THE CARTAN RING OF A FINITE GROUP

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Abstract. We determine the structure of the Cartan ring of a finite group G, defined as the Grothendieck ring of G modulo its ideal generated by projective modules, and we identify the ideal of the Cartan ring arising from all relatively Q-projective modules for a p-subgroup Q of G.

Let F be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. We denote the Grothendieck ring of the group algebra FG by $\mathrm{G}_0(FG)$. Then $\mathrm{G}_0(FG)$ is finitely generated and free, considered as an abelian group. The isomorphism classes $[S]$ of the simple FG-modules S form a basis of $\mathrm{G}_0(FG)$, and the multiplication in $\mathrm{G}_0(FG)$ is induced by the tensor product of FG-modules. For an arbitrary FG-module V we set $[V] := a_1[S_1] + \cdots a_2[S_2]$, if the simple module S_i appears with multiplicity a_i as composition factor in V. It is well-known that the classes $[P]$ of the projective FG-modules P span an ideal $\mathrm{K}_0(FG)$ of $\mathrm{G}_0(FG)$. We call the quotient

$$\mathrm{G}_0(FG)/\mathrm{K}_0(FG) =: \mathrm{Cart}(FG)$$

the *Cartan ring* of FG. Its structure, as an abelian group, is described by the Cartan matrix of FG. Brauer has shown (cf. [3, Theorem 3.6.32]) that $\mathrm{Cart}(FG)$ is a finite abelian p-group; more precisely,

$$\mathrm{Cart}(FG) \cong \prod_{s \in G_{p'}/\sim_G} \mathbf{Z}/p^{d_s}\mathbf{Z} \qquad \text{(as a group)}.$$

Here the direct product ranges over a set of representatives for the conjugacy classes of p'-elements s in G and, for $s \in G_{p'}$, p^{d_s} is the order of a Sylow p-subgroup D_s of the centralizer $C_G(s)$. Often d_s is called the *defect* of s (and of the conjugacy class of G containing s). In this note we will determine the *ring* structure of $\mathrm{Cart}(FG)$.

In the following, we will denote the field of p-adic numbers by $\mathbf{Q}_p$ and the ring of p-adic integers by $\mathbf{Z}_p$. We fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and denote a primitive $|G|_{p'}$-th root of 1 in $\overline{\mathbf{Q}}_p$ by ζ. (For a positive integer n, n_p and $n_{p'}$ denote the p-part and the p'-part, respectively, of n, so that $n = n_p n_{p'}$.)

In this paper, $\mathrm{Irr}(G)$ will denote the set of irreducible characters $G \longrightarrow \overline{\mathbf{Q}}_p$. Moreover, $\mathrm{IBr}(G)$ will denote the set of irreducible Brauer characters $G_{p'} \longrightarrow \mathbf{Z}_p[\zeta]$. (Recall that $G_{p'}$ denotes the set of p'-elements in G.) Also, $\mathrm{Pim}(G)$ will denote the set of indecomposable projective characters $G_{p'} \longrightarrow \mathbf{Z}_p[\zeta]$.

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For a subring R of $\overline{\mathbf{Q}}_p$, we will denote the set of R-linear combinations of $\mathrm{Irr}(G)$ by $R\mathrm{Irr}(G)$. In a similar way, we define $R\mathrm{IBr}(G)$ and $R\mathrm{Pim}(G)$. Then $R\mathrm{IBr}(G)$ is a commutative R-algebra, and $R\mathrm{Pim}(G)$ is an ideal in $R\mathrm{IBr}(G)$. There is an isomorphism of rings

$$\mathrm{G}_0(FG) \longrightarrow \mathbf{Z}\mathrm{IBr}(G)$$

taking the class $[M]$ of an FG-module M to its Brauer character. This isomorphism maps $\mathrm{K}_0(FG)$ onto $\mathbf{Z}\mathrm{Pim}(G)$ and induces an isomorphism of rings

$$\mathrm{Cart}(FG) = \mathrm{G}_0(FG)/\mathrm{K}_0(FG) \cong \mathbf{Z}\mathrm{IBr}(G)/\mathbf{Z}\mathrm{Pim}(G) =: \mathrm{Cart}(G).$$

Since $\mathrm{Cart}(G)$ is a finite p-group, we also have an isomorphism of rings

$$\mathrm{Cart}(G) \cong \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Cart}(G) \cong \mathbf{Z}_p\mathrm{IBr}(G)/\mathbf{Z}_p\mathrm{Pim}(G).$$

Our first result is well-known (cf. [2, pp. 81-82], for example). It is an ingredient in one of the usual proofs of Brauer's Induction Theorem.

Lemma 1. *Let K be the conjugacy class of a p-regular element $s \in G$, and denote the characteristic function of K by $\epsilon_K : G_{p'} \longrightarrow \mathbf{Z}_p[\zeta]$ (i.e., $\epsilon_K(g) = 1$ for $g \in K$, and $\epsilon_K(g) = 0$ otherwise). Then $\epsilon_K \in \mathbf{Z}_p[\zeta]\mathrm{IBr}(G)$ and $|C_G(s)|\epsilon_K \in \mathbf{Z}_p[\zeta]\mathrm{Pim}(G)$.*

Proof. Let D be a Sylow p-subgroup of $C_G(s)$, and let $\lambda_1, \ldots, \lambda_n$ denote the irreducible characters of $\langle s\rangle \times D$ containing D in their kernel. Then

$$\phi := \sum_{i=1}^{n} \lambda_i(s^{-1})\mathrm{Ind}_{\langle s\rangle\times D}^{G}(\lambda_i) \in \mathbf{Z}_p[\zeta]\mathrm{Irr}(G);$$

Moreover, we have $\phi(g) \in \mathbf{Z}$ for $g \in G$, $\phi(g) = 0$ if g is not conjugate to an element in the coset sD, and $\phi(s) = |C_G(s) : D| \not\equiv 0 \pmod p$. Thus the restriction of $|C_G(s) : D|^{-1}\phi$ to $G_{p'}$ belongs to $\mathbf{Z}_p[\zeta]\mathrm{IBr}(G)$ and equals ϵ_K.

Similarly, let $\mu_1, \ldots, \mu_n$ denote the irreducible characters of $\langle s\rangle$. Then

$$\Phi := \sum_{i=1}^{n} \mu_i(s^{-1})\mathrm{Ind}_{\langle s\rangle}^{G}(\mu_i) \in \mathbf{Z}_p[\zeta]\mathrm{Irr}(G);$$

moreover, we have $\Phi(g) \in \mathbf{Z}$ for $g \in G$, $\Phi(g) = 0$ if $g \notin K$, and $\Phi(s) = |C_G(s)|$. The restriction of Φ to $G_{p'}$ belongs to $\mathbf{Z}_p[\zeta]\mathrm{Pim}(G)$ and equals $|C_G(s)|\epsilon_K$.

For $s \in G_{p'}$, the map

$$t_s : \mathbf{Z}_p[\zeta]\mathrm{IBr}(G) \longrightarrow \mathbf{Z}_p[\zeta], \quad \phi \longmapsto \phi(s),$$

is an epimorphism of rings. By Lemma 1, these epimorphisms induce an isomorphism of rings

$$t : \mathbf{Z}_p[\zeta]\mathrm{IBr}(G) \longrightarrow \prod_{s\in G_{p'}/\sim_G} \mathbf{Z}_p[\zeta], \quad \phi \longmapsto (\phi(s) : s \in G_{p'}/\sim_G).$$

Lemma 1 and standard properties of projective characters imply that

$$t(\mathbf{Z}_p[\zeta]\mathrm{Pim}(G)) = \prod_{s\in G_{p'}/\sim_G} |C_G(s)|\mathbf{Z}_p[\zeta].$$

We conclude that t induces isomorphisms of rings

$$\mathbf{Z}_p[\zeta] \otimes_{\mathbf{Z}_p} \mathrm{Cart}(G) \cong \mathbf{Z}_p[\zeta]\mathrm{IBr}(G)/\mathbf{Z}_p[\zeta]\mathrm{Pim}(G) \cong \prod_{s\in G_{p'}/\sim_G} \mathbf{Z}_p[\zeta]/|C_G(s)|\mathbf{Z}_p[\zeta].$$

In order to descend from $\mathbf{Z}_p[\zeta] \otimes_{\mathbf{Z}_p} \mathrm{Cart}(G)$ to $\mathrm{Cart}(G)$ we use Galois theory. In the following, Γ denotes the Galois group of the Galois extension $\mathbf{Q}_p(\zeta)|\mathbf{Q}_p$. Since $\mathbf{Q}_p(\zeta)|\mathbf{Q}_p$ is unramified, Γ is isomorphic to the subgroup of $(\mathbf{Z}/|G|_{p'}\mathbf{Z})^\times$ generated by $p + |G|_{p'}\mathbf{Z}$. The corresponding isomorphism maps $p + |G|_{p'}\mathbf{Z}$ to the automorphism $\sigma \in \Gamma$ sending ζ to ζ^p. Note that $G \times \Gamma$ acts on $G_{p'}$ in such a way that

$$(x, \sigma)y := xy^p x^{-1} \quad \text{for} \quad x, y \in G.$$

We will denote by $G_{p'}/\sim_{G\times\Gamma}$ the set of orbits, and we will consider Γ as a subgroup of $G \times \Gamma$. Then the action of $G \times \Gamma$ on $G_{p'}$ induces an action of Γ on the set $G_{p'}/\sim_G$ of p-regular conjugacy classes of G.

Since Γ acts on $\mathbf{Z}_p[\zeta]$ and on $G_{p'}/\sim_G$, it also acts via ring automorphisms on the ring of class functions $f\colon G_{p'}/\sim_G \to \mathbf{Z}_p[\zeta]$ such that $(\sigma(f))(s) = \sigma(f(\sigma^{-1}(s)))$ for every $s \in G_{p'}$. Note that the Brauer characters are fixed under this action so that

$$\sigma\Big(\sum_{\phi\in \mathrm{IBr}(G)} a_\phi \phi\Big) = \sum_{\phi\in \mathrm{IBr}(G)} \sigma(a_\phi)\phi\,,$$

with $a_\phi \in \mathbf{Z}_p[\zeta]$ for $\phi \in \mathrm{IBr}(G)$. Thus $\mathbf{Z}_p\mathrm{IBr}(G)$ is the ring of Γ-fixed points:

$$\mathbf{Z}_p\mathrm{IBr}(G) = (\mathbf{Z}_p[\zeta]\mathrm{IBr}(G))^\Gamma := \{\psi \in \mathbf{Z}_p[\zeta]\mathrm{IBr}(G) : \sigma(\psi) = \psi\}.$$

The action of Γ on $\mathbf{Z}_p[\zeta]\mathrm{IBr}(G)$ restricts to an action of Γ on $\mathbf{Z}_p\mathrm{Pim}(G)$ with

$$\sigma\Big(\sum_{\Phi\in \mathrm{Pim}(G)} b_\Phi \Phi\Big) = \sum_{\Phi\in \mathrm{Pim}(G)} \sigma(b_\Phi)\Phi\,,$$

with $b_\Phi \in \mathbf{Z}_p[\zeta]$ for $\Phi \in \mathrm{Pim}(G)$, and we have

$$\mathbf{Z}_p\mathrm{Pim}(G) = (\mathbf{Z}_p[\zeta]\mathrm{Pim}(G))^\Gamma := \{\Psi \in \mathbf{Z}_p[\zeta]\mathrm{Pim}(G) : \sigma(\Psi) = \Psi\}.$$

The Γ-action on $\prod_{s\in G_{p'}/\sim_G} \mathbf{Z}_p[\zeta]$ induced by the ring isomorphism t is obviously given by

$$\sigma(a) = (\sigma(a_{\sigma^{-1}(s)}) : s \in G_{p'}/\sim_G)$$

for $a = (a_s : s \in G_{p'}/\sim_G) \in \prod_{s\in G_{p'}/\sim_G} \mathbf{Z}_p[\zeta]$.

Since $t\colon \mathbf{Z}_p[\zeta]\mathrm{IBr}(G) \longrightarrow \prod_{s\in G_{p'}/\sim_G} \mathbf{Z}_p[\zeta]$ is Γ-equivariant, it restricts to an isomorphism between Γ-fixed rings:

$$t^\Gamma : \mathbf{Z}_p\mathrm{IBr}(G) \longrightarrow \Big(\prod_{s\in G_{p'}/\sim_G} \mathbf{Z}_p[\zeta]\Big)^\Gamma.$$

It is clear that

$$t^\Gamma(\mathbf{Z}_p\mathrm{Pim}(G)) = \Big(\prod_{s\in G_{p'}/\sim_G} |C_G(s)|\mathbf{Z}_p[\zeta]\Big)^\Gamma.$$

For $s \in G_{p'}$, the set

$$\Gamma_s := \{\gamma \in \Gamma : \gamma(s) \sim_G s\}$$

is a subgroup of Γ and, by Frobenius reciprocity, we have an isomorphism of rings

$$\Big(\prod_{s\in G_{p'}/\sim_G} \mathbf{Z}_p[\zeta]\Big)^\Gamma \cong \prod_{s\in G_{p'}/\sim_{G\times\Gamma}} \mathbf{Z}_p[\zeta]^{\Gamma_s};$$

here the product on the right hand side ranges over a set of representatives for the $G \times \Gamma$-orbits on $G_{p'}$. Under the above ring isomorphism, $(\prod_{s\in G_{p'}/\sim_G} |C_G(s)|\mathbf{Z}_p[\zeta])^\Gamma$ corresponds to $\prod_{s\in G_{p'}/\sim_{G\times\Gamma}} |C_G(s)|\mathbf{Z}_p[\zeta]^{\Gamma_s}$. For $s \in G_{p'}$, let

$$m_s := |\Gamma : \Gamma_s| = [\mathbf{Q}_p(\zeta)^{\Gamma_s} : \mathbf{Q}_p],$$

so that m_s is the number of conjugacy classes of G contained in the $G \times \Gamma$-orbit of s. Then $\mathbf{Q}_p(\zeta)^{\Gamma_s}$ is the unique unramified extension of $\mathbf{Q}_p$ of degree m_s in $\overline{\mathbf{Q}}_p$, and $\mathbf{Z}_p[\zeta]^{\Gamma_s}$ is its ring of integers. We conclude:

Theorem 2. *For $s \in G_{p'}$, let R_s denote the ring of integers in the unique unramified extension field K_s of $\mathbf{Q}_p$ of degree m_s. Then the Cartan ring* $\mathrm{Cart}(G)$ *is isomorphic (as a ring) to*

$$\prod_{s \in G_{p'}/\sim_{G\times\Gamma}} R_s/|C_G(s)|R_s.$$

It is easy to see that

$$R_s = \mathbf{Z}_p[\phi(s) : \phi \in \mathrm{IBr}(G)]$$

for $s \in G_{p'}$. Note that if $s \in G_{p'}$ lies in a class of defect zero, then $R_s/|C_G(s)|R_s$ is the zero-ring and can be omitted from the above direct product.

Let us record some consequences of the above structure theorem:

Corollary 3. *Let $s_1, \ldots s_r \in G_{p'}$ be representatives of the $G \times \Gamma$-orbits of elements in $G_{p'}$ which lie in conjugacy classes of positive defect, let $d_1, \ldots, d_r$ denote the defects of their respective conjugacy classes, and let $m_i := |\Gamma : \Gamma_{s_i}|$. Then the following hold:*

(i) The ring $\mathrm{Cart}(G)$ *is the direct product of r uniserial rings A_i $(i = 1, \ldots, r)$ and each A_i has Loewy length d_i.*

(ii) The Jacobson radical $J(\mathrm{Cart}(G))$ *equals* $p\mathrm{Cart}(G)$, *and* $\mathrm{Cart}(G)/p\mathrm{Cart}(G)$ *is the direct product of r finite fields A_i/pA_i $(i = 1, \ldots, r)$; each field A_i/pA_i contains p^{m_i} elements.*

(iii) $\mathrm{Cart}(G)$ *has precisely* $(d_1 + 1)\cdots(d_r + 1)$ *ideals*

$$\prod_{i=1}^{r} p^{n_i} A_i \quad (0 \le n_i \le d_i \quad \text{for} \quad i = 1, \ldots, r).$$

For every p-subgroup Q of G, the relatively Q-projective FG-modules yield an ideal I_Q of $\mathrm{Cart}(G)$. ($I_Q = 0$ for $Q = 1$, and $I_Q = \mathrm{Cart}(G)$ for every Sylow p-subgroup Q of G.) We will identify these ideals using the following result from [1].

Proposition 4. ([1, Théorème 3]) *Let Q be a p-subgroup of G. Then the ring isomorphism t maps the $\mathbf{Z}_p$-span of the Brauer characters of relatively Q-projective FG-modules onto the ideal $\prod_{s \in G_{p'}/\sim_G} c_s^Q R_s$ in $\prod_{s \in G_{p'}/\sim_G} R_s$, where*

$$c_s^Q := \gcd\{|C_G(s) : C_G(s) \cap xQx^{-1}| : x \in G\}\,.$$

The following result follows now in the same way as we derived Theorem 2, by taking Γ-fixed points.

Theorem 5 *Let Q be a p-subgroup of G. Then with the notation of Corollary 3, the ideal I_Q of* $\mathrm{Cart}(G)$ *is given by*

$$I_Q = \prod_{i=1}^{r} p^{n_i^Q} A_i$$

with $p^{n_i^Q} = (c_{s_i}^Q)_p$.

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