{mon}$ together with a morphism $\varepsilon: \mathcal{J}(M_0) \rightarrow \mathcal{F}$ in $\mathcal{B}_F(G)$ such that

$$\mathcal{J}(M_*) \xrightarrow{\varepsilon} \mathcal{F} \longrightarrow 0$$

is an exact sequence in $\mathcal{B}_F(G)$; i.e., for each $(H,\varphi) \in \mathcal{M} = \mathcal{M}_F(G)$, the sequence

$$\cdots \xrightarrow{\partial_1^{((H,\varphi))}} M_1^{((H,\varphi))} \xrightarrow{\partial_0^{((H,\varphi))}} M_0^{((H,\varphi))} \xrightarrow{\varepsilon_{(H,\varphi)}} \mathcal{F}_{(H,\varphi)} \longrightarrow 0 \quad (3.1.a)$$

of $FN_G(H,\varphi)$ -modules is exact.

A monomial resolution as above is called *finite*, if there exists $n \in \mathbb{N}$ such that $M_{n+i} = 0$ for all $i \in \mathbb{N}$. And it is called *split*, if the chain complex in (3.1.a) is split exact in $FN_G(H,\varphi)\text{mod}$ for each $(H,\varphi) \in \mathcal{M}$, i.e., the kernel of each map is an $FN_G(H,\varphi)$ -direct summand in the corresponding term of the sequence.

A *monomial resolution* of an FG -module V is a monomial resolution of $\mathcal{I}(V)$.

3.2 Remark Let $V \in {}_{FG}\text{mod}$ and let $\mathcal{J}(M_*) \rightarrow \mathcal{I}(V)$ be a monomial resolution of V .

(a) Recall from Proposition 2.11 that

$$\begin{aligned} \text{Hom}_{FG}(\mathcal{V}(M_0), V) &\cong \text{Hom}_{\mathcal{B}_F(G)}(\mathcal{J}(M_0), \mathcal{I}(V)) \\ \varepsilon &\mapsto (\text{Br}_{(H,\varphi)}^V \circ \varepsilon_{(H,\varphi)})_{(H,\varphi) \in \mathcal{M}} \end{aligned}$$

so that the exact chain complex (3.1.a) is given by

$$\dots \xrightarrow{\partial_1^{((H,\varphi))}} M_1^{((H,\varphi))} \xrightarrow{\partial_0^{((H,\varphi))}} M_0^{((H,\varphi))} \xrightarrow{\varepsilon^{(H,\varphi)}} \overline{V}(H,\varphi) \longrightarrow 0 \quad (3.2.a)$$

if $\varepsilon \in \text{Hom}_{FG}(\mathcal{V}(M_0), V)$ is the corresponding FG -homomorphism. In particular, with $(H, \varphi) = (1, 1)$, the sequence

$$\dots \xrightarrow{\mathcal{V}(\partial_1)} \mathcal{V}(M_1) \xrightarrow{\mathcal{V}(\partial_0)} \mathcal{V}(M_0) \xrightarrow{\varepsilon} V \longrightarrow 0$$

is an exact chain complex of FG -modules, explaining the term ‘monomial resolution’. It will follow from Corollary 3.10 that every finite monomial resolution of V is automatically a split monomial resolution. In particular, since V then is a direct summand of $\mathcal{V}(M_0)$, finite monomial resolutions exist, if at all, only for trivial source modules.

(b) Note that the above definition of a monomial resolution of $V \in {}_{FG}\text{triv}$ differs from the one given in [Bo97]. There, we used the generalized fixed points $V^{(H,\varphi)}$ to obtain a sheaf and resolve it. This was appropriate for an interpretation of the canonical Brauer induction formula as being induced by a monomial resolution on the level of Grothendieck groups, since the induction formula for complex representations involves the dimensions of the subspaces $V^{(H,\varphi)}$. Here, we would like to find a similar interpretation for the canonical induction formula for trivial source modules. However, in this induction formula, the dimensions of $\overline{V}(H,\varphi)$ play the corresponding role.

In the sequel we explain the connection between monomial resolutions and the canonical induction formula for trivial source modules in more detail. For this purpose we need to recall the following notions and facts from [Bo98b, Sections 1, 2] and [Bo97, Section 5]. Let $T(G)$ (resp. $T_+^{\text{ab}}(G)$) denote the Grothendieck group of the category ${}_{FG}\text{triv}$ (resp. ${}_{FG}\text{mon}$) with respect to direct sums. Then $T(G)$ can be regarded as the free abelian group on the set of isomorphism classes $[V]$ of indecomposable trivial source FG -modules V . If a module $V \in {}_{FG}\text{triv}$ decomposes as $V = V_1 \oplus \dots \oplus V_n$ into indecomposable direct summands V_1, \dots, V_n , then we write $[V] := [V_1] + \dots + [V_n] \in T(G)$. Moreover, $T_+^{\text{ab}}(G)$ can be regarded as the free abelian group on the set of G -orbits $[H, \varphi]_G$ of elements $(H, \varphi) \in \mathcal{M}$, cf. [Bo97, 5.1]. More precisely, if $M \in {}_{FG}\text{mon}$ is such that G acts transitively on the lines of M and if (H, φ) is the stabilizing pair of a line of M , then we set $[M] := [H, \varphi]_G \in T_+^{\text{ab}}(G)$. And if $M = M_1 \oplus \dots \oplus M_n$ with ‘transitive’ $M_i \in {}_{FG}\text{mon}$, then we set $[M] := [M_1] + \dots + [M_n] \in T_+^{\text{ab}}(G)$. Tensor products induce ring structures on $T(G)$ and $T_+^{\text{ab}}(G)$. Restriction from G to H induces a ring homomorphism $\text{res}_+^G: T_+^{\text{ab}}(G) \rightarrow T_+^{\text{ab}}(H)$ for every $H \leq G$. The subring $T^{\text{ab}}(G)$ of $T(G)$ generated by the elements $[F_\varphi]$, $\varphi \in \widehat{G}$, is isomorphic to the group ring $\mathbb{Z}\widehat{G}$. There is a ring homomorphism

$$\pi_G: T_+^{\text{ab}}(G) \rightarrow T^{\text{ab}}(G), \quad [H, \varphi]_G \mapsto \begin{cases} \varphi, & \text{if } H = G, \\ 0, & \text{if } H < G. \end{cases}$$

Note that

$$(\pi_H \circ \text{res}_+^G)([M]) = \sum_{\varphi \in \widehat{H}} (\dim_F M^{((H,\varphi))}) \cdot \varphi \quad (3.2.b)$$

for all $M \in {}_{FG}\text{mon}$ and all $(H, \varphi) \in \mathcal{M}$.

If $V \in {}_{FG}\text{triv}$ and $\mathcal{J}(M_*) \rightarrow \mathcal{I}(V)$ is a finite split monomial resolution of V , then the *Lefschetz element*

$$\Lambda(M_*) := \sum_{n \in \mathbb{N}_0} (-1)^n [M_n] \in T_+^{\text{ab}}(G)$$

of M_* coincides with the image

$$a_G([V]) \in T_+^{\text{ab}}(G)$$

of $[V] \in T(G)$ under the canonical induction formula $a_G: T(G) \rightarrow T_+^{\text{ab}}(G)$ introduced in [Bo98b, Examples 1.8(c), 6.13]. In fact, $a_G([V])$ is characterized as the element in $\mathbb{Q} \otimes T_+^{\text{ab}}(G)$ satisfying

$$(\pi_H \circ \text{res}_+^G)(a_G([V])) = \sum_{\varphi \in \widehat{H}} (\dim_F \overline{V}(H, \varphi)) \cdot \varphi \in T^{\text{ab}}(H), \quad (3.2.c)$$

for all $H \leq G$, since $\dim_F \overline{V}(H, \varphi)$ is the multiplicity of F_φ as a direct summand in $\text{Res}_H^G(V)$, cf. Lemma 1.2. On the other hand, by Equation (3.2.b), also

$$\begin{aligned} (\pi_H \circ \text{res}_+^G)(\Lambda(M_*)) &= \sum_{n \in \mathbb{N}_0} (-1)^n (\pi_H \circ \text{res}_+^G)([M_n]) \\ &= \sum_{n \in \mathbb{N}_0} (-1)^n \sum_{\varphi \in \widehat{H}} (\dim_F M_n^{((H, \varphi))}) \cdot \varphi \\ &= \sum_{\varphi \in \widehat{H}} \left(\sum_{n \in \mathbb{N}_0} (-1)^n \dim_F M_n^{((H, \varphi))} \right) \cdot \varphi \\ &= \sum_{\varphi \in \widehat{H}} \dim_F \overline{V}(H, \varphi) \cdot \varphi, \end{aligned}$$

where the last equality follows from the exactness of the chain complex (3.2.a).

It will turn out that not every trivial source FG -module can have monomial resolutions in the sense of Definition 3.1. Those trivial source modules which have monomial resolutions are characterized by admitting the following form of filtration.

3.3 Definition Let $V \in {}_{FG}\text{mod}$. A *Brauer filtration* of V is a collection of F -subspaces $V(H, \varphi)$ of V , $(H, \varphi) \in \mathcal{M}$, such that

- (i) $gV(H, \varphi) = V^g(H, \varphi)$ for all $g \in G$ and $(H, \varphi) \in \mathcal{M}$.
- (ii) $V(H, \varphi) \subseteq V(I, \psi)$ for all $(I, \psi) \leq (H, \varphi)$ in \mathcal{M} .
- (iii) $V(H, \varphi) \oplus \ker(\text{Br}_{(H, \varphi)}^V) = V^{(H, \varphi)}$ for all $(H, \varphi) \in \mathcal{M}$.

3.4 Remark (a) Not every trivial source module has a Brauer filtration as the counter example in [Bc98, 6.2.5] shows.

(b) Note that each Brauer filtration of $V \in {}_{FG}\text{mod}$ gives rise to a sheaf $(V(H, \varphi))_{(H, \varphi) \in \mathcal{M}}$ in $\text{Sh}_F(G)$ such that its image under the functor \mathcal{B} is isomorphic to $\mathcal{I}(V)$ (cf. Remark 2.12(a)). In fact, the isomorphism is given by the restricted Brauer maps $\text{Br}_{(H, \varphi)}^V: V(H, \varphi) \rightarrow \overline{V}(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$. Two Brauer filtrations $V(H, \varphi)$ and $V'(H, \varphi)$ of V give rise to isomorphic sheaves, if and only if there exists an FG -module isomorphism $f: V \rightarrow V$ such that $f(V(H, \varphi)) = V'(H, \varphi)$ for all $(H, \varphi) \in \mathcal{M}$. This raises the question: Are any two Brauer filtrations of V isomorphic in the above sense?

3.5 Proposition *If $V \in {}_{FG}\text{mod}$ has a monomial resolution then V has a Brauer filtration.*

Proof Assume that V has a monomial resolution

$$\cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\varepsilon} V \longrightarrow 0.$$

Then we claim that the subspaces $V(H, \varphi) := \varepsilon(M_0^{((H, \varphi))})$, $(H, \varphi) \in \mathcal{M}$, of V form a Brauer filtration of V . By the G -linearity of ε , property (i) in Definition 3.3 holds. Moreover, property (ii) holds obviously by construction. Since the composition

$$M_0^{((H, \varphi))} \xrightarrow{\varepsilon} V^{(H, \varphi)} \xrightarrow{\text{Br}_{(H, \varphi)}^V} \overline{V}(H, \varphi)$$

is surjective, we obtain $V(H, \varphi) + \ker(\text{Br}_{(H, \varphi)}^V) = V^{(H, \varphi)}$, for all $(H, \varphi) \in \mathcal{M}$. Now assume that $v \in V(H, \varphi) \cap \ker(\text{Br}_{(H, \varphi)}^V)$. Then $v = \varepsilon(m_0)$ for some $m_0 \in M_0^{((H, \varphi))}$ and $\text{Br}_{(H, \varphi)}^V(\varepsilon^{(H, \varphi)}(m_0)) = 0 \in \overline{V}(H, \varphi)$. By the exactness of (3.2.a) there exists $m_1 \in M_1^{((H, \varphi))}$ such that $m_0 = \partial_1(m_1)$. Thus, $v = \varepsilon(m_0) = \varepsilon(\partial_1(m_1)) = 0$ since (3.2.a) is also a chain complex for $(H, \varphi) = (1, 1)$. \square

If $V \in {}_{FG}\text{mod}$ has a monomial resolution, we call the Brauer filtration $V(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$, defined in the preceding proof the Brauer filtration *induced* by the monomial resolution.

3.6 Lemma Let $(I, \psi) \in \mathcal{M}$, let $(H, \varphi) \in \mathcal{M}_F(N_G(I, \psi))$ such that $\varphi|_{H \cap I} = \psi|_{H \cap I}$, and let $\lambda \in \widehat{HI}$ denote the simultaneous extension of $\psi \in \widehat{I}$ and $\varphi \in \widehat{H}$.

(a) If $V \in {}_F G \text{ mod}$ satisfies $V = V^{(I, \psi)}$, then

$$V^{(IH, \lambda)} = V^{(H, \varphi)}, \quad \ker(\text{Br}_{(IH, \lambda)}^V) = \ker(\text{Br}_{(H, \varphi)}^V),$$

and

$$\overline{V}(IH, \lambda) = \overline{V}(H, \varphi) = \overline{V^{(I, \psi)}}(H, \varphi).$$

(b) If $\mathcal{F}, \mathcal{G} \in \text{Sh}_F(G)$ (resp. $\mathcal{F}, \mathcal{G} \in \mathcal{B}_F(G)$), and $\varepsilon \in \text{Hom}_{\text{Sh}_F(G)}(\mathcal{G}, \mathcal{F})$ (resp. $\varepsilon \in \text{Hom}_{\mathcal{B}_F(G)}(\mathcal{G}, \mathcal{F})$), then the diagram

$$\begin{array}{ccc} \mathcal{G}_{(IH, \lambda)} & \xrightarrow{\varepsilon_{(IH, \lambda)}} & \mathcal{F}_{(IH, \lambda)} \\ \text{gts}_{(I, \psi)}^{(IH, \lambda)} \downarrow & & \downarrow \text{fts}_{(I, \psi)}^{(IH, \lambda)} \\ \overline{\mathcal{G}}_{(I, \psi)}(IH, \lambda) & \xrightarrow{\overline{\varepsilon}_{(I, \psi)}(IH, \lambda)} & \overline{\mathcal{F}}_{(I, \psi)}(IH, \lambda) \\ \text{id} \downarrow & & \downarrow \text{id} \\ \overline{\mathcal{G}}_{(I, \psi)}(H, \varphi) & \xrightarrow{\overline{\varepsilon}_{(I, \psi)}(H, \varphi)} & \overline{\mathcal{F}}_{(I, \psi)}(H, \varphi) \end{array}$$

commutes.

Proof (a) Obviously, $V^{(IH, \lambda)} = V^{(H, \varphi)}$.

Next we show that $\ker(\text{Br}_{(IH, \lambda)}^V) = \ker(\text{Br}_{(H, \varphi)}^V)$. Let $J < IH$ with $p \mid [IH : J]$ and let $v \in V^{(J, \lambda)} = V^{(IJ, \lambda)}$. Then

$$\text{tr}_{(J, \lambda)}^{(IH, \lambda)}(v) = \text{tr}_{(IJ, \lambda)}^{(IH, \lambda)}(\text{tr}_{(J, \lambda)}^{(IJ, \lambda)}(v)) = \text{tr}_{(IJ, \lambda)}^{(IH, \lambda)}([IJ : J]v) = [IJ : J] \text{tr}_{(H \cap IJ, \varphi)}^{(H, \varphi)}(v).$$

If $p \mid [IJ : J]$, then $\text{tr}_{(J, \lambda)}^{(IJ, \lambda)}(v) = 0$. If $p \nmid [IJ : J]$, then $p \mid [IH : IJ] = [H : H \cap IJ]$, and we obtain $\text{tr}_{(J, \lambda)}^{(IH, \lambda)}(v) \in \ker(\text{Br}_{(H, \varphi)}^V)$. Conversely, let $J < H$ with $p \mid [H : J]$ and $v \in V^{(J, \varphi)} = V^{(IJ, \lambda)}$. Then

$$\text{tr}_{(J, \varphi)}^{(H, \varphi)}(v) = \text{tr}_{(IJ \cap H, \varphi)}^{(H, \varphi)}(\text{tr}_{(J, \varphi)}^{(IJ \cap H, \varphi)}(v)) = \text{tr}_{(IJ \cap H, \varphi)}^{(H, \varphi)}([IJ \cap H : J]v) = [IJ \cap H : J] \text{tr}_{(IJ, \lambda)}^{(IH, \lambda)}(v).$$

If $p \mid [IJ \cap H : J]$ then $\text{tr}_{(J, \varphi)}^{(H, \varphi)}(v) = 0$. If $p \nmid [IJ \cap H : J]$, then $p \mid [H : IJ \cap H] = [IH : IJ]$ and we obtain $\text{tr}_{(J, \varphi)}^{(H, \varphi)}(v) \in \ker(\text{Br}_{(IH, \lambda)}^V)$. The last assertion in part (a) is now immediate.

(b) This is obvious with (a). \square

3.7 Proposition Assume that $V \in {}_F G \text{mod}$ has a Brauer filtration $V(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$, and denote by $\mathcal{F} \in \text{Sh}_F(G)$ the associated sheaf.

(a) For any $(H, \varphi) \in \mathcal{M}$ one has

$$\ker(\text{Br}_{(H, \varphi)}^V) = \sum_{\substack{(I, \psi) \leq (H, \varphi) \\ p \mid [H : I]}} \text{tr}_{(I, \psi)}^{(H, \varphi)}(V(I, \psi))$$

(b) For any pairs $(I, \psi) \trianglelefteq (H, \varphi)$ and $(J, \lambda) \leq (H, \varphi)$ in \mathcal{M} with $p \mid [H : J]$ and any $v \in V(J, \lambda)$ one has

$$\text{tr}_{(J, \lambda)}^{(H, \varphi)}(v) \in \ker(\text{Br}_{(H, \varphi)}^{V(I, \psi)}) \oplus \ker(\text{Br}_{(I, \psi)}^V)^{(H, \varphi)}.$$

(c) For any pair $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} one has

$$V(I, \psi)^{(H, \varphi)} \cap \ker(\text{Br}_{(H, \varphi)}^V) = \ker(\text{Br}_{(H, \varphi)}^{V(I, \psi)}) \quad (3.7.a)$$

and

$$V(I, \psi)^{(H, \varphi)} = V(H, \varphi) \oplus \ker(\text{Br}_{(H, \varphi)}^{V(I, \psi)}). \quad (3.7.b)$$

In particular,

$$\mathcal{F}\text{ts}_{(I, \psi)}^{(H, \varphi)} = \text{Br}_{(H, \varphi)}^{V(I, \psi)} : V(H, \varphi) \rightarrow \overline{V(I, \psi)}(H, \varphi)$$

is bijective.

(d) For any $(I, \psi) \in \mathcal{M}$, the $FN_G(I, \psi)$ -module $W := \overline{V}(I, \psi)$ has the Brauer filtration

$$W(H, \varphi) := \begin{cases} \text{Br}_{(I, \psi)}^V(V(HI, \varphi * \psi)), & \text{if } \psi|_{H \cap I} = \varphi|_{H \cap I}, \\ 0, & \text{if } \psi|_{H \cap I} \neq \varphi|_{H \cap I}, \end{cases}$$

for $(H, \varphi) \in \mathcal{M}_F(N_G(I, \psi))$, where $\varphi * \psi \in \widehat{HI}$ denotes the unique simultaneous extension of $\varphi \in \widehat{H}$ and $\psi \in \widehat{I}$.

Proof (a) This follows immediately by induction on $|H|$ and use of the transitivity of the trace maps.

(b) We have

$$\text{tr}_{(J, \lambda)}^{(H, \varphi)}(v) = \text{tr}_{(IJ, \varphi)}^{(H, \varphi)}(\text{tr}_{(J, \lambda)}^{(IJ, \varphi)}(v)) = \text{tr}_{(IJ, \varphi)}^{(H, \varphi)}(\text{tr}_{(I \cap J, \lambda)}^{(I, \psi)}(v))$$

with

$$\text{tr}_{(I \cap J, \lambda)}^{(I, \psi)}(v) \in (V(I, \psi))^{(IJ, \varphi)} = V(I, \psi)^{(IJ, \varphi)} \oplus \ker(\text{Br}_{(I, \psi)}^V)^{(IJ, \varphi)}.$$

Since $p \mid [H : J]$, we have $p \mid [H : IJ]$ or $p \mid [IJ : J] = [I : I \cap J]$. So, if $p \mid [H : IJ]$, then

$$\begin{aligned} \text{tr}_{(J, \lambda)}^{(H, \varphi)}(v) &\in \text{tr}_{(IJ, \varphi)}^{(H, \varphi)}(V(I, \psi)^{(IJ, \varphi)}) + \text{tr}_{(IJ, \varphi)}^{(H, \varphi)}(\ker(\text{Br}_{(I, \psi)}^V)^{(IJ, \varphi)}) \\ &\subseteq \ker(\text{Br}_{(H, \varphi)}^{V(I, \psi)}) + \ker(\text{Br}_{(I, \psi)}^V)^{(H, \varphi)}, \end{aligned}$$

and if $p \mid [I : I \cap J]$, then

$$\begin{aligned} \text{tr}_{(J, \lambda)}^{(H, \varphi)}(v) &= \text{tr}_{(IJ, \varphi)}^{(H, \varphi)}(\text{tr}_{(I \cap J, \psi)}^{(I, \psi)}(v)) \in \text{tr}_{(IJ, \varphi)}^{(H, \varphi)}(\ker(\text{Br}_{(I, \psi)}^V)^{(IJ, \varphi)}) \\ &\subseteq \ker(\text{Br}_{(I, \psi)}^V)^{(H, \varphi)}. \end{aligned}$$

(c) Obviously, the right hand side of Equation (3.7.a) is contained in the left hand side. So let $v \in V(I, \psi)^{(H, \varphi)} \cap \ker(\text{Br}_{(H, \varphi)}^V)$. Then, by part (a) and (b), we can write $v = x + y$ with $x \in \ker(\text{Br}_{(H, \varphi)}^{V(I, \psi)})$ and $y \in \ker(\text{Br}_{(I, \psi)}^V)^{(H, \varphi)}$. Since

$$V^{(H, \varphi)} = (V(I, \psi))^{(H, \varphi)} = V(I, \psi)^{(H, \varphi)} \oplus \ker(\text{Br}_{(I, \psi)}^V)^{(H, \varphi)}$$

is a direct sum and since $x, v \in V(I, \psi)^{(H, \varphi)}$, we obtain $y = 0$ and $v = x \in \ker(\text{Br}_{(H, \varphi)}^{V(I, \psi)})$.

The last equation together with the equation

$$V^{(H, \varphi)} = V(H, \varphi) \oplus \ker(\text{Br}_{(H, \varphi)}^V)$$

implies

$$V(I, \psi)^{(H, \varphi)} = V(H, \varphi) \oplus (V(I, \psi)^{(H, \varphi)} \cap \ker(\text{Br}_{(H, \varphi)}^V)),$$

since $V(H, \varphi) \subseteq V(I, \psi)^{(H, \varphi)}$. Now, Equation (3.7.a) implies Equation (3.7.b).

(d) Obviously part (i) and (ii) of Definition 3.3 are satisfied for $W(H, \varphi)$, $(H, \varphi) \in \mathcal{M}_F(N_G(I, \psi))$. In order to show part (iii), let $(H, \varphi) \in \mathcal{M}_F(N_G(I, \psi))$. If $\psi|_{H \cap I} \neq \varphi|_{H \cap I}$, then $W^{(H, \varphi)} = 0$ and there is nothing to show. So we assume that $\psi|_{H \cap I} = \varphi|_{H \cap I}$. We have to show that

$$\text{Br}_{(I, \psi)}^V(V(HI, \varphi * \psi)) \oplus \ker(\text{Br}_{(H, \varphi)}^{\overline{V}(I, \psi)}) = \overline{V}(I, \psi)^{(H, \varphi)}.$$

Since $\text{Br}_{(I, \psi)}^V: V(I, \psi) \rightarrow \overline{V}(I, \psi)$ is an isomorphism, it suffices to show that

$$V(HI, \varphi * \psi) \oplus \ker(\text{Br}_{(H, \varphi)}^{V(I, \psi)}) = V(I, \psi)^{(H, \varphi)}.$$

By Lemma 3.6(a) we have

$$V(I, \psi)^{(H, \varphi)} = V(I, \psi)^{(HI, \varphi * \psi)} \quad \text{and} \quad \ker(\text{Br}_{(H, \varphi)}^{V(I, \psi)}) = \ker(\text{Br}_{(HI, \varphi * \psi)}^{V(I, \psi)})$$

so that we may assume that $(I, \psi) \leq (H, \varphi)$. Now the result follows immediately from Equation (3.7.b). \square

Recall that a chain complex

$$\cdots \longrightarrow V_{n+1} \xrightarrow{\partial_{n+1}} V_n \xrightarrow{\partial_n} V_{n-1} \longrightarrow \cdots$$

of R -modules (for an arbitrary ring R) is called *split*, if at each degree $n \in \mathbb{Z}$, the R -submodule $\text{im}(\partial_{n+1})$ is an R -module direct summand of $\ker(\partial_n)$ and $\ker(\partial_n)$ is an R -module direct summand of V_n . The above chain complex is called *split exact* if it is exact and split.

Moreover, we call a chain complex

$$\cdots \longrightarrow \mathcal{F}_{n+1} \xrightarrow{\partial_{n+1}} \mathcal{F}_n \xrightarrow{\partial_n} \mathcal{F}_{n-1} \longrightarrow \cdots$$

in $\mathcal{B}_F(G)$ or $\text{Sh}_F(G)$ *locally split*, if its evaluation at each $(H, \varphi) \in \mathcal{M}$ is split as a chain complex of $FN_G(H, \varphi)$ -modules.

3.8 Lemma *Let*

$$0 \longrightarrow V_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} V_0 \longrightarrow 0 \tag{3.8.a}$$

be a chain complex in $FG\text{mod}$.

(i) *If $V_0, \dots, V_{n-1} \in FG\text{triv}$ and if the chain complex*

$$0 \longrightarrow \overline{V}_n(P) \xrightarrow{\overline{\partial}_n(P)} \cdots \xrightarrow{\overline{\partial}_1(P)} \overline{V}_0(P) \longrightarrow 0 \tag{3.8.b}$$

is exact for all p -subgroups P of G , then also $V_n \in FG\text{triv}$ and the chain complex (3.8.a) is split exact.

(ii) *If $V_1, \dots, V_n \in FG\text{triv}$ and if the chain complex (3.8.b) is exact for all p -subgroups P of G , then also $V_0 \in FG\text{triv}$ and the chain complex (3.8.a) is split exact.*

(iii) *If $V_0, \dots, V_n \in FG\text{triv}$ and if the chain complex*

$$\overline{V}_n(P) \xrightarrow{\overline{\partial}_n(P)} \cdots \xrightarrow{\overline{\partial}_1(P)} \overline{V}_0(P) \longrightarrow 0 \tag{3.8.c}$$

is exact for all p -subgroups P of G , then the chain complex (3.8.a) is split. In particular, $\ker(\partial_n) \in {}_{FG}\text{triv}$.

(iv) If $V_0, \dots, V_n \in {}_{FG}\text{triv}$ and if the chain complex

$$0 \longrightarrow \overline{V}_n(P) \xrightarrow{\overline{\partial}_n(P)} \dots \xrightarrow{\overline{\partial}_1(P)} \overline{V}_0(P) \quad (3.8.d)$$

is exact for all p -subgroups P of G , then the chain complex (3.8.a) is split. In particular, $\text{im}(\partial_1) \in {}_{FG}\text{triv}$.

Proof (i) Since the chain complex (3.8.b) is exact for $P = 1$, the chain complex (3.8.a) is exact. Since $\overline{\partial}_1(P)$ is surjective for all p -subgroups P of G , Proposition 6.4 in [Bc98] implies that the kernel $W_1 := \ker(\partial_1) = \text{im}(\partial_2)$ is an FG -direct summand of V_1 and therefore a trivial source FG -module. We obtain a chain complex

$$0 \longrightarrow V_n \xrightarrow{\partial_n} V_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_3} V_2 \xrightarrow{\partial_2} W_1 \longrightarrow 0 \quad (3.8.e)$$

such that

$$0 \rightarrow \overline{V}(P) \xrightarrow{\overline{\partial}_n(P)} \dots \xrightarrow{\overline{\partial}_3(P)} \overline{V}_2(P) \xrightarrow{\overline{\partial}_2(P)} \overline{W}_1(P) \longrightarrow 0$$

is exact for all p -subgroups P of G . We can now argue by induction with the shorter chain complex (3.8.e), the cases $n = 1$ and $n = 0$ being true for trivial reasons.

(ii) This is proved similarly, again using [Bc98, Proposition 6.4] in the dual formulation.

(iii) and (iv) are proved in a similar way as (i) and (ii). \square

3.9 Lemma Let $\mathcal{F} \in \text{Sh}_F(G)$ (resp. $\mathcal{F} \in \mathcal{B}_F(G)$) have the property that $\mathcal{F}\text{ts}_{(I,\psi)}^{(H,\varphi)} : \mathcal{F}_{(H,\varphi)} \rightarrow \overline{\mathcal{F}_{(I,\psi)}}(H,\varphi)$ is bijective for every pair $(I,\psi) \trianglelefteq (H,\varphi)$ of elements in \mathcal{M} such that H/I is a p -group. Moreover, let

$$M_n \xrightarrow{\partial_n} M_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} M_0$$

be a sequence of morphisms in ${}_{FG}\text{mon}$ and let $\varepsilon \in \text{Hom}_{\text{Sh}_F(G)}(\mathcal{J}(M_0), \mathcal{F})$ (resp. $\varepsilon \in \text{Hom}_{\mathcal{B}_F(G)}(\mathcal{J}(M_0), \mathcal{F})$).

(a) If

$$0 \longrightarrow \mathcal{J}(M_n) \xrightarrow{\mathcal{J}(\partial_n)} \dots \xrightarrow{\mathcal{J}(\partial_1)} \mathcal{J}(M_0) \xrightarrow{\varepsilon} \mathcal{F} \longrightarrow 0$$

is exact, then $\mathcal{F}_{(H,\varphi)} \in {}_{FN_G(H,\varphi)}\text{triv}$ for all $(H,\varphi) \in \mathcal{M}$.

(b) If $\mathcal{F}_{(H,\varphi)} \in {}_{FN_G(H,\varphi)}\text{triv}$ for all $(H,\varphi) \in \mathcal{M}$ and if

$$\mathcal{J}(M_n) \xrightarrow{\mathcal{J}(\partial_n)} \dots \xrightarrow{\mathcal{J}(\partial_1)} \mathcal{J}(M_0) \xrightarrow{\varepsilon} \mathcal{F} \longrightarrow 0$$

is exact, then it is locally split.

Proof (a) We fix $(I,\psi) \in \mathcal{M}$ and a p -subgroup P of $N_G(I,\psi)$. By Lemma 3.8(ii) it suffices to show that the chain complex

$$0 \longrightarrow \overline{M_n^{((I,\psi))}}(P) \xrightarrow{\overline{\partial_n^{((I,\psi))}}(P)} \dots \xrightarrow{\overline{\partial_1^{((I,\psi))}}(P)} \overline{M_0^{((I,\psi))}}(P) \xrightarrow{\overline{\varepsilon_{(I,\psi)}}(P)} \overline{\mathcal{F}_{(I,\psi)}}(P) \longrightarrow 0$$

is exact. But, by Lemma 3.6, this chain complex is isomorphic to the chain complex

$$0 \longrightarrow M_n^{((IP,\varphi))} \xrightarrow{\partial_n^{((IP,\varphi))}} \dots \xrightarrow{\partial_1^{((IP,\varphi))}} M_0^{((IP,\varphi))} \xrightarrow{\varepsilon_{(IP,\varphi)}} \mathcal{F}_{(IP,\varphi)} \longrightarrow 0,$$

where $\varphi \in \widehat{IP}$ denotes the unique extension of $\psi \in \widehat{I}$ to IP . By the hypothesis of the lemma, this sequence is exact.

(b) This is proved in a similar way as part (a) using Lemma 3.8(iii) and Lemma 3.6. \square

We obtain the following immediate corollary.

3.10 Corollary *Let $V \in {}_{FG}\text{mod}$.*

- (i) *If V has a finite monomial resolution, then $V \in {}_{FG}\text{triv}$.*
- (ii) *If $V \in {}_{FG}\text{triv}$, then every monomial resolution of V is split.*

Proof (i) This follows from Lemma 3.5, Proposition 3.7, and Lemma 3.9(a).
(ii) This follows from Lemma 3.9(b). □

In the next section we will show that $V \in {}_{FG}\text{mod}$ has a monomial resolution if and only if it has a Brauer filtration, and that it has a finite monomial resolution if and only if it additionally is a trivial source module. We will construct the resolution in the category $\text{Sh}_F(G)$ and thereby obtain a resolution in the sense of Definition 3.1.

4 Monomial resolutions of sheaves in $\text{Sh}_F(G)$

Similar to Definition 3.1 we define monomial resolutions of sheaves in $\text{Sh}_F(G)$.

4.1 Definition Let $\mathcal{F} \in \text{Sh}_F(G)$. A monomial resolution of \mathcal{F} is a chain complex

$$M_*: \quad \cdots \xrightarrow{\partial_2} M_2 \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0$$

in ${}_{FG}\text{mon}$ together with a morphism $\varepsilon: \mathcal{J}(M_0) \rightarrow \mathcal{F}$ in $\text{Sh}_F(G)$ such that

$$\mathcal{J}(M_*) \xrightarrow{\varepsilon} \mathcal{F} \longrightarrow 0$$

is an exact sequence in $\text{Sh}_F(G)$; i.e., for each $(H, \varphi) \in \mathcal{M} = \mathcal{M}_F(G)$, the sequence

$$\cdots \xrightarrow{\partial_1^{((H, \varphi))}} M_1^{((H, \varphi))} \xrightarrow{\partial_0^{((H, \varphi))}} M_0^{((H, \varphi))} \xrightarrow{\varepsilon_{(H, \varphi)}} \mathcal{F}_{(H, \varphi)} \longrightarrow 0 \quad (4.1.a)$$

of $FN_G(H, \varphi)$ -modules is exact.

4.2 Remark Note that if $M_* \xrightarrow{\varepsilon} \mathcal{F}$ is a monomial resolution of a sheaf $\mathcal{F} \in \text{Sh}_F(G)$ as in Definition 4.1, then it is at the same time a monomial resolution of the Brauer sheaf $\mathcal{B}(\mathcal{F}) \in \mathcal{B}_F(G)$ in the sense of Definition 3.1, cf. Remark 2.12(b).

In particular, if $V \in {}_{FG}\text{mod}$ has a Brauer filtration $V(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$, and if $\mathcal{F} \in \text{Sh}_F(G)$ is the resulting sheaf, then any monomial resolution of \mathcal{F} in the sense of Definition 4.1 provides a monomial resolution of $\mathcal{I}(V)$ and V in the sense of Definition 3.1, cf. Remark 3.4(b).

Conversely, every monomial resolution of $V \in {}_{FG}\text{mod}$ in the sense of Definition 3.1 comes this way. In fact, as pointed out in the proof of Proposition 3.5, a monomial resolution of V induces a Brauer filtration of V and is at the same time a monomial resolution of the induced sheaf in $\text{Sh}_F(G)$ in the sense of Definition 4.1.

4.3 Our next aim is to classify the sheaves $\mathcal{F} \in \text{Sh}_F(G)$ that have monomial resolutions, resp. (finite) split monomial resolutions. In order to proceed we need some more notation.

With the notation from Remark 3.2(b) we obtain a map

$$\rho_G^{T^{\text{ab}}} := (\pi_H \circ \text{res}_+^G)_{H \leq G}: T_+^{\text{ab}}(G) \rightarrow (T^{\text{ab}})^+(G) := \left(\prod_{H \leq G} T^{\text{ab}}(H) \right)^G$$

which was studied in [Bo98b, 2.3]. This map is called the *mark homomorphism*. It is an injective ring homomorphism with finite cokernel.

The following conditions on a sheaf $\mathcal{F} \in \text{Sh}_F(G)$ will turn out to be important:

- (i) For all $(H, \varphi) \in \mathcal{M}$, the $FN_G(H, \varphi)$ -module $\mathcal{F}_{(H, \varphi)}$ is a trivial source module.

(ii) For all $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} such that H/I is a p -group, the transitivity map

$$\mathcal{F}ts_{(I, \psi)}^{(H, \varphi)} = \text{Br}_{(H, \varphi)}^{\mathcal{F}_{(I, \psi)}} \circ \text{res}_{(I, \psi)}^{(H, \varphi)} : \mathcal{F}_{(H, \varphi)} \rightarrow (\mathcal{F}_{(I, \psi)})^{(H, \varphi)} \rightarrow \overline{\mathcal{F}_{(I, \psi)}}(H, \varphi)$$

is bijective.

(iii) For all $(I, \psi) \in \mathcal{M}$, all intermediate groups $I \leq H \leq N_G(I, \psi)$ such that H/I is a cyclic p' -group, and every p' -element $h \in H$ such that hI generates H/I , one has

$$s_h(\mathcal{F}_{(I, \psi)}) = \sum_{\substack{\varphi \in \widehat{H} \\ \varphi|_I = \psi}} s_h(\mathcal{F}_{(H, \varphi)}),$$

where $s_h(M)$ is the Brauer character of an FH -module M evaluated at h .

(iv) The element

$$\left(\sum_{\varphi \in \widehat{H}} \dim_F \mathcal{F}_{(H, \varphi)} \cdot \varphi \right)_{H \leq G} \in (T^{\text{ab}})^+(G)$$

is in the image of $\rho_G^{T^{\text{ab}}}$.

4.4 Lemma (a) For each $M \in {}_F G \text{mon}$, the sheaf $\mathcal{J}(M) \in \text{Sh}_F(G)$ satisfies conditions (i)–(iv) in 4.3.

(b) If $V \in {}_F G \text{triv}$ has a Brauer filtration, then the resulting sheaf $\mathcal{F} \in \text{Sh}_F(G)$ satisfies conditions (i)–(iv) in 4.3.

Proof (a) Condition (i) is clearly satisfied and condition (ii) was shown to hold in Proposition 2.6(b). Next let $(I, \psi) \in \mathcal{M}$, $H \leq N_G(I, \psi)$, and $h \in H$ be as in 4.3(iii). Then

$$M^{((I, \psi))} = \bigoplus_{\substack{\varphi \in \widehat{H} \\ \varphi|_I = \psi}} M^{((H, \varphi))} \oplus \bigoplus_{\substack{(I, \psi) \leq (U, \mu) \\ H \not\leq U}} M((U, \mu)),$$

and s_h vanishes on the last direct sum, since, for $(U, \mu) \geq (I, \psi)$ with $H \not\leq U$, the FH -submodule generated by a line in $M((U, \mu))$ is induced from the proper subgroup $U \cap H$ of H . From Equation (3.2.b) we immediately obtain condition (iv).

(b) Let $V(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$, be a Brauer filtration of V . Then $V(H, \varphi) \cong \overline{V}(H, \varphi) \in {}_F N_G(H, \varphi) \text{triv}$ by Lemma 1.2(iii), and condition (i) is satisfied. Condition (ii) is satisfied by Lemma 1.10(vi). Condition (iii) is satisfied, since

$$\overline{V}(I, \psi) = \bigoplus_{\substack{\varphi \in \widehat{H} \\ \varphi|_I = \psi}} \overline{V}(I, \psi)^{(H, \varphi)} \cong \bigoplus_{\substack{\varphi \in \widehat{H} \\ \varphi|_I = \psi}} \overline{V}(H, \varphi)$$

by Lemma 1.10(vii), whenever $(I, \psi) \trianglelefteq (H, \varphi)$ are elements in \mathcal{M} such that H/I is a p' -group. Finally, condition (iv) is satisfied, since the canonical induction formula for trivial source modules is integral, i.e., $a_G([V]) \in T_+^{\text{ab}}(G)$, and since $\rho_G^{T^{\text{ab}}}(a_G([V]))$ is given by Equation (3.2.c). \square

4.5 Lemma Let $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$ be a short exact sequence in $\text{Sh}_F(G)$. If \mathcal{G} and \mathcal{H} satisfy conditions (i)–(iv) in 4.3, then also \mathcal{F} does and the sequence is locally split.

Proof Obviously, conditions (iii) and (iv) are inherited by \mathcal{F} from \mathcal{G} and \mathcal{H} . Next we show that the map $\beta_{(I, \psi)} : \mathcal{G}_{(I, \psi)} \rightarrow \mathcal{H}_{(I, \psi)}$ splits for all $(I, \psi) \in \mathcal{M}$. In fact, let $P \leq N_G(I, \psi)$ be a p -subgroup, set $H := IP$, and let $\varphi \in \widehat{H}$ denote the unique extension of ψ to H . The vertical maps in the commutative diagram in 3.6(b) for the morphism β are isomorphisms by condition (ii). Therefore, with $\beta_{(H, \varphi)}$ also $\beta_{(I, \psi)}(P)$ is surjective. By Lemma 3.8(ii), the map $\beta_{(I, \psi)}$ splits as $FN_G(I, \psi)$ -module homomorphism. In particular, $\mathcal{F}_{(I, \psi)}$ is isomorphic to a direct summand of $\mathcal{G}_{(I, \psi)}$. Thus, \mathcal{F} satisfies condition (i) and the sequence $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$ is locally split. Finally, we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}_{(H,\varphi)} & \longrightarrow & \mathcal{G}_{(H,\varphi)} & \longrightarrow & \mathcal{H}_{(H,\varphi)} \longrightarrow 0 \\
& & \downarrow \mathcal{F}\text{ts}_{(I,\psi)}^{(H,\varphi)} & & \downarrow \mathcal{G}\text{ts}_{(I,\psi)}^{(H,\varphi)} & & \downarrow \mathcal{H}\text{ts}_{(I,\psi)}^{(H,\varphi)} \\
0 & \longrightarrow & \overline{\mathcal{F}}_{(I,\psi)}(H,\varphi) & \longrightarrow & \overline{\mathcal{G}}_{(I,\psi)}(H,\varphi) & \longrightarrow & \overline{\mathcal{H}}_{(I,\psi)}(H,\varphi) \longrightarrow 0
\end{array}$$

for all $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} . Since the top row is split exact for all $(H, \varphi) \in \mathcal{M}$, also the bottom row is exact. If H/I is a p -group, then the middle and right hand vertical maps are isomorphisms. This implies that also the left hand vertical map is an isomorphism, and \mathcal{F} satisfies condition (ii). \square

4.6 Proposition *Let $M \in {}_F G\text{mon}$ and let $\mathcal{F} \in \text{Sh}_F(G)$. Then the map*

$$\begin{aligned}
\text{Hom}_{\text{Sh}_F(G)}(\mathcal{J}(M), \mathcal{F}) &\rightarrow \left(\prod_{(H,\varphi) \in \mathcal{M}} \text{Hom}_F(M((H,\varphi)), \mathcal{F}_{(H,\varphi)}) \right)^G \\
(f_{(H,\varphi)})_{(H,\varphi) \in \mathcal{M}} &\mapsto (f_{(H,\varphi)}|_{M((H,\varphi))})_{(H,\varphi) \in \mathcal{M}}
\end{aligned}$$

is an isomorphism of F -vector spaces, where G acts on the direct product on the right hand side by the obvious conjugation. If \mathcal{R} is a set of representatives for the G -conjugacy classes of \mathcal{M} , then the image of the above map is isomorphic to

$$\prod_{(H,\varphi) \in \mathcal{R}} \text{Hom}_{F N_G(H,\varphi)}(M((H,\varphi)), \mathcal{F}_{(H,\varphi)}).$$

Proof Clearly, the map is F -linear and assumes values in the G -fixed points of the above direct product. We first show the injectivity of the map. Assume that $f_{(H,\varphi)}|_{M((H,\varphi))} = 0$ for all $(H, \varphi) \in \mathcal{M}$. Then $f_{(H,\varphi)} = 0$ for all $(H, \varphi) \in \mathcal{M}$, since for all $(H', \varphi') \in \mathcal{M}$ with $(H, \varphi) \leq (H', \varphi')$ one has

$$\begin{aligned}
f_{(H,\varphi)}(M((H', \varphi'))) &= f_{(H,\varphi)}(\text{res}_{(H,\varphi)}^{(H', \varphi')} (M((H', \varphi')))) \\
&= \text{res}_{(H,\varphi)}^{(H', \varphi')} (f_{(H', \varphi')} (M((H', \varphi')))) = 0.
\end{aligned}$$

Next assume that $g_{(H,\varphi)} \in \text{Hom}_F(M((H,\varphi)), \mathcal{F}_{(H,\varphi)})$, $(H, \varphi) \in \mathcal{M}$, is a G -equivariant family of maps. We define

$$f_{(H,\varphi)}: M((H,\varphi)) \rightarrow \mathcal{F}_{(H,\varphi)}, \quad (H, \varphi) \in \mathcal{M},$$

on the direct summand $M((H', \varphi'))$, for $(H', \varphi') \in \mathcal{M}$ with $(H, \varphi) \leq (H', \varphi')$, as

$$f_{(H,\varphi)}|_{M((H', \varphi'))} := \text{res}_{(H,\varphi)}^{(H', \varphi')} \circ g_{(H', \varphi')}.$$

Now it is easy to see that $f_{(H,\varphi)}$, $(H, \varphi) \in \mathcal{M}$, is in $\text{Hom}_{\text{Sh}_F(G)}(\mathcal{J}(M), \mathcal{F})$ and that it is mapped to the family $g_{(H,\varphi)}$, $(H, \varphi) \in \mathcal{M}$.

The last assertion in the proposition is obvious. \square

Recall from [Be84, §2.13] that for each p -hypo-elementary subgroup U of G (i.e., $U/O_p(U)$ is cyclic) and any p' -element $u \in U$ such that $U/O_p(U) = \langle uO_p(U) \rangle$, one has a ring homomorphism

$$s_{(U,u)}: T(G) \rightarrow \mathbb{C},$$

called the (U, u) -species, which is defined on $[W] \in T(G)$ for $W \in {}_F G\text{triv}$ as follows: Set $P := O_p(U)$ and decompose $\text{Res}_U^G(W) = W_1 \oplus W_2$ such that $\text{Res}_P^U(W_1) \cong F \oplus \cdots \oplus F$ and

such that each indecomposable direct summand of W_2 has vertex strictly smaller than P (this is equivalent to $F \nmid \text{Res}_P^U(W_2)$). Then $W_1 \cong \overline{W}(P)$ as FU -modules and $s_{(U,u)}([W])$ is defined as the Brauer character of W_1 evaluated at u . Recall that two trivial source FG -modules W and \widetilde{W} are isomorphic if and only if $s_{(U,u)}([W]) = s_{(U,u)}([\widetilde{W}])$ for all (U, u) as above.

4.7 Proposition *If $(H, \varphi) \in \mathcal{M}$ and $s_{(U,u)}$ is a species for $T(G)$, then*

$$s_{(U,u)}(\text{ind}_H^G([F_\varphi])) = \sum_{\substack{g \in U \backslash G/H \\ U \leqslant^g H}} \varphi(g^{-1}ug).$$

In particular, if $N_G(H, \varphi) = G$, then

$$s_{(U,u)}(\text{ind}_H^G([F_\varphi])) = \begin{cases} [G : H]\varphi(u), & \text{if } U \leqslant H, \\ 0, & \text{if } U \not\leqslant H. \end{cases}$$

Proof Let $P := O_p(U)$. Then

$$\text{Res}_U^G(\text{Ind}_H^G(F_\varphi)) \cong \bigoplus_{g \in U \backslash G/H} \text{Ind}_{U \cap {}^g H}^U(\text{Res}_{U \cap {}^g H}^{{}^g H}(F_{({}^g \varphi)}))$$

implies

$$\overline{\text{Ind}_H^G(F_\varphi)}(P) \cong \bigoplus_{\substack{g \in U \backslash G/H \\ P \leqslant^g H}} \text{Ind}_{U \cap {}^g H}^U(F_{({}^g \varphi)})$$

as FU -modules, and

$$s_{(U,u)}(\text{ind}_H^G([F_\varphi])) = \sum_{\substack{g \in U \backslash G/H \\ P \leqslant^g H}} s_u(\text{Ind}_{U \cap {}^g H}^U(F_{({}^g \varphi)})) = \sum_{\substack{g \in U \backslash G/H \\ U \leqslant^g H}} ({}^g \varphi)(u),$$

since s_u is trivial on FU -modules induced from proper subgroups not containing u . \square

For $\mathcal{F} \in \text{Sh}_F(G)$ and $(H, \varphi) \in \mathcal{M}$ we denote by $d_{\mathcal{F}}(H, \varphi)$ the maximal number $n \in \mathbb{N}_0$ for which there exists a strictly ascending chain

$$(H, \varphi) = (H_0, \varphi_0) < \cdots < (H_n, \varphi_n)$$

in \mathcal{M} with $\mathcal{F}_{(H_n, \varphi_n)} \neq 0$. If $\mathcal{F}_{(H', \varphi')} = 0$ for all $(H', \varphi') \in \mathcal{M}$ with $(H, \varphi) \leqslant (H', \varphi')$, then we set $d_{\mathcal{F}}(H, \varphi) := -1$. By $d_{\mathcal{F}}$ we denote the maximum of all $d_{\mathcal{F}}(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$.

4.8 Lemma *Let $\mathcal{F} \in \text{Sh}_F(G)$ satisfy conditions (i)–(iv) in 4.3 and assume that $(I, \psi) \in \mathcal{M}$ satisfies $d_{\mathcal{F}}(I, \psi) = 0$, i.e., (I, ψ) is maximal in \mathcal{M} under the condition $\mathcal{F}_{(I, \psi)} \neq 0$. Then*

$$\mathcal{F}_{(I, \psi)} \cong \text{Ind}_I^{N_G(I, \psi)}(F_\psi \oplus \cdots \oplus F_\psi)$$

as $FN_G(I, \psi)$ -modules.

Proof In view of Proposition 4.7 and since \mathcal{F} satisfies condition (i), it suffices to show that there exists a positive integer k such that

$$s_{(U,u)}(\mathcal{F}_{(I, \psi)}) = \begin{cases} k[N_G(I, \psi) : I]\psi(u), & \text{if } U \leqslant I, \\ 0, & \text{if } U \not\leqslant I, \end{cases}$$

for each species $s_{(U,u)}$ of $T(N_G(I, \psi))$. Let $U \leqslant N_G(I, \psi)$ be p -hypoelementary, let $P := O_p(U)$, and let $u \in U$ be a p' -element such that $\langle uP \rangle = U/P$. By the definition of $s_{(U,u)}$ we have

$$s_{(U,u)}(\mathcal{F}_{(I, \psi)}) = s_u(\overline{\mathcal{F}_{(I, \psi)}}(P)).$$

We first show that $s_{(U,u)}(\mathcal{F}_{(I,\psi)}) = 0$ if $U \not\leq I$. Let $\varphi \in \widehat{IP}$ denote the unique extension of $\psi \in \widehat{I}$ to IP . Then, by condition (ii) in 4.3 and by Lemma 3.6(a), one has

$$\overline{\mathcal{F}_{(I,\psi)}}(P) = \overline{\mathcal{F}_{(I,\psi)}}(IP, \varphi) \cong \mathcal{F}_{(IP,\varphi)}.$$

If $P \not\leq I$, then $\mathcal{F}_{(IP,\varphi)} = 0$ by the maximality assumption. Thus we may assume that $P \leq I$ and so $\mathcal{F}_{(IP,\varphi)} = \mathcal{F}_{(I,\psi)}$ and $s_{(U,u)}(\mathcal{F}_{(I,\psi)}) = s_u(\mathcal{F}_{(I,\psi)})$. Since $U \not\leq I$, we have $\mathcal{F}_{(H,\varphi)} = 0$ for each extension $\hat{\varphi}$ of φ to $H := IU$. Since H/I is a cyclic p' -group, condition (iii) implies $s_u(\mathcal{F}_{(I,\psi)}) = 0$.

If $U \leq I$, then obviously $s_{(U,u)}(\mathcal{F}_{(I,\psi)}) = l \cdot \psi(u)$ with $l := \dim_F \mathcal{F}_{(I,\psi)}$, and it suffices to show that $[N_G(I, \psi) : I]$ divides l . From condition (iv) in 4.3 we obtain

$$\left(\sum_{\varphi \in \widehat{H}} \dim_F \mathcal{F}_{(H,\varphi)} \cdot \varphi \right)_{H \leq G} = \rho_G^{T^{\text{ab}}} \left(\sum_{[H,\varphi]_G \in \mathcal{M}/G} \alpha_{[H,\varphi]_G} \cdot [H,\varphi]_G \right)$$

for certain integers $\alpha_{[H,\varphi]_G}$. The maximality of (I, ψ) with $\dim_F \mathcal{F}_{(I,\psi)} \neq 0$ and the definitions of res_+ and π imply that $\alpha_{[H,\varphi]_G} = 0$ for all (H, φ) such that $(I, \psi) < {}^g(H, \varphi)$ for some $g \in G$. From the definition of π and res_+ it now follows that the only contribution to the basis element ψ in $T^{\text{ab}}(I) \subseteq (\prod_{H \leq G} T^{\text{ab}}(H))^G$ comes from $(\pi_I \circ \text{res}_+^G)(\alpha_{[I,\psi]_G} \cdot [I, \psi]_G)$ and equals $[N_G(I, \psi) : I] \cdot \alpha_{[I,\psi]_G}$. Thus, $\dim_F \mathcal{F}_{(I,\psi)} = [N_G(I, \psi) : I] \cdot \alpha_{[I,\psi]_G}$, and the proof is complete. \square

4.9 Proposition (a) For every $\mathcal{F} \in \text{Sh}_F(G)$ there exist $M \in {}_{FG}\text{mon}$ and an epimorphism $\varepsilon: \mathcal{J}(M) \rightarrow \mathcal{F}$.

(b) For every $\mathcal{F} \in \text{Sh}_F(G)$ satisfying conditions (i)–(iv) in 4.3, there exist $M \in {}_{FG}\text{mon}$ and an epimorphism $\varepsilon: \mathcal{J}(M) \rightarrow \mathcal{F}$ such that $\varepsilon_{(H,\varphi)}: M^{((H,\varphi))} \rightarrow \mathcal{F}_{(H,\varphi)}$ is an isomorphism for all $(H, \varphi) \in \mathcal{M}$ with $d_{\mathcal{F}}(H, \varphi) \leq 0$. In particular, if $\mathcal{G} := \ker(\varepsilon) \in \text{Sh}_F(G)$, then $d_{\mathcal{G}}(I, \psi) \leq d_{\mathcal{F}}(I, \psi) - 1$ for all $(I, \psi) \in \mathcal{M}$ with $d_{\mathcal{F}}(I, \psi) \geq 0$, and $d_{\mathcal{G}} \leq d_{\mathcal{F}} - 1$ if $\mathcal{F} \neq 0$.

Proof Let \mathcal{R} denote a set of representatives for the G -conjugacy classes of \mathcal{M} . For each $(H, \varphi) \in \mathcal{R}$ we will construct an FH -module $W_{(H,\varphi)} \cong F_{\varphi} \oplus \cdots \oplus F_{\varphi}$ together with a map $\alpha_{(H,\varphi)} \in \text{Hom}_{FH}(W_{(H,\varphi)}, \mathcal{F}_{(H,\varphi)})$. We can view $W_{(H,\varphi)}$ as object in ${}_{FH}\text{mon}$ by choosing an arbitrary decomposition into one-dimensional subspaces. We set

$$M := \bigoplus_{(H,\varphi) \in \mathcal{R}} \text{Ind}_H^G(W_{(H,\varphi)}) \in {}_{FG}\text{mon}$$

(see [Bo97, Remark 1.5(e)] for the obvious construction of the induction of objects in ${}_{FH}\text{mon}$) and observe that

$$M((H, \varphi)) = \bigoplus_{g \in N_G(H, \varphi)/H} g \otimes_{FH} W_{(H,\varphi)} \subseteq \text{Ind}_H^G(W_{(H,\varphi)}).$$

Furthermore, we set

$$\beta_{(H,\varphi)} := \text{Ind}_H^{N_G(H,\varphi)}(\alpha_{(H,\varphi)}): M((H, \varphi)) \rightarrow \mathcal{F}_{(H,\varphi)}$$

and define $\varepsilon \in \text{Hom}_{\text{Sh}_F(G)}(\mathcal{J}(M), \mathcal{F})$ as the corresponding map under the isomorphism in Proposition 4.6. Note that if each $\beta_{(H,\varphi)}$, $(H, \varphi) \in \mathcal{R}$, is surjective, then ε is an epimorphism.

(a) For $(H, \varphi) \in \mathcal{R}$ we may set $W_{(H,\varphi)} := \text{Res}_H^{N_G(H,\varphi)}(\mathcal{F}_{(H,\varphi)})$ and define $\alpha_{(H,\varphi)}: W_{(H,\varphi)} \rightarrow \mathcal{F}_{(H,\varphi)}$ as the identity. With $\alpha_{(H,\varphi)}$ also $\beta_{(H,\varphi)}$ is surjective and the result follows.

(b) For all $(H, \varphi) \in \mathcal{R}$ with $d_{\mathcal{F}}(H, \varphi) > 0$ we proceed as in (a). For $(H, \varphi) \in \mathcal{R}$ with $d_{\mathcal{F}}(H, \varphi) = -1$ we set $W_{(H,\varphi)} := 0$. Finally, for $(H, \varphi) \in \mathcal{R}$ with $d_{\mathcal{F}}(H, \varphi) = 0$, Lemma 4.8 implies the existence of an FH -submodules $W_{(H,\varphi)}$ of $\mathcal{F}_{(H,\varphi)}$ with

$$W_{(H,\varphi)} \cong F_{\varphi} \oplus \cdots \oplus F_{\varphi} \quad \text{and} \quad \mathcal{F}_{(H,\varphi)} = \bigoplus_{g \in N_G(H,\varphi)/H} gW_{(H,\varphi)}.$$

We define $\alpha_{(H,\varphi)}: W_{(H,\varphi)} \rightarrow \mathcal{F}_{(H,\varphi)}$ as the inclusion. Then $\beta_{(H,\varphi)}: M((H,\varphi)) = M^{((H,\varphi))} \rightarrow \mathcal{F}_{(H,\varphi)}$ is an isomorphism. Now it is easy to see that the morphism $\varepsilon: \mathcal{J}(M) \rightarrow \mathcal{F}$, defined as above, has the desired properties. \square

4.10 Theorem *For every $\mathcal{F} \in \text{Sh}_F(G)$ the following hold:*

- (a) \mathcal{F} has a monomial resolution.
- (b) \mathcal{F} has a split monomial resolution if and only if \mathcal{F} satisfies conditions (i) and (ii) in 4.3. Moreover, in this case, every monomial resolution of \mathcal{F} is split.
- (c) \mathcal{F} has a finite split monomial resolution if and only if \mathcal{F} satisfies conditions (i)–(iv) in 4.3.

Proof (a) This follows from iterated application of Proposition 4.9 to \mathcal{F} and the kernel \mathcal{G} of $\varepsilon: \mathcal{J}(M) \rightarrow \mathcal{F}$ and by observing that the composition $\mathcal{J}(N) \rightarrow \mathcal{G} \subseteq \mathcal{J}(M)$ is of the form $\mathcal{J}(\partial)$ for some $\partial \in {}_{FG}\text{mon}(N, M)$ by [Bo97, Proposition 3.2 and 3.9].

(b) Let $M_* \xrightarrow{\varepsilon} \mathcal{F}$ be a split monomial resolution. Then, for each $(H, \varphi) \in \mathcal{M}$, the sequence $M_*^{((H,\varphi))} \rightarrow \mathcal{F}_{(H,\varphi)} \rightarrow 0$ is split exact so that $\mathcal{F}_{(H,\varphi)}$ is isomorphic to a direct summand of $M_0^{((H,\varphi))}$ as $FN_G(H, \varphi)$ -module. Thus, with $M_0^{((H,\varphi))}$ also $\mathcal{F}_{(H,\varphi)}$ is a trivial source $FN_G(H, \varphi)$ -module, and \mathcal{F} satisfies condition (i) in 4.3. Moreover, if $(I, \psi) \trianglelefteq (H, \varphi)$ in \mathcal{M} with H/I a p -group, then we obtain a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & M_1^{((H,\varphi))} & \longrightarrow & M_0^{((H,\varphi))} & \longrightarrow & \mathcal{F}_{(H,\varphi)} \longrightarrow 0 \\
& & \downarrow M_1 \text{ts}_{(I,\psi)}^{(H,\varphi)} & & \downarrow M_0 \text{ts}_{(I,\psi)}^{(H,\varphi)} & & \downarrow \mathcal{F} \text{ts}_{(I,\psi)}^{(H,\varphi)} \\
\cdots & \longrightarrow & \overline{M_1^{((I,\psi))}}(H, \varphi) & \longrightarrow & \overline{M_0^{((I,\psi))}}(H, \varphi) & \longrightarrow & \overline{\mathcal{F}_{(I,\psi)}}(H, \varphi) \longrightarrow 0
\end{array}$$

with split exact top row. Since $M_*^{((I,\psi))} \rightarrow \mathcal{F}_{(I,\psi)} \rightarrow 0$ is split exact, also the bottom row is split exact. By Proposition 2.6(b), the maps $M_i \text{ts}_{(I,\psi)}^{(H,\varphi)}$, $i \geq 0$, are isomorphisms. This implies that also $\mathcal{F} \text{ts}_{(I,\psi)}^{(H,\varphi)}$ is an isomorphism, and \mathcal{F} satisfies condition (ii) in 4.3.

Conversely, if \mathcal{F} satisfies conditions (i) and (ii) in 4.3, and if $M_* \xrightarrow{\varepsilon} \mathcal{F}$ is any monomial resolution (which exists by part (a)), then it is also split by Lemma 3.9(b).

(c) If $M_* \xrightarrow{\varepsilon} \mathcal{F}$ is a finite split monomial resolution, then \mathcal{F} satisfies conditions (i) and (ii) in 4.3 by part (b). Moreover, condition (iii) in 4.3 follows from the fact that, for any exact sequence $0 \rightarrow V_n \rightarrow \cdots \rightarrow V_0 \rightarrow 0$ in ${}_{FG}\text{mod}$ and any element $g \in G$, one has $\sum_{i=0}^n (-1)^i s_g(V_i) = 0$, and from Lemma 4.4(a). Also, the element associated to \mathcal{F} in $(T^{\text{ab}})^+(G)$ is the alternating sum of the elements associated to $\mathcal{J}(M_i)$, $i \in \mathbb{N}_0$, so that condition (iv) in 4.3 holds for \mathcal{F} by Lemma 4.4(b).

Conversely, assume that \mathcal{F} satisfies conditions (i)–(iv) in 4.3. By induction on $d_{\mathcal{F}}$ we show that \mathcal{F} has a finite split monomial resolution. In fact, if $d_{\mathcal{F}} = -1$, then $\mathcal{F} = 0$ and the assertion is true with $M_* = 0$. If $d_{\mathcal{F}} \geq 0$, then, by Proposition 4.9(b), there exists $M_0 \in {}_{FG}\text{mon}$ and an epimorphism $\varepsilon: \mathcal{J}(M_0) \rightarrow \mathcal{F}$ such that $d_{\ker(\varepsilon)} < d_{\mathcal{F}}$. By Lemma 4.5, the sheaf $\mathcal{G} := \ker(\varepsilon) \in \text{Sh}_F(G)$ again satisfies conditions (i)–(iv) and, by induction, \mathcal{G} has a finite split monomial resolution $0 \rightarrow M_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} M_1, \mathcal{J}(M) \xrightarrow{\delta_1} \mathcal{G}$. By [Bo97, Proposition 3.2 and 3.9], the composition

$$\mathcal{J}(M_1) \xrightarrow{\delta_1} \mathcal{G} \subseteq \mathcal{J}(M_0)$$

is of the form $\mathcal{J}(\partial_1)$ for a unique map $\partial_1 \in {}_{FG}\text{mon}(M_1, M_0)$, and the sequence $0 \rightarrow \mathcal{J}(M_n) \rightarrow \cdots \rightarrow \mathcal{J}(M_0) \xrightarrow{\varepsilon} \mathcal{F}$ is a finite monomial resolution of \mathcal{F} . \square

The following theorem follows now by combining the previous results.

4.11 Theorem *Let $V \in {}_F G \text{mod}$.*

(a) *V has a monomial resolution (in the sense of Definition 3.1) if and only if V has a Brauer filtration.*

(b) *The following are equivalent:*

(i) *V has a split monomial resolution.*

(ii) *V has a finite monomial resolution.*

(iii) *V has a finite split monomial resolution.*

(iv) *$V \in {}_F G \text{triv}$ and V has a Brauer filtration.*

Moreover, if (i)–(iv) hold, then every monomial resolution of V is split.

Proof (a) If V has a monomial resolution, then V has a Brauer filtration by Proposition 3.5. Conversely, if V has a Brauer filtration, then, by Remark 4.2 and Theorem 4.10(a), V has a monomial resolution.

(b) (i) \Rightarrow (iv): This follows from Proposition 3.5 and the splitting of the surjection $M_0 \xrightarrow{\varepsilon} V$, if $M_* \xrightarrow{\varepsilon} V$ is a monomial resolution of V .

(iv) \Rightarrow (iii): Let $\mathcal{F} \in \text{Sh}_F(G)$ be the sheaf associated to a Brauer filtration of V . Then, by Remark 4.2, it suffices to construct a finite split monomial resolution of \mathcal{F} in the sense of Definition 4.1. It follows from Lemma 4.4(b) and Theorem 4.10(c) that this is possible.

(iii) \Rightarrow (ii) and (iii) \Rightarrow (i) are trivial.

(ii) \Rightarrow (iv): This follows from Corollary 3.10(i) and from Proposition 3.5.

Finally, if (i)–(iv) hold, then every monomial resolution of V is split by Corollary 3.10(ii). \square

5 Functoriality, examples, and open questions

Next we want to examine if a homomorphism $f: V \rightarrow W$ in ${}_F G \text{mod}$ can be extended to a chain map between monomial resolutions of V and W .

5.1 Proposition *Let $V, W \in {}_F G \text{mod}$ and assume that*

$$M_* \xrightarrow{\delta} V \quad \text{and} \quad N_* \xrightarrow{\varepsilon} W$$

are monomial resolutions of V and W . Let $\mathcal{F}, \mathcal{G} \in \text{Sh}_F(G)$ be the sheaves defined by the Brauer filtrations of V and W which are induced by the above monomial resolutions, i.e. $\mathcal{F}_{(H, \varphi)} := V(H, \varphi) := \delta(M_0^{(H, \varphi)})$ and $\mathcal{G}_{(H, \varphi)} := W(H, \varphi) := \varepsilon(N_0^{(H, \varphi)})$ for $(H, \varphi) \in \mathcal{M}$.

(a) *Let $f \in \text{Hom}_{{}_F G}(V, W)$. There exists a chain map $f_*: M_* \rightarrow N_*$ of chain complexes in ${}_F G \text{mon}$ such that the diagram*

$$\begin{array}{ccc} M_* & \xrightarrow{\delta} & V \\ f_* \downarrow & & \downarrow f \\ N_* & \xrightarrow{\varepsilon} & W \end{array} \quad (5.1.a)$$

commutes if and only if $f(V(H, \varphi)) \subseteq W(H, \varphi)$ for all $(H, \varphi) \in \mathcal{M}$. In this case f_ is determined by the commutativity of Diagram (5.1.a) up to homotopy.*

(b) Let $f_*: M_* \rightarrow N_*$ be a chain map of chain complexes in $FG\text{mon}$, then there exists a unique $f \in \text{Hom}_{FG}(V, W)$ such that Diagram (5.1.a) commutes. Moreover, $f(V(H, \varphi)) \subseteq W(H, \varphi)$ for all $(H, \varphi) \in \mathcal{M}$.

(c) Part (a) and (b) define a bijection between the set of homomorphisms $f \in \text{Hom}_{FG}(V, W)$ inducing a sheaf morphism $\mathcal{F} \rightarrow \mathcal{G}$, i.e., $f(V(H, \varphi)) \subseteq W(H, \varphi)$ for all $(H, \varphi) \in \mathcal{M}$, and the set of homotopy classes of chain maps $f_*: M_* \rightarrow N_*$.

Proof (a) Assume that f_* exists. Then

$$f(V(H, \varphi)) = (f \circ \delta)(M_0^{((H, \varphi))}) = (\varepsilon \circ f_0)(M_0^{((H, \varphi))}) \subseteq \varepsilon(N_0^{((H, \varphi))}) = W(H, \varphi)$$

for all $(H, \varphi) \in \mathcal{M}$. Conversely, if $f(V(H, \varphi)) \subseteq W(H, \varphi)$, for all $(H, \varphi) \in \mathcal{M}$, then f induces a morphism $f': \mathcal{F} \rightarrow \mathcal{G}$ in $\text{Sh}_F(G)$. Since $\mathcal{J}(M_*) \xrightarrow{\delta'} \mathcal{F}$ and $\mathcal{J}(N_*) \xrightarrow{\varepsilon'} \mathcal{G}$ are projective resolutions (with $\delta'_{(H, \varphi)} := \delta|_{M_0^{((H, \varphi))}}$ and ε defined similarly), f' can be extended to a chain map $f_*: \mathcal{J}(M_*) \rightarrow \mathcal{J}(N_*)$. Further, since $\mathcal{J}: FG\text{mon} \rightarrow \text{Sh}_F(G)$ is fully faithful, there exists $f_*: M_* \rightarrow N_*$ which makes Diagram (5.1.a) commutative. Moreover, f'_* is unique up to homotopy, and, since \mathcal{J} is fully faithful, also f_* is.

(b) Since $\delta': \mathcal{J}(M_*) \rightarrow \mathcal{F}$ and $\varepsilon': \mathcal{J}(N_*) \rightarrow \mathcal{G}$ are projective resolutions, the morphisms δ' and ε' induce isomorphisms $\bar{\delta}': H_0(\mathcal{J}(M_*)) \cong \mathcal{F}$ and $\bar{\varepsilon}': H_0(\mathcal{J}(N_*)) \cong \mathcal{G}$. Using these we obtain a commutative diagram

$$\begin{array}{ccc} H_0(\mathcal{J}(M_*)) & \xrightarrow{\bar{\delta}'} & \mathcal{F} \\ \downarrow H_0(\mathcal{J}(f_*)) & & \downarrow f' \\ H_0(\mathcal{J}(N_*)) & \xrightarrow{\bar{\varepsilon}'} & \mathcal{G} \end{array} \quad (5.1.b)$$

for a unique sheaf morphism $f': \mathcal{F} \rightarrow \mathcal{G}$. This implies that $f := f'_{(1,1)}: V \rightarrow W$ makes Diagram (5.1.a) commutative. If also $g \in \text{Hom}_{FG}(V, W)$ makes Diagram (5.1.a) commutative, then the induced morphism $g': \mathcal{F} \rightarrow \mathcal{G}$ (see the proof of part (a)) makes Diagram (5.1.b) commutative. This implies $f' = g'$ and also $f = g$, since $g = g'_{(1,1)}$.

(c) This is obvious with (a) and (b). \square

For $V, W \in FG\text{mod}$ with two Brauer filtrations $V(H, \varphi)$ and $W(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$, we define a morphism of the Brauer filtrations as an FG -module homomorphism $f: V \rightarrow W$ satisfying $f(V(H, \varphi)) = W(H, \varphi)$ for all $(H, \varphi) \in \mathcal{M}$. This defines a category $FG\text{triv}^\circ$ whose objects are trivial source FG -modules together with a fixed Brauer filtration.

5.2 Corollary Let $V \in FG\text{mod}$ and let $M_* \xrightarrow{\delta} V$ and $N_* \xrightarrow{\varepsilon} V$ be two monomial resolutions. Then M_* and N_* are homotopy equivalent chain complexes in $FG\text{mon}$ if and only if the induced Brauer filtrations of V are isomorphic. Taking the induced Brauer filtration defines a bijection between the homotopy equivalence classes of monomial resolutions of V and the isomorphism classes of Brauer filtrations of V . In particular, any two monomial resolutions of V are homotopy equivalent if and only if any two Brauer filtrations of V are isomorphic.

Proof Let $\mathcal{F}, \mathcal{G} \in \text{Sh}_F(G)$ denote the sheaves defined by the induced filtrations of the two monomial resolutions. If M_* and N_* are homotopy equivalent, then $H_0(\mathcal{J}(M_*)) (\cong \mathcal{F})$ and $H_0(\mathcal{J}(N_*)) (\cong \mathcal{G})$ are isomorphic in $\text{Sh}_F(G)$. Conversely, if $f': \mathcal{F} \rightarrow \mathcal{G}$ is an isomorphism in $\text{Sh}_F(G)$, then $f := f'_{(1,1)}$ can be extended to a chain map $f_*: M_* \rightarrow N_*$ as in Proposition 5.1, and also its inverse $g = f^{-1}: V \rightarrow V$ can be extended to g_* . Thus, $g_* \circ f_*$ and $f_* \circ g_*$ extend the identity map of V . Now the uniqueness statement in Proposition 5.1(a) implies that $g_* \circ f_*$ and

$f_* \circ g_*$ are both homotopic to the identity of V . The remaining statements are now immediate. \square

5.3 Remark (a) It is not difficult to construct examples of $V \in {}_F G \text{mod}$ having two different Brauer filtrations, for instance, G the dihedral group of order 8 and V the transitive permutation module with point stabilizer a non-central subgroup of order 2 in characteristic 2. But we know of no example which has two non-isomorphic Brauer filtrations. Moreover, one can easily construct examples of $V \in {}_F G \text{mod}$ together with two different Brauer filtrations which admit more than one isomorphism between them. Thus, there is no hope that the monomial resolution is functorial, even if one restricts one's attention to those trivial source modules which have a Brauer filtration.

(b) These problems are resolved if one works with the category ${}_F G \text{triv}^\circ$ instead of ${}_F G \text{triv}$. The monomial resolution then defines a functor from ${}_F G \text{triv}^\circ$ to the homotopy category of ${}_F G \text{mod}$.

5.4 Proposition *Let $V \in {}_F G \text{triv}$ be indecomposable with vertex contained in $P := O_p(G)$. Then V has a Brauer filtration, unique up to isomorphism.*

Proof First we prove the existence of a Brauer filtration on V . By the Mackey formula, $\text{Res}_P^G(V)$ has a direct summand $W \cong \text{Ind}_Q^P(F)$, where Q is a vertex of V . The subgroup $H := PN_G(Q)$ is the inertia group of W and, by [HW75, Satz 2.2], the module V is of the form $V = \bigoplus_{g \in G/H} gU \cong \text{Ind}_H^G(U)$ for some indecomposable direct summand $U \in {}_F H \text{triv}$ of $\text{Res}_H^G(V)$ with the property that $\text{Res}_P^H(U)$ is isomorphic to a direct sum of k copies of W , for some $k \in \mathbb{N}$. By Lemma 1.2(a), we may write

$$U^Q = U(Q) \oplus \ker(\text{Br}_Q^U)$$

for some $U(Q) \in {}_F N_G(Q) \text{triv}$. We set

$$U({}^g Q) := gU(Q)$$

for each $g \in G$. Then we claim that

$$U = \bigoplus_{x \in P/N_P(Q)} U({}^x Q).$$

To prove the claim we consider the FP -module homomorphism $f: \text{Ind}_{N_P(Q)}^P(U(Q)) \rightarrow U$ induced by the $FN_P(Q)$ -linear embedding $U(Q) \rightarrow U$. It suffices to show that f is injective, since $\dim_F(U) = k \cdot \dim_F(W) = k \cdot [P : Q]$ and $\dim_F(U(Q)) = k \cdot \dim_F \overline{\text{Ind}_Q^P(F)}(Q) = k \cdot [N_P(Q) : Q]$. The FP -module $\tilde{U} := \text{Ind}_{N_P(Q)}^P(U(Q))$ is isomorphic to the direct sum of k copies of $\text{Ind}_Q^P(F)$, since $W(Q) \cong \text{Ind}_Q^{N_P(Q)}(F)$. We assume that f is not injective. Then there exists an element $0 \neq x \in \text{soc}(\tilde{U})$ with $f(x) = 0$. Thus, $x = \text{tr}_Q^P(1 \otimes y)$ for some $y \in U(Q)$ and $0 = f(x) = \text{tr}_Q^P(f(1 \otimes y)) = \text{tr}_Q^P(y)$. Hence,

$$0 = \text{Br}_Q^U(\text{tr}_Q^P(y)) = \sum_{g \in Q \backslash P/Q} \text{Br}_Q^U(\text{tr}_{Q \cap {}^g Q}^Q(y)) = \sum_{g \in N_P(Q)/Q} \text{Br}_Q^U(y) = \text{Br}_Q^U(\text{tr}_Q^{N_P(Q)}(y)),$$

so that $\text{tr}_Q^{N_P(Q)}(y) \in \ker(\text{Br}_Q^U) \cap U(Q) = 0$ and therefore $x = \text{tr}_{N_P(Q)}^P(1 \otimes \text{tr}_Q^{N_P(Q)}(y)) = 0$. This proves our claim. Our claim implies

$$V = \bigoplus_{g \in G/H} gU = \bigoplus_{g \in G/N_G(Q)} U({}^g Q).$$

If $R \leq G$ is a p -subgroup, we set

$$V(R) := \bigoplus_{\substack{g \in G/N_G(Q) \\ R \leq {}^g Q}} U({}^g Q).$$

Then obviously

$$gV(R) = V({}^gR) \quad \text{and} \quad V(R) \subseteq V(R')$$

for all $g \in G$ and all $R' \leq R$. In particular, $V(R) \in {}_{FN_G(R)}\text{triv}$, since we have the decomposition

$$V = V(R) \oplus \tilde{V}(R)$$

into $FN_G(R)$ -submodules with $\tilde{V}(R)$ the direct sum of those $U({}^gQ)$ with $g \in G/N_G(Q)$ such that $R \not\leq {}^gQ$. By Lemma 1.2(b), we have

$$\begin{aligned} \overline{V}(R) &= \overline{\bigoplus_{g \in G/N_G(Q)} U({}^gQ)(R)} \cong \bigoplus_{g \in R \backslash G/N_G(Q)} \overline{\text{Ind}_{R \cap {}^gQ}^R(U({}^gQ))}(R) \\ &\cong \bigoplus_{\substack{g \in G/N_G(Q) \\ R \leq {}^gQ}} \overline{U({}^gQ)}(R) \cong \bigoplus_{\substack{g \in G/N_G(Q) \\ R \leq {}^gQ}} U({}^gQ). \end{aligned}$$

Thus,

$$V^R = V(R) \oplus \ker(\text{Br}_R^V),$$

since $\tilde{V}(R)^R = \ker(\text{Br}_R^V)$.

Now let $(H, \varphi) \in \mathcal{M}$ and let R be a Sylow p -subgroup of H . If $R \not\leq P$, then we set $V(H, \varphi) := 0$. If $R \leq P$, then $R = H \cap P$ is normal in H and we set

$$V(H, \varphi) := V(R)^{(H, \varphi)}.$$

Then it is immediate that

$$gV(H, \varphi) = g(V(R)^{(H, \varphi)}) = (gV(R))^{g(H, \varphi)} = V({}^gR)^{g(H, \varphi)}$$

and

$$V(H, \varphi) = V(R)^{(H, \varphi)} \subseteq V(R)^{(I, \psi)} \subseteq V(S)^{(I, \psi)} = V(I, \psi)$$

for all $g \in G$ and $(I, \psi) \leq (H, \varphi)$, where $S = P \cap I$ is the Sylow p -subgroup of I . Finally, we have

$$V^{(H, \varphi)} = V(R)^{(H, \varphi)} \oplus \tilde{V}(R)^{(H, \varphi)}$$

and claim that $\tilde{V}(R)^{(H, \varphi)} = \ker(\text{Br}_{(H, \varphi)}^V)$. In fact, since $p \nmid [H : R]$, we have

$$\tilde{V}(R)^{(H, \varphi)} = \text{tr}_{(R, 1)}^{(H, \varphi)}(\tilde{V}(R)^R) = \text{tr}_{(R, 1)}^{(H, \varphi)}\left(\sum_{R' < R} \text{tr}_{R'}^R(\tilde{V}(R)^{R'})\right) \subseteq \ker(\text{Br}_{(H, \varphi)}^V).$$

On the other hand, by Proposition 1.9, we have $\dim_F \overline{V}(H, \varphi) = \dim_F \overline{V}(R)^{(H, \varphi)} = \dim_F V(R)^{(H, \varphi)}$. This completes the proof of the existence of a Brauer filtration on V .

Next we turn to the uniqueness part of the proposition. Let $V'(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$, be a second Brauer filtration on V . We will compare it with the family $V(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$, constructed above. Since $V(Q) \cong \overline{V}(Q) \cong V'(Q, 1)$, as $FN_G(Q)$ -modules, we may choose an $FN_G(Q)$ -isomorphism $f_Q: V(Q) \rightarrow V'(Q, 1)$. Since $V = \bigoplus_{g \in G/N_G(Q)} gV(Q)$, this isomorphism induces an FG -homomorphism $f: V \rightarrow V$. We claim that f is an isomorphism. In fact, assume that $\ker(f) \neq 0$. Then $\ker(f)^P \neq 0$, since $\ker(f)$ contains a simple FP -submodule, and we choose $0 \neq u \in \ker(f)^P$. Let T be a set of representatives for $G/PN_G(Q)$. Since

$$V = \bigoplus_{x \in T} x \cdot U = \bigoplus_{x \in T} x \cdot \left(\bigoplus_{y \in P/N_P(Q)} yV(Q) \right),$$

we obtain

$$V^P = \bigoplus_{x \in T} x \cdot \left(\bigoplus_{y \in P/N_P(Q)} yV(Q) \right)^P.$$

Thus, we may write

$$u = \sum_{x \in T} x \cdot \mathrm{tr}_{N_P(Q)}^P(v_x)$$

with elements $v_x \in V(Q)^{N_P(Q)}$, $x \in T$. Since $V(Q) = U(Q)$ is isomorphic to a sum of copies of $\mathrm{Ind}_Q^{N_P(Q)}(F)$ as $FN_P(Q)$ -modules, we can write

$$v_x = \mathrm{tr}_Q^{N_P(Q)}(w_x)$$

with elements $w_x \in V(Q)$, $x \in T$. For $g \in T$ we have

$$\begin{aligned} 0 &= \mathrm{Br}_{gQ}^V(f(u)) = \mathrm{Br}_{gQ}^V\left(\sum_{x \in T} x \cdot \mathrm{tr}_{N_P(Q)}^P(f(v_x))\right) \\ &= \sum_{x \in T} \mathrm{Vcon}_{(Q,1)}^g\left(\mathrm{Br}_Q^V(g^{-1}x \cdot \mathrm{tr}_{N_P(Q)}^P(f(v_x)))\right). \end{aligned}$$

If $g \neq x$, then with $z := g^{-1}x$ we obtain

$$\begin{aligned} \mathrm{Br}_Q^V(z \cdot \mathrm{tr}_{N_P(Q)}^P(f(v_x))) &= \mathrm{Br}_Q^V(\mathrm{tr}_{zQ}^P(f(zw_x))) \\ &= \mathrm{Br}_Q^V\left(\sum_{h \in Q \setminus P/zQ} \mathrm{tr}_{Q \cap {}^h z Q}^Q(f(hzw_x))\right) = 0, \end{aligned}$$

since $z \notin PN_G(Q)$ implies ${}^h z Q \neq Q$. Therefore, we have

$$\begin{aligned} 0 &= \mathrm{Br}_Q^V(\mathrm{tr}_{N_P(Q)}^P(f(v_g))) = \mathrm{Br}_Q^V(\mathrm{tr}_Q^P(f(w_g))) \\ &= \mathrm{Br}_Q^V\left(\sum_{h \in Q \setminus P/Q} \mathrm{tr}_{Q \cap {}^h Q}^Q(f(w_g))\right) = \mathrm{Br}_Q^V(\mathrm{tr}_Q^{N_P(Q)}(f(w_g))). \end{aligned}$$

Since $\mathrm{tr}_Q^{N_P(Q)}(f(w_g)) \in V'(Q, 1)$ and since Br_Q^V is injective on $V'(Q, 1)$, we obtain

$$0 = \mathrm{tr}_Q^{N_P(Q)}(f(w_g)) = f_Q(\mathrm{tr}_Q^{N_P(Q)}(w_g)).$$

The injectivity of $f_Q: V(Q) \rightarrow V'(Q, 1)$ now implies $v_g = \mathrm{tr}_Q^{N_P(Q)}(w_g) = 0$ and the contradiction $u = 0$. This proves the claim that f is an isomorphism.

Let $R \leq G$ be a p -subgroup. Then

$$\begin{aligned} f(V(R)) &= f\left(\bigoplus_{\substack{g \in G/N_G(Q) \\ R \leq {}^g Q}} gV(Q)\right) = \sum_{\substack{g \in G/N_G(Q) \\ R \leq {}^g Q}} g \cdot f(V(Q)) \\ &= \sum_{\substack{g \in G/N_G(Q) \\ R \leq {}^g Q}} gV'(Q, 1) = \sum_{\substack{g \in G/N_G(Q) \\ R \leq {}^g Q}} V'({}^g Q, 1) \subseteq V'(R, 1). \end{aligned}$$

Moreover, if $(H, \varphi) \in \mathcal{M}$ and R is a Sylow p -subgroup of H , we have

$$f(V(H, \varphi)) = f(V(R)^{(N_H(R), \varphi)}) \subseteq V'(R, 1)^{(N_H(R), \varphi)} = V'(H, \varphi)$$

by Proposition 1.9. This completes the proof. \square

The following examples are special cases of the situation in the last proposition.

5.5 Example (Projective modules) Let $W \in {}_{FG}\mathrm{mod}$ and let $W(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$, be a Brauer filtration. Then for each $(H, \varphi) \in \mathcal{M}$, with H a p' -group, one has $W(H, \varphi) = W^{(H, \varphi)}$, since $\ker(\mathrm{Br}_{(H, \varphi)}^W) = 0$ in this case. Moreover, if W is projective, then $\overline{W}(H, \varphi) = 0$ whenever

H is not a p' -group, since $\overline{W}(H, \varphi) \neq 0$ implies $F_\varphi \mid \text{Res}_H^G(W)$ and this implies that W has a direct summand with vertex containing a Sylow p -subgroup of H .

Now let $V \in {}_{FG}\text{mod}$ be projective. Then $V \in {}_{FG}\text{triv}$ and V has a unique Brauer filtration, namely

$$V(H, \varphi) := \begin{cases} V^{(H, \varphi)}, & \text{if } H \text{ is a } p'\text{-group,} \\ 0, & \text{otherwise,} \end{cases}$$

for $(H, \varphi) \in \mathcal{M}$. By Theorem 4.11, V has a finite split monomial resolution, and by Corollary 5.2, any two monomial resolutions are homotopy equivalent. If also $W \in {}_{FG}\text{mod}$ and $f \in \text{Hom}_{FG}(V, W)$, then, by the above considerations, for any Brauer filtration $W(H, \varphi)$, $(H, \varphi) \in \mathcal{M}$, of W one has $f(V(H, \varphi)) \subseteq W(H, \varphi)$. Thus, $f: V \rightarrow W$ can be extended to $f_*: M_* \rightarrow N_*$, whenever M_* and N_* are monomial resolutions of V and W , respectively. This shows that the construction of monomial resolutions and extensions of FG -module homomorphisms defines a functor from the category of finitely generated projective FG -modules to the homotopy category of $FG\text{mon}$.

5.6 Example (p' -groups) Assume that G is a p' -group. Then the ring $T(G)$ can be identified with the character ring $R(G)$ and the ring $T_+^{\text{ab}}(G)$ can be identified with the ring $R_+^{\text{ab}}(G)$ (cf. [Bo98b]). Under these identifications, the canonical induction formula for trivial source modules coincides with the canonical Brauer induction formula for complex characters. Since every FG -module is projective, we can apply the results of the previous example. Also the resulting monomial resolutions are defined exactly in the same way as in [Bo97] for complex representations, since the generalized Brauer construction \overline{V} coincides with just the generalized fixed point construction $V^{(H, \varphi)}$.

Finally, we state several questions we were unable to answer.

5.7 Questions (a) *Are any two Brauer filtrations of a trivial source FG -module V isomorphic?* If the answer is positive, monomial resolutions are unique up to homotopy equivalence. The authors do not expect that this question has a positive answer.

(b) *Can the class of trivial source FG -modules which have a Brauer filtration be characterized more directly?*

(c) In his work on projective resolutions of Mackey functors (see [Bc98]), S. Bouc was naturally led to consider the class of trivial source FG -modules V having a Brauer filtration only on the pairs $(P, 1)$, where P is a p -subgroup. *Can any such Brauer filtration on the p -subgroups be extended to a Brauer filtration on all pairs?* Or in a weaker formulation: *Is it equivalent for a trivial source module to have a Brauer filtration and to have a filtration in Bouc's sense?*

(d) *Which are the trivial source modules that have a monomial resolution of length 0?* If this is the case for a trivial source module, then its canonical induction formula has only nonnegative coefficients. In the case of a p' -group G , these two conditions are equivalent (see [Bo90, Remark 2.13(d)]), since the canonical induction formula in this case coincides with the canonical Brauer induction formula introduced in [Bo90]. In fact, the modules with this property are just direct sums of one-dimensional modules. This leads to the question: *Which are the trivial source modules whose canonical induction formula has only nonnegative coefficients? Do they all have monomial resolutions of length 0?*

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