Central idempotents of the bifree and left-free double Burnside ring*

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Abstract

We determine the blocks, i.e., the primitive central idempotents, of the bifree double Burnside ring $B^\Delta(G,G)$ and the left-free double Burnside ring $B^\triangleleft(G,G)$, as well as the primitive central idempotents of the algebras arising from scalar extension to $\mathbb{Q}$.

1 Introduction

The aim of this paper is to find the primitive central idempotents of the subrings $B^\Delta(G,G)$ and $B^\triangleleft(G,G)$ of the double Burnside ring $B(G,G)$ of a finite group G, as well as the primitive central idempotents of the algebras $\mathbb{Q}B^\Delta(G,G)$ and $\mathbb{Q}B^\triangleleft(G,G)$. Recall that the double Burnside ring $B(G,G)$ is the Grothendieck group of the category of finite (G,G)-bisets with respect to disjoint unions equipped with the multiplication induced by tensoring (G,G)-bisets over G. The subrings $B^\Delta(G,G) \subseteq B^\triangleleft(G,G) \subseteq B(G,G)$ arise from considering bifree (G,G)-bisets and left-free (G,G)-bisets. These classes of bisets are of particular interest, since via the theory of biset functors introduced by Bouc they are related to globally defined Mackey functors or globally defined Mackey functors with inflation (or deflation) maps as extra structures. The bifree subring of a p-group S is also related to fusion systems on S, cf. [RS] and [BD1], and the left-free subring is related to stable homotopy classes of selfmaps of the p-completion of the classifying space BG of G, cf. [MP] and [AKO]. For generalities on double Burnside rings and biset functors we refer the reader to Bouc's book [Bc2].

For the bifree double Burnside ring we derive the following result. Let $\hat\Sigma_G$ denote a set of representatives of the isomorphism classes of subgroups of G.

1.1 Theorem *The primitive central idempotents e_U of $B^\Delta(G,G)$ are parametrized by the elements $U \in \hat\Sigma_G$ such that U is a perfect group. The primitive central idempotents $e_{(U,\chi)}$ of $\mathbb{Q}B^\Delta(G,G)$ are*

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parametrized by a set $\mathcal{E}_G$ of pairs (U,χ) with $U \in \hat{\Sigma}_G$ and certain irreducible characters $\chi \in \mathrm{Irr}_{\mathbb{Q}}(\mathrm{Out}(U))$, i.e., characters of irreducible modules of the outer automorphism group of U over $\mathbb{Q}$. More precisely, in $\mathbb{Q}B^{\Delta}(G,G)$, one has $e_U = \sum_{\chi} e_{(V,\chi)}$, for each perfect $U \in \hat{\Sigma}_G$, where V is in $\hat{\Sigma}_G$ such that $V^{(\infty)} \cong U$ and χ is in $\mathrm{Irr}_{\mathbb{Q}}(\mathrm{Out}(V))$ such that $(V,\chi) \in \mathcal{E}_G$.

Here, $V^{(\infty)}$ denotes the smallest normal subgroup of V with solvable quotient. A precise definition of $\mathcal{E}_G$ can be found in Remark 3.9. Theorem 1.1 is a special case of Theorem 3.8 which determines the primitive central idempotents of the ring $RB^{\Delta}(G,G)$, i.e., the scalar extension of $B^{\Delta}(G,G)$ to R, for any integral domain R, and of Remark 3.9 in which more general fields K are considered in place of $\mathbb{Q}$.

For the left-free double Burnside ring we have the following result:

1.2 Theorem *The center of $B^{\triangleleft}(G,G)$ is connected, i.e., 0 and 1 are the only central idempotents of $B^{\triangleleft}(G,G)$. The primitive central idempotents of $\mathbb{Q}B^{\triangleleft}(G,G)$ are contained in $\mathbb{Q}B^{\Delta}(G,G)$. They are the sums $\sum_{(U,\chi)\in\mathcal{E}} e_{(U,\chi)}$, where $\mathcal{E} \subseteq \mathcal{E}_G$ is an equivalence class of $\mathcal{E}_G$ with respect to the transitive and symmetric closure of the relation defined on two elements (U,χ), $(U',\chi') \in \mathcal{E}_G$ by*

$$e_{(U,\chi)}\mathbb{Q}B^{\triangleleft}(G,G)e_{(U',\chi')} \neq \{0\}\,. \tag{1}$$

The relation defined by (1) is reformulated in explicit character theoretic terms in Lemma 5.6. Theorem 1.2 is a special case of Theorem 5.5 and Corollary 5.7, in which scalar extensions $RB^{\triangleleft}(G,G)$ and $KB^{\triangleleft}(G,G)$ for a larger class of integral domains R (replacing $\mathbb{Z}$) and fields K (replacing $\mathbb{Q}$) are considered.

The paper is arranged as follows. Section 2 introduces the notation used in the paper. In Section 3 we consider the bifree double Burnside ring $B^{\Delta}(G,G)$. The main theorem of this section, Theorem 3.8 describes the primitive central idempotents of $RB^{\Delta}(G,G)$ for an arbitrary integral domain R. Remark 3.9 summarizes the case where R is a field with some restrictions on the characteristic. Section 4 uses the same methods as Section 3 to show that the double Burnside ring $B^{\mathcal{F}}(S,S)$ associated to a fusion system $\mathcal{F}$ on a p-group S has connected center, cf. Theorem 4.3. The ring $B^{\mathcal{F}}(S,S)$ was introduced in [BD1]; it is a subring of $B^{\Delta}(S,S)$. In Section 5 the primitive central idempotents of $RB^{\triangleleft}(G,G)$ are studied for certain integral domains R. The main results are Theorem 5.5 and Corollary 5.7 which imply Theorem 1.2. Section 6 is solely devoted to the technical proof of Lemma 5.6 which uses a variety of results and notations from [BD1]. In Section 7 we consider the example where G is a cyclic group and an elementary abelian group. We also pose two questions related to the sum of character values on double cosets that is related to the condition describing the relation on $\mathcal{E}_G$ leading to the block decomposition of $\mathbb{Q}B^{\triangleleft}(G,G)$.

2 Notation

Throughout, G denotes a finite group.

2.1 Generalities. The cardinality of a set X is denoted by $|X|$.

We denote by $H \leqslant G$ that H is a subgroup of G and by $H < G$ that H is a proper subgroup of G. Similarly, $H \trianglelefteq G$ (resp. $H \triangleleft G$) denotes that H is a normal (resp. proper normal) subgroup of G. The trivial subgroup of G will often be denoted by 1. The group of automorphisms of G is denoted by $\mathrm{Aut}(G)$. For an element g of G, we denote by $c_g \in \mathrm{Aut}(G)$ the automorphism $x \mapsto gxg^{-1}$ of G. By

$\mathrm{Inn}(G)$ we denote the set of the automorphism c_g, $g \in G$, and by $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ we denote the outer automorphism group of G. For $H \leqslant G$ and $g \in G$ we also write ${}^gH$ instead of gHg^{-1}. If two subgroups H and K of G are conjugate we write $H =_G K$ and if H is conjugate to a subgroup of K we write $H \leqslant_G K$.

If X is a left G-set and $x \in X$, we write $\mathrm{stab}_G(x)$ for the stabilizer of x in G, and X^G for the set of G-fixed points of X.

2.2 The (double) Burnside ring. Recall that the Burnside ring $B(G)$ of G is the Grothendieck group of the category of finite left G-sets with respect to the disjoint union of G-sets. The multiplication on $B(G)$ is induced by taking the direct product of G-sets. The element in $B(G)$ associated to a finite left G-set X is denoted by $[X]$. If H runs through a set of representatives of the conjugacy classes of subgroups of G, then the elements $[G/H] \in B(G)$, associated to the transitive G-sets G/H, form a $\mathbb{Z}$-basis of $B(G)$. For any subgroup H of G we have a ring homomorphism $\Phi_H \colon B(G) \to \mathbb{Z}$ determined by $\Phi_H([X]) = |X^H|$, for any finite left G-set X. We refer the reader to [CR, §80A] or [Bc2, Chapter 2] for the basic facts on the Burnside ring.

For two finite groups G and H, the double Burnside group $B(G,H)$ is the Grothendieck group of the category of finite (G,H)-bisets X, i.e., finite sets with a left G-action and a right H-action which commute with each other, with respect to disjoint unions. As a special case, we obtain the double Burnside ring $B(G,G)$ whose multiplication is induced by taking the tensor product $X \times_G Y$ of two (G,G)-bisets X and Y. This is the set of G-orbits $x \times_G y$ of elements $(x,y) \in X \times Y$ under the G-action $g(x,y) := (xg^{-1}, gy)$. We often identify (G,H)-biset structures on a set X with left $G \times H$-set structures on the same set X via $(g,h)x = gxh^{-1}$ for $x \in X$, $g \in G$ and $h \in H$. With this identification we can identify $B(G,H)$ and $B(G \times H)$ as additive groups. Note that the abelian group $B(G \times G)$ now has two ring structures, the first given by the direct product construction, the second by the tensor product construction on $B(G,G)$. We denote the first just by "$\cdot$" and the second by "$\cdot_G$". For more details we refer the reader to [Bc2, Chapter 2].

If G and H are finite groups and if $L \leqslant G \times H$ is a subgroup, we denote by $p_1 \colon G \times H \to G$ and $p_2 \colon G \times H \to H$ the two projection maps. Moreover, we set $k_1(L) := \{g \in G \mid (g,1) \in L\}$ and $k_2(L) := \{h \in H \mid (1,h) \in L\}$. Then $k_i(L) \trianglelefteq p_i(L)$ for $i = 1,2$, and $\eta(L) \colon p_2(L)/k_2(L) \to p_1(L)/k_1(L)$, defined by $gk_2(L) \mapsto hk_2(L)$ whenever $(g,h) \in L$, is a well-defined isomorphism. This way one obtains a bijection between the set of subgroups L of $G \times H$ and the quintuples $(P_1, K_1, \eta, P_2, K_2)$ with $K_1 \trianglelefteq P_1 \leqslant G$, $K_2 \trianglelefteq P_2 \leqslant H$, and $\eta \colon P_2/K_2 \xrightarrow{\sim} P_1/K_1$, cf. [Bc2, Lemma 2.3.25]. With this notation, a (G,G)-biset X is left-free (resp. bifree) if and only if each stabilizer L of an element of X satisfies $k_1(L) = 1$ (resp. $k_1(L) = 1 = k_2(L)$). Thus, $B^{\triangleleft}(G,G)$ (resp. $B^{\Delta}(G,G)$) is free over the basis elements $[G \times G/L]$, where L runs through a set of representatives of $G \times G$-conjugacy classes of subgroups of $G \times G$ with $k_1(L) = 1$ (resp. $k_1(L) = 1 = k_2(L)$). For an isomorphism $\phi \colon V \xrightarrow{\sim} U$ between subgroups of G we set $\Delta(U,\phi,V) := \{(\phi(v), v) \mid v \in V\}$, the subgroup corresponding to $(U,1,\phi,V,1)$. If $U = V$ and $\phi = \mathrm{id}_U$ we also write $\Delta(U)$.

3 Central idempotents of $RB^{\Delta}(G,G)$

Throughout this section G denotes a finite group and R denotes a commutative ring. We denote by Σ_G the set of subgroups of G, by $\tilde{\Sigma}_G \subseteq \Sigma_G$ a set of representatives of the conjugacy classes of Σ_G, and by $\hat{\Sigma}_G \subseteq \tilde{\Sigma}_G$ a set of representatives of the isomorphism classes of Σ_G.

3.1 Lemma *Let R be an integral domain and let X be a transitive G-set. If no prime divisor of $|X|$ is invertible in R then the RG-permutation module RX is indecomposable.*

Proof We may assume that $|X| \neq 1$. Assume that $RX = M \oplus N$ is a direct sum decomposition into RG-submodules M and N, and assume that $M \neq \{0\}$. Then M and N are finitely generated projective R-modules and they have a well-defined R-rank, cf. [DI, Theorem I.4.12]. Let $x \in X$, set $H := \mathrm{stab}_G(x)$, and let p be a prime divisor of $|X| = [G : H]$. Since $pR \neq R$, there exists a maximal ideal P of R such that $p \in P$. Then $F := R/P$ is a field of characteristic p. Let $\bar{F}$ denote an algebraic closure of F. Then $\bar{F}X \cong \bar{F} \otimes_R M \oplus \bar{F} \otimes_R N$, where $\bar{F}X$, $\bar{F} \otimes_R M$ and $\bar{F} \otimes_R N$ are relatively H-projective $\bar{F}G$-modules. By a result of Green, the p-part $[G : H]_p = |X|_p$ of $|X|$ divides

$$\dim_{\bar{F}} \bar{F} \otimes_R M = \dim_F F \otimes_R M = \mathrm{rk}_{R_P}(R_P \otimes_R M) = \mathrm{rk}_R(M)\,.$$

Since p is arbitrary, we conclude that $|X|$ divides $\mathrm{rk}_R(M)$. But

$$0 \neq \mathrm{rk}_R(M) \leqslant \mathrm{rk}_R(M) + \mathrm{rk}_R(N) = \mathrm{rk}_R(M \oplus N) = \mathrm{rk}_R(RX) = |X|,$$

which implies $\mathrm{rk}_R(M) = |X|$ and $\mathrm{rk}_R(N) = 0$. Thus, $N = 0$ and $M = RX$. □

3.2 Remark Let Λ be a ring and let $1_\Lambda = e_1 + \cdots + e_n$ be a decomposition of 1_Λ into primitive pairwise orthogonal idempotents $e_1, \ldots, e_n$ of Λ. Then every central idempotent e of Λ is equal to the subsum $e = \sum_{i \in I} e_i$, where I denotes the set of all elements $i \in \{1, \ldots, n\}$ satisfying $e_i e = e_i$. In fact, let $i \in \{1, \ldots, n\}$ be arbitrary. Then ee_i and $(1_\Lambda - e)e_i$ are orthogonal idempotents with $ee_i + (1-e)e_i = e_i$. Since e_i is primitive, we obtain $ee_i = e_i$ or $ee_i = 0$. By multiplying the equation $1_\Lambda = e_1 + \cdots + e_n$ on both sides with e, we now obtain the desired expression for e.

3.3 Proposition *Let X be a finite G-set and let R be an integral domain. Assume that, for every $x \in X$ and for every prime divisor p of $[G : \mathrm{stab}_G(x)]$, one has $\{0\} \neq pR \neq R$. Then the ring $\mathrm{End}_{RG}(RX)$ has no central idempotents different from 0 and 1.*

Proof Let K denote the field of fractions of R. We decompose X into G-orbits, $X = X_1 \coprod \cdots \coprod X_n$, and obtain decompositions

$$RX = RX_1 \oplus \cdots \oplus RX_n \quad \text{and} \quad KX = KX_1 \oplus \cdots \oplus KX_n \tag{2}$$

into RG-submodules and KG-submodules, respectively. We decompose KX_i, for each $i = 1, \ldots, n$, into indecomposable KG-submodules:

$$KX_i = V_i^{(1)} \oplus \cdots \oplus V_i^{(r_i)}\,. \tag{3}$$

We may assume that $V_i^{(1)} \cong K$, the trivial KG-module. In fact, the hypothesis on R and X implies that $|X_i| \neq 0$ in K. This implies that $\iota\colon K \to KX_i$, $1 \mapsto |X_i|^{-1} \sum_{x \in X_i} x$, and $\pi\colon KX_i \to K$, $x \mapsto 1$, are KG-module homomorphisms with $\pi \circ \iota = \mathrm{id}_K$, so that K is isomorphic to a direct summand of KX_i. Let $e_i \in \mathrm{End}_{RG}(RX)$ denote the idempotent which is the projection onto the i-th component in the first decomposition in (2). Then e_i is primitive in $\mathrm{End}_{RG}(RX)$, by Lemma 3.1. We view $\mathrm{End}_{RG}(RX)$ as a subring of $\mathrm{End}_{KG}(KX)$ via the canonical embedding and decompose e_i in $\mathrm{End}_{KG}(KX)$ further into primitive idempotents corresponding to the decomposition in (3):

$$e_i = e_i^{(1)} + \cdots + e_i^{(r_i)}\,.$$

Altogether we have a primitive decomposition

$$1 = (e_1^{(1)} + e_1^{(2)} + \cdots + e_1^{(r_1)}) + \cdots + (e_n^{(1)} + \cdots + e_n^{(r_n)}) \tag{4}$$

in $\mathrm{End}_{KG}(KX)$. Now let e be a non-zero central idempotent of $\mathrm{End}_{RG}(RX)$. Since $1 = e_1 + \cdots + e_n$ is a primitive decomposition of 1 in $\mathrm{End}_{RG}(RX)$, we have $e = \sum_{i\in I} e_i$ for some $\emptyset \neq I \subseteq \{1, \ldots, n\}$, by Remark 3.2. Since e is also a central idempotent of $\mathrm{End}_{KG}(KX)$, it is also a subsum of the decomposition in (4). Since $I \neq \emptyset$, there exists an element $i \in I$, and we have $e_i e = e_i$. This implies that $e_i^{(1)} e = e_i^{(1)}$. For every $j \in \{1, \ldots, n\}$ there exists an isomorphism $\alpha\colon KX \to KX$ such that $\alpha e_i^{(1)} \alpha^{-1} = e_j^{(1)}$. The equation $e_i^{(1)} e = e_i^{(1)}$ implies

$$e_j^{(1)} = \alpha e_i^{(1)} \alpha^{-1} = \alpha e_i^{(1)} e \alpha^{-1} = \alpha e_i^{(1)} \alpha^{-1} e = e_j^{(1)} e\,.$$

This implies that $e_j e \neq 0$, and Remark 3.2 implies that $j \in I$. Thus $I = \{1, \ldots, n\}$ and $e = 1$. □

Recall that we have a ring homomorphism $\Delta\colon B(G) \to B^\Delta(G,G)$, given by $\Delta([G/U]) = [G\times G/\Delta(U)]$. The following lemma was proved in [BP].

3.4 Lemma *Let $a \in B(G)$ and let $U \leqslant G$. Then*

$$\Phi_{\Delta(U)}(\Delta(a)) = |C_G(U)| \cdot \Phi_U(a)\,.$$

3.5 Remark Recall from [BD1, 5.3–5.5] that the map

$$\sigma_G\colon B^\Delta(G,G) \to \prod_{U\in\hat{\Sigma}_G} \mathrm{End}_{\mathbb{Z}\mathrm{Out}(U)}(\mathbb{Z}\overline{\mathrm{Inj}}(U,G))\,,$$
$$a \mapsto \Big([\mu] \mapsto \sum_{[\lambda]\in\overline{\mathrm{Inj}}(U,G)} \frac{\Phi_{\Delta(\lambda(U),\lambda\mu^{-1},\mu(U))}(a)}{|C_G(\lambda(U))|} \cdot [\lambda]\Big)_{U\in\hat{\Sigma}_G}$$

is a well-defined injective ring homomorphism with image of finite index which induces an R-algebra homomorphism

$$\sigma_G\colon RB^\Delta(G,G) \to \prod_{U\in\hat{\Sigma}_G} \mathrm{End}_{R\mathrm{Out}(U)}(R\overline{\mathrm{Inj}}(U,G))$$

for every commutative ring R. The latter homomorphism is an isomorphism if $|G|$ is invertible in R. In particular, if $a \in Z(B^\Delta(G,G))$ then $\sigma_G(a)$ is central in $\prod_{U\in\hat{\Sigma}_G} \mathrm{End}_{\mathbb{Z}\mathrm{Out}(U)}(\mathbb{Z}\overline{\mathrm{Inj}}(U,G))$ and $\prod_{U\in\hat{\Sigma}_G} \mathrm{End}_{R\mathrm{Out}(U)}(R\overline{\mathrm{Inj}}(U,G))$. Here, for $U \in \hat{\Sigma}_G$, $\mathrm{Inj}(U,G)$ denotes the $(G, \mathrm{Aut}(U))$-biset of injective group homomorphisms from U to G with $g \cdot \lambda \cdot \omega := c_g \circ \lambda \circ \omega$ for $g \in G$, $\lambda \in \mathrm{Inj}(U,G)$, and $\omega \in \mathrm{Aut}(U)$. Finally, $\overline{\mathrm{Inj}}(U,G) := G\backslash\mathrm{Inj}(U,G)$ is the set of G-orbits with the induced right action of $\mathrm{Out}(U)$. The G-orbit of $\lambda \in \mathrm{Inj}(U,G)$ is denoted by $[\lambda]$. We fix a group $U \in \hat{\Sigma}_G$. Let $U_1, \ldots, U_r \in \tilde{\Sigma}_G$ be the representatives of the G-conjugacy classes of all subgroups of G which are isomorphic to U. Then the right $\mathrm{Out}(U)$-set $\overline{\mathrm{Inj}}(U,G)$ decomposes into orbits,

$$\overline{\mathrm{Inj}}(U,G) = \coprod_{i=1}^{r} \overline{\mathrm{Inj}}(U,G)_i\,,$$

where $\mathrm{Inj}(U,G)_i$ denotes the set of elements $\lambda \in \mathrm{Inj}(U,G)$ such that $\lambda(U)$ is G-conjugate to U_i, and $\overline{\mathrm{Inj}}(U,G)_i$ denotes the set of G-orbits formed by such elements. In particular, we obtain a decomposition into $R\mathrm{Out}(U)$-submodules:

$$R\overline{\mathrm{Inj}}(U,G) = \bigoplus_{i=1}^{r} R\overline{\mathrm{Inj}}(U,G)_i\,. \tag{5}$$

Finally, note that, for every $i = 1, \ldots, r$, the map $\lambda \mapsto [\lambda]$ induces a bijection

$$N_G(U_i)\backslash\mathrm{Inj}(U,U_i) \xrightarrow{\sim} \overline{\mathrm{Inj}}(U,G)_i$$

of $\mathrm{Out}(U)$-sets, and that $N_G(U_i)\backslash\mathrm{Inj}(U,U_i) \cong A_i\backslash\mathrm{Aut}(U)$ as $\mathrm{Out}(U)$-sets, where $A_i := \lambda^{-1}B_i\lambda \leqslant \mathrm{Aut}(U)$ denotes the subgroup corresponding to the image B_i of the map $N_G(U_i) \to \mathrm{Aut}(U_i)$, $g \mapsto c_g$, under any isomorphism $\lambda\colon U \xrightarrow{\sim} U_i$.

3.6 Remark In this remark we assume that R is an integral domain with field of fractions K such that $|G|$ is invertible in K. We denote by π the set of prime divisors of $|G|$ which are not invertible in R. By Θ_G^π we denote the set of π-perfect subgroups of G, i.e., the subgroups U of G with the property that U has no factor group of prime order $p \in \pi$. For any group G we denote by $G^{(\pi)}$ the smallest normal subgroup of G with solvable factor group of π-order, i.e., of order only divisible by primes from π. Clearly, $G^{(\pi)}$ is π-perfect. Thus, $U^{(\pi)} \in \Theta_G^\pi$ for every $U \leqslant G$. We further define $\tilde{\Theta}_G^\pi := \Theta_G^\pi \cap \tilde{\Sigma}_G$ and $\hat{\Theta}_G^\pi := \Theta_G^\pi \cap \hat{\Sigma}_G$. Thus, $\hat{\Theta}_G^\pi \subseteq \tilde{\Theta}_G^\pi$ are sets of representatives of the isomorphism classes and of the conjugacy classes of π-perfect subgroups of G, respectively.

Recall from [Bc1, Corollary 3.3.6] the following description of the primitive idempotents of $RB(G)$ and of $KB(G)$. For each $U \in \Sigma_G$, let $e_U \in KB(G)$ denote the unique element with the property that $\Phi_{U'}(e_U) = 1$ if $U =_G U'$, and $\Phi_{U'}(e_U) = 0$ if $U' \neq_G U$. Then the elements e_U, $U \in \tilde{\Sigma}_G$, form a set of primitive, pairwise orthogonal idempotents of $KB(G)$ whose sum is equal to 1.

For every $U \in \Theta_G^\pi$ set

$$\varepsilon_U^{(\pi)} := \sum_{\substack{V \in \tilde{\Sigma}_G \\ V^{(\pi)} =_G U}} e_V\,.$$

Then the elements $\varepsilon_U^{(\pi)}$, $U \in \tilde{\Theta}_G^\pi$, are primitive, pairwise orthogonal idempotents of $RB(G)$ whose sum is equal to 1.

3.7 Proposition *Assume that K is a field such that $|G|$ is invertible in K. Further assume the notation established in Remarks 3.5 and 3.6. Let $V \leqslant G$, and let $(f_U)_{U\in\hat{\Sigma}_G}$ denote the image of $e_V \in KB(G)$ under the K-algebra homomorphism*

$$KB(G) \xrightarrow{\ \Delta\ } KB^\Delta(G,G) \xrightarrow{\ \sigma_G\ } \prod_{U\in\hat{\Sigma}_G} \mathrm{End}_{K\mathrm{Out}(U)}(K\overline{\mathrm{Inj}}(U,G))\,.$$

If $U \in \hat{\Sigma}_G$ is not isomorphic to V then $f_U = 0$. If $U \in \hat{\Sigma}_G$ is isomorphic to V then f_U is equal to the projection onto the direct summand of $K\overline{\mathrm{Inj}}(U,G)$ with respect to the decomposition in (5), with K in place of R, which is indexed by the G-conjugacy class containing V.

Proof Let $U \in \hat{\Sigma}_G$ and let $\mu \in \mathrm{Inj}(U, G)$. Then

$$f_U([\mu]) = \sum_{[\lambda] \in \overline{\mathrm{Inj}}(U,G)} \frac{\Phi_{\Delta(\lambda(U), \lambda\mu^{-1}, \mu(U))}(\Delta(e_V))}{|C_G(\lambda(U))|} [\lambda] \,.$$

Suppose that $f_U([\mu]) \neq 0$. We will show that then $\mu(U)$ is G-conjugate to V and that $f_U([\mu]) = [\mu]$.

Since e_V is a linear combination of elements of the form $[G/W]$ with $W \leqslant_G V$, the idempotent $\Delta(e_V)$ is a linear combination of elements of the form $[G \times G/\Delta(W)]$ with $W \leqslant_G V$. Thus, $\Phi_{\Delta(\lambda(U), \lambda\mu^{-1}, \mu(U))}([G \times G/\Delta(W)]) \neq 0$ for some $W \leqslant_G V$, and therefore $\Delta(\lambda(U), \lambda\mu^{-1}, \mu(U)) \leqslant_{G \times G} \Delta(W)$. Hence, $\Delta(\lambda(U), \lambda\mu^{-1}, \mu(U)) =_{G \times G} \Delta(X)$ for some $X \leqslant W \leqslant_G V$, and

$$0 \neq \Phi_{\Delta(\lambda(U), \lambda\mu^{-1}, \mu(U))}(\Delta(e_V)) = \Phi_{\Delta(X)}(\Delta(e_V)) = |C_G(X)| \Phi_X(e_V) \,,$$

by Lemma 3.4. Thus, $X =_G V$ and $\Delta(\lambda(U), \lambda\mu^{-1}, \mu(U)) =_{G \times G} \Delta(V)$. Let $g, h \in G$ be such that

$$\Delta(V) = {}^{(g,h)}\Delta(\lambda(U), \lambda\mu^{-1}, \mu(U)) = \{(g\lambda(u)g^{-1}, h\mu(u)h^{-1}) \mid u \in U\} \,.$$

Since $[\lambda] = [c_g\lambda]$ and $[\mu] = [c_h\mu]$, we may assume that $\Delta(V) = \Delta(\lambda(U), \lambda\mu^{-1}, \mu(U)) = \{(\lambda(u), \mu(u)) \mid u \in U\}$. Thus, $\lambda = \mu$, $\lambda(U) = V$, and

$$f_U([\mu]) = \frac{\Phi_{\Delta(V)}(\Delta(e_V))}{|C_G(V)|}[\mu] = \Phi_V(e_V)[\mu] = [\mu] \,,$$

again by Lemma 3.4. □

3.8 Theorem *Let R be an integral domain with field of fractions K such that $|G|$ is invertible in K. Let π denote the set of prime divisors of $|G|$ which are not invertible in R. Assume the notation from Remark 3.6. The primitive central idempotents of $RB^\Delta(G, G)$ are parametrized by isomorphism classes of π-perfect subgroups of G. More precisely, for $W \in \hat{\Theta}_G^\pi$, set*

$$\hat{\varepsilon}_W^{(\pi)} := \sum_{W \cong V \in \tilde{\Theta}_G} \varepsilon_V^{(\pi)} \in RB(G) \,.$$

Then the elements $\Delta(\hat{\varepsilon}_W^{(\pi)})$, $W \in \hat{\Theta}_G^\pi$, are primitive, pairwise orthogonal idempotents of $Z(RB^\Delta(G, G))$ whose sum is equal to 1.

Proof We will make use of the commutative diagram

$$\begin{array}{ccccc}
RB(G) & \xrightarrow{\Delta} & RB^\Delta(G,G) & \xrightarrow{\sigma_G} & \prod_{U \in \hat{\Sigma}_G} \mathrm{End}_{R\mathrm{Out}(U)}(R\overline{\mathrm{Inj}}(U,G)) \\
\downarrow & & \downarrow & & \downarrow \\
KB(G) & \xrightarrow{\Delta} & KB^\Delta(G,G) & \xrightarrow{\sigma_G} & \prod_{U \in \hat{\Sigma}_G} \mathrm{End}_{K\mathrm{Out}(U)}(K\overline{\mathrm{Inj}}(U,G))
\end{array}$$

whose vertical maps are the canonical embeddings. All maps in the diagram are injective and the map σ_G of the bottom row is an isomorphism.

(a) First we show that each element $\Delta(\hat{\varepsilon}_W^{(\pi)})$, $W \in \hat{\Theta}_G^\pi$, is a central idempotent of $RB^\Delta(G,G)$. For $W \in \hat{\Theta}_G^\pi$, the element $\hat{\varepsilon}_W^{(\pi)} := \sum_{W \cong V \in \tilde{\Theta}_G} \varepsilon_V^{(\pi)}$ is an idempotent of $RB(G)$, by Remark 3.6. Therefore, $\Delta(\hat{\varepsilon}_W^{(\pi)})$ is an idempotent of $RB^\Delta(G,G)$. To see that it is central in $RB^\Delta(G,G)$ it suffices to show that $\sigma_G(\Delta(\hat{\varepsilon}_W^{(\pi)}))$ is central in $\prod_{U \in \hat{\Sigma}_G} \mathrm{End}_{K\mathrm{Out}(U)}(K\overline{\mathrm{Inj}}(U,G))$. But, by Proposition 3.7, the U-component of $\sigma_G(\Delta(\hat{\varepsilon}_W^{(\pi)}))$ is equal to the identity map if $U^{(\pi)} \cong W$ and it is equal to 0 if $U^{(\pi)} \not\cong W$. So clearly, this element is central.

(b) Next we show that each element $\Delta(\hat{\varepsilon}_W^{(\pi)}))$, $W \in \hat{\Theta}_G^\pi$, is primitive in $Z(RB^\Delta(G,G))$. Let $W \in \hat{\Theta}_G^\pi$ and let e be a primitive central idempotent of $RB^\Delta(G,G)$ with $e \cdot_G \Delta(\hat{\varepsilon}_W^{(\pi)}) = e$. Then $\sigma_G(e)$ is a central idempotent of $\prod_{U \in \tilde{\Sigma}_G} \mathrm{End}_{R\mathrm{Out}(U)}(R\overline{\mathrm{Inj}}(U,G))$. By Proposition 3.3 there exists a subset $\Xi \subset \Sigma_G$ which is closed under taking isomorphic subgroups such that, with $\hat{\Xi} := \Xi \cap \hat{\Sigma}_G$, the U-component of $\sigma_G(e)$ is equal to the identity if $U \in \hat{\Xi}_G$ and equal to 0 if $U \notin \hat{\Xi}_G$. Now Proposition 3.7 implies that

$$\sigma_G(e) = \sigma_G(\Delta(\sum_{U \in \tilde{\Xi}} e_U))\,,$$

where $\tilde{\Xi} = \Xi \cap \tilde{\Sigma}_G$. This implies that

$$e = \Delta(\sum_{U \in \tilde{\Xi}} e_U) \in RB^\Delta(G,G) \cap \Delta(KB(G)) = \Delta(RB(G))\,.$$

The injectivity of Δ implies that $\sum_{U \in \tilde{\Xi}} e_U \in RB(G)$. Since $e \neq 0$ and since $e \cdot_G \Delta(\hat{\varepsilon}_W^{(\pi)}) = e$, we know that Ξ contains a subgroup U of G satisfying $U^{(\pi)} \cong W$. Moreover, since $\sum_{U \in \tilde{\Xi}} e_U$ is an idempotent in $RB(G)$, we obtain that Ξ contains a subgroup which is isomorphic to W. Since Ξ is closed under taking isomorphic subgroups, Ξ contains all subgroups of G which are isomorphic to W. Again, since $\sum_{U \in \tilde{\Xi}} e_U$ is an element of $RB(G)$, Ξ contains all subgroups U of G with $U^{(\pi)} \cong W$. This implies that $e \cdot_G \Delta(\hat{\varepsilon}_W^{(\pi)}) = \Delta(\hat{\varepsilon}_W^{(\pi)})$ and therefore, $e = e \cdot_G \Delta(\hat{\varepsilon}_W^{(\pi)}) = \Delta(\hat{\varepsilon}_W^{(\pi)})$. Thus, $\Delta(\hat{\varepsilon}_W^{(\pi)})$ is primitive in $Z(RB^\Delta(G,G))$.

(c) Finally,

$$\sum_{W \in \hat{\Theta}_G^\pi} \Delta(\hat{\varepsilon}_W^{(\pi)}) = \Delta\big(\sum_{W \cong V \in \tilde{\Theta}_G^\pi} \varepsilon_V^{(\pi)}\big) = \Delta(\sum_{U \in \tilde{\Sigma}_G} e_U) = \Delta(1) = 1\,,$$

and the proof is complete. □

In the following remark we will determine the primitive central idempotents of $KB^\Delta(G,G)$ for certain fields K. This will be used in Section 5. Euler's totient function will be denoted by φ.

3.9 Remark Let K be a field such that $|G|$, $|\mathrm{Out}(U)|$ and $\varphi(|\mathrm{Out}(U)|)$, for $U \leqslant G$, are invertible in K.

(a) Recall from Remark 3.5 that the map

$$\sigma_G \colon KB^\Delta(G,G) \xrightarrow{\sim} \prod_{U \in \hat{\Sigma}_G} \mathrm{End}_{K\mathrm{Out}(U)}(K\overline{\mathrm{Inj}}(U,G)) \qquad (6)$$

is an isomorphism of K-algebras. Moreover, for each $U \in \hat{\Sigma}_G$, the $K\mathrm{Out}(U)$-module $K\overline{\mathrm{Inj}}(U,G)$ is semisimple. Let $\mathcal{E}_G$ denote the set of pairs (U,χ) with $U \in \hat{\Sigma}_G$ and $\chi \in \mathrm{Irr}_K(\mathrm{Out}(U))$ such that χ occurs as a constituent in the character of $K\overline{\mathrm{Inj}}(U,G)$. Then, by the above isomorphism, the primitive central idempotents of $KB^\Delta(G,G)$ are given by the elements $e_{(U,\chi)}$, $(U,\chi) \in \mathcal{E}_G$, where, for $U' \in \hat{\Sigma}_G$, the U'-component of $\sigma_G(e_{(U,\chi)})$ is equal to 0 if $U' \neq U$ and equal to the map $a \mapsto a \cdot e_\chi$, for $a \in K\overline{\mathrm{Inj}}(U,G)$. Here, e_χ denotes the primitive idempotent of $K\mathrm{Out}(U)$ associated to the irreducible character χ. Note that one has

$$e_\chi = \frac{\chi(1)}{s^2 r |\mathrm{Out}(U)|} \sum_{\overline{\omega} \in \mathrm{Out}(U)} \chi(\overline{\omega}^{-1})\, \overline{\omega} \in K\mathrm{Out}(U)\,, \tag{7}$$

if $\chi = s(\psi_1 + \cdots + \psi_r)$ is a decomposition of χ into absolutely irreducible characters over some extension field of K. Note that s is invertible in K, since s divides $|\mathrm{Out}(U)|$ in the case that $\mathrm{char}(K) = 0$ and since $s = 1$ if $\mathrm{char}(K) \neq 0$. Also note that r is invertible in K, since r is the degree of a subextension of the extension $K(\zeta)/K$, where ζ is a root of unity of order $|\mathrm{Out}(U)|$.

(b) For $U \in \hat{\Sigma}_G$, let $\tilde{\Sigma}_G(U)$ denote the set of elements $V \in \tilde{\Sigma}_G$ with $V \cong U$. We can rewrite the decomposition (5) in Remark 3.5 as indexed over $\tilde{\Sigma}_G(U)$:

$$K\overline{\mathrm{Inj}}(U,G) = \bigoplus_{V \in \tilde{\Sigma}_G(U)} K\overline{\mathrm{Inj}}(U,G)_V\,. \tag{8}$$

Then, for $(U,\chi) \in \mathcal{E}_G$ and each $V \in \tilde{\Sigma}_G(U)$, the element $e_{(U,\chi,V)}$, defined by requiring that, for $U' \in \hat{\Sigma}_G$, the U'-component of $\sigma_G(e_{(U,\chi,V)})$ is equal to 0 if $U' \neq U$, and that the U-component is equal to 0 in all components of the decomposition (8) different from V, and finally equal to "multiplication with e_χ" in the V-component. This leads to a decomposition

$$e_{(U,\chi)} = \sum_{V \in \tilde{\Sigma}_G(U)} e_{(U,\chi,V)} \tag{9}$$

of the primitive idempotent of $Z(KB^\Delta(G,G))$ as a sum of pairwise orthogonal idempotents in $KB^\Delta(G,G)$.

4 Central idempotents of $RB^{\mathcal{F}}(S,S)$ for a fusion system $\mathcal{F}$ on a p-group S

Throughout this section we fix a p-group S and a (not necessarily saturated) fusion system $\mathcal{F}$ on S. For definitions and basic results on fusion systems we refer the reader to [AKO]. In [BD1], a subring $B^{\mathcal{F}}(S,S)$ of $B^\Delta(S,S)$ was constructed which is defined as the $\mathbb{Z}$-span of the standard basis elements $[S \times S/\Delta(P,\phi,Q)]$, where $\phi\colon Q \xrightarrow{\sim} P$ runs through all isomorphisms in the category $\mathcal{F}$. We call $B^{\mathcal{F}}(S,S)$ the *double Burnside ring of* $\mathcal{F}$. In this section we will show that $B^{\mathcal{F}}(S,S)$ has no central idempotents different from 0 and 1.

4.1 Remark In this remark we recall some notation and some results from [BD1]. Again we denote by Σ_S the set of subgroups of S, by $\tilde{\Sigma}_S \subseteq \Sigma_S$ a set of representatives of the S-conjugacy classes of subgroups of S, and by $\hat{\Sigma}_S^{\mathcal{F}} \subseteq \tilde{\Sigma}_S$ a set of representatives of the $\mathcal{F}$-isomorphism classes of subgroups of S. It was shown in [BD1, Theorem 5.7] (see also [BD1, Subsection 7.11]) that the map

$$\tilde{\sigma}_S^{\mathcal{F}} \colon B^{\mathcal{F}}(S,S) \to \prod_{P\in\hat{\Sigma}_S^{\mathcal{F}}} \mathrm{End}_{\mathbb{Z}\mathrm{Out}_{\mathcal{F}}(P)}(\mathbb{Z}\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S))\,,$$

$$a \mapsto \Big([\psi] \mapsto \sum_{[\phi]\in\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)} \frac{\Phi_{\Delta(\phi(P),\phi\psi^{-1},\psi(P))}(a)}{|C_S(P)|}\cdot[\phi]\Big)_P\,,$$

is a well-defined injective ring homomorphism with finite cokernel which induces an R-algebra homomorphism

$$\tilde{\sigma}_S^{\mathcal{F}} \colon RB^{\mathcal{F}}(S,S) \to \prod_{P\in\hat{\Sigma}_S^{\mathcal{F}}} \mathrm{End}_{R\mathrm{Out}_{\mathcal{F}}(P)}(R\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S))$$

for every commutative ring R. If p is invertible in R then the latter homomorphism is an isomorphism. Here, $\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)$ denotes the set of S-orbits of $\mathrm{Hom}_{\mathcal{F}}(P,S)$ under the action $x\cdot\phi := c_x\circ\phi$ for $x\in S$ and $\phi\in\mathrm{Hom}_{\mathcal{F}}(P,S)$. The set $\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)$ has a right action of the group $\mathrm{Out}_{\mathcal{F}}(P) := \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$, which is given by composition. The S-orbit of $\phi\in\mathrm{Hom}_{\mathcal{F}}(P,S)$ is denoted by $[\phi]$.

We fix a subgroup $P\in\hat{\Sigma}_S^{\mathcal{F}}$. Assume that $P_1,\ldots,P_r\in\tilde{\Sigma}_S^{\mathcal{F}}$ are representatives of the conjugacy classes of subgroups of S which are $\mathcal{F}$-isomorphic to P. Then the right $\mathrm{Out}_{\mathcal{F}}(P)$-set $\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)$ decomposes into orbits

$$\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S) = \coprod_{i=1}^{r} \overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)_i\,,$$

where $\mathrm{Hom}_{\mathcal{F}}(P,S)_i$ denotes the set of elements $\phi\in\mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $\phi(P)$ is S-conjugate to P_i, and $\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)_i$ denotes the set of S-orbits of $\mathrm{Hom}_{\mathcal{F}}(P,S)_i$. In particular, we obtain a decomposition

$$R\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S) = \bigoplus_{i=1}^{r} R\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)_i \tag{10}$$

into $R\mathrm{Out}_{\mathcal{F}}(P)$-submodules.

The following proposition can be proved in a completely analogous way as Proposition 3.7.

4.2 Proposition *Let K be a field of characteristic different from p and assume the notations from Remarks 4.1 and 3.6. Furthermore, let $Q\leqslant S$ and let $(f_P)_{P\in\hat{\Sigma}_S^{\mathcal{F}}}$ denote the image of $e_Q\in KB(S)$ under the K-algebra homomorphism*

$$KB(S)\xrightarrow{\ \Delta\ } KB^{\mathcal{F}}(S,S)\xrightarrow{\ \tilde{\sigma}_S^{\mathcal{F}}\ } \prod_{P\in\hat{\Sigma}_S^{\mathcal{F}}} \mathrm{End}_{K\mathrm{Out}_{\mathcal{F}}(P)}(K\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S))\,.$$

If $P\in\hat{\Sigma}_S^{\mathcal{F}}$ is not $\mathcal{F}$-isomorphic to Q then $f_P=0$. If $P\in\hat{\Sigma}_S^{\mathcal{F}}$ is $\mathcal{F}$-isomorphic to Q then the endomorphism f_P is equal to the projection onto the direct summand of $K\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)$ with respect to the decomposition in (10), with K in place of R, which is indexed by the S-conjugacy class containing Q.

For every $P\leqslant S$ we denote by $\mathrm{Aut}_S(P)\leqslant\mathrm{Aut}(P)$ the image of the map $N_S(P)\to\mathrm{Aut}(P)$, $g\mapsto c_g$.

4.3 Theorem *Let R be an integral domain with the following property: One has $\{0\} \neq pR \neq R$ and for every isomorphism $\phi\colon P \xrightarrow{\sim} Q$ in the category $\mathcal{F}$ and every prime divisor q of $[\mathrm{Aut}_{\mathcal{F}}(P) : (\mathrm{Aut}_{\mathcal{F}}(P) \cap \mathrm{Aut}_S(Q)^{\phi})]$ one has $\{0\} \neq qR \neq R$. Then the center of the ring $RB^{\mathcal{F}}(S,S)$ is connected. In particular, when $R = \mathbb{Z}$, the center of $B^{\mathcal{F}}(S,S)$ is connected.*

Proof The proof is similar to the proof of Theorem 3.8. Let K denote the field of fractions of R. We will use the diagram

$$\begin{array}{ccccc} RB(S) & \xrightarrow{\Delta} & RB^{\mathcal{F}}(S,S) & \xrightarrow{\tilde{\sigma}_S^{\mathcal{F}}} & \prod_{P\in\hat{\Sigma}_S^{\mathcal{F}}} \mathrm{End}_{R\mathrm{Out}_{\mathcal{F}}(P)}(R\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)) \\ \downarrow & & \downarrow & & \downarrow \\ KB(S) & \xrightarrow{\Delta} & KB^{\mathcal{F}}(S,S) & \xrightarrow{\tilde{\sigma}_S^{\mathcal{F}}} & \prod_{P\in\hat{\Sigma}_S^{\mathcal{F}}} \mathrm{End}_{K\mathrm{Out}_{\mathcal{F}}(P)}(K\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)) \end{array}$$

Again, each map in the diagram is injective and the map $\tilde{\sigma}_S^{\mathcal{F}}$ of the bottom row is an isomorphism, since $\mathrm{char}(K) \neq p$.

Let e be a non-zero central idempotent of $RB^{\mathcal{F}}(S,S)$. We will show that $e = 1$. The element $\tilde{\sigma}_S^{\mathcal{F}}(e)$ is a central idempotent in $\prod_{P\in\hat{\Sigma}_S^{\mathcal{F}}} \mathrm{End}_{R\mathrm{Out}_{\mathcal{F}}(P)}(R\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S))$. We want to invoke Proposition 3.3 and need to determine the stabilizer of $[\phi]$ for $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. It follows from an easy calculation that $\mathrm{stab}_{\mathrm{Aut}_{\mathcal{F}}(P)}([\phi]) = \mathrm{Aut}_{\mathcal{F}}(P) \cap \mathrm{Aut}_S(Q)^{\phi}$, where $Q := \phi(P)$. Thus, by Proposition 3.3, there exists a subset $\Xi \subseteq \Sigma_S$ which is closed under taking $\mathcal{F}$-isomorphic subgroups, such that, with $\hat{\Xi} := \Xi \cap \hat{\Sigma}_S^{\mathcal{F}}$, one has:

$$\tilde{\sigma}_S^{\mathcal{F}}(e) = (\delta_{P\in\hat{\Xi}})_{P\in\hat{\Sigma}_S^{\mathcal{F}}}\,,$$

where $\delta_{P\in\hat{\Xi}}$ denotes the identity map if $P \in \hat{\Xi}$ and the zero-map if not. Now Proposition 4.2 implies that

$$\tilde{\sigma}_S^{\mathcal{F}}(e) = \tilde{\sigma}_S^{\mathcal{F}}(\Delta(\sum_{P\in\tilde{\Xi}} e_P))\,,$$

where $\tilde{\Xi} := \Xi \cap \tilde{\Sigma}_S$. By the injectivity of $\tilde{\sigma}_S^{\mathcal{F}}$, we have

$$e = \Delta(\sum_{P\in\tilde{\Xi}} e_P) \in RB^{\mathcal{F}}(S,S) \cap \Delta(KB(S)) = \Delta(RB(S))\,.$$

Thus, the idempotent $\sum_{P\in\tilde{\Xi}} e_P$ of $KB(S)$ is contained in $RB(S)$. But since $pR \neq R$, this implies that $\Xi = \Sigma_S$ and that $\sum_{P\in\tilde{\Xi}} e_P = 1$, by Remark 3.6. ∎

5 Central idempotents of $RB^{\lhd}(G,G)$

In this section we will use again the notation from Section 3. Thus, R denotes a commutative ring, G denotes a finite group, and $\hat{\Sigma}_G \subseteq \tilde{\Sigma}_G \subseteq \Sigma_G$ denote sets of representatives of isomorphism classes, resp. representatives of conjugacy classes in the set Σ_G of subgroups of G.

The goal of this section is to show that the Grothendieck ring $B^{\lhd}(G,G)$ of left-free (G,G)-bisets has no central idempotent different from 0 and 1. We will prove the same result for a class of scalar extensions $RB^{\lhd}(G,G)$ from $\mathbb{Z}$ to R for certain commutative rings R. First, we will use the following result, see Theorem 6.4(c) from [BD1].

5.1 Proposition *Let K be a field such that $|G|$ and $|\mathrm{Out}(U)|$, $U \leqslant G$, are invertible in K. Then one has a decomposition $KB^{\lhd}(G,G) = KB^{\Delta}(G,G) \oplus J$, where J denotes the Jacobson radical of $KB^{\lhd}(G,G)$.*

By the above proposition, the following lemma will apply to $KB^{\lhd}(G,G)$ and its subalgebra $KB^{\Delta}(G,G)$, for fields K as in Proposition 5.1. We denote the Jacobson radical of a ring Λ by $J(\Lambda)$.

5.2 Lemma *Let Λ be a ring and let Γ be a (not necessarily unitary) subring of Λ such that $\Lambda = \Gamma \oplus J(\Lambda)$. Then each central idempotent of Λ is contained in Γ.*

Proof Let $e \in Z(\Lambda)$ be an idempotent and write $e = f + x$ with $f \in \Gamma$ and $x \in J(\Lambda)$. Then $f^2 \equiv e^2 = e \equiv f \mod J(\Lambda)$ implies $f^2 - f \in \Gamma \cap J(\Lambda)$ so that f is an idempotent. Now also $e - ef = ex$ and $f - ef = -xf$ are idempotents and contained in $J(\Lambda)$. Thus, $e - ef = 0 = f - ef$ and $e = f \in \Gamma$. □

Proposition 5.1 and Lemma 5.2 imply the following corollary.

5.3 Corollary *Let K be a field such that $|G|$ and $|\mathrm{Out}(U)|$, for $U \leqslant G$, are invertible in K. Then every central idempotent of $KB^{\lhd}(G,G)$ is already contained in $KB^{\Delta}(G,G)$.*

In order to determine the primitive central idempotents of $KB^{\lhd}(G,G)$, for appropriate fields K, the following lemma will be useful.

5.4 Lemma *Let Λ be a ring and assume that $1 = \sum_{i\in I} e_i$ is a decomposition of $1 \in \Lambda$ into a finite sum of non-zero pairwise orthogonal (not necessarily central) idempotents of Λ with the property that for each central idempotent f of Λ and each $i \in I$ one has $e_i f \in \{e_i, 0\}$. Denote by $\sim$ the symmetric and reflexive relation on I defined by $i \sim j$ if and only if $e_i\Lambda e_j \neq 0$ or $e_j \Lambda e_i \neq 0$, and denote by $\approx$ the transitive closure of $\sim$; that is, $i \approx j$ if and only if there exists a sequence $i = i_0, i_1, \ldots, i_n = j$ in I such that $i_{k-1} \sim i_k$ for all $k = 1, \ldots, n$. Then $\approx$ is an equivalence relation. If $I_1, \ldots, I_s$ denote the equivalence classes of I with respect to $\approx$ then the elements $f_k := \sum_{i\in I_k} e_i$, $k = 1, \ldots, s$, are primitive pairwise orthogonal central idempotents of Λ with $f_1 + \cdots + f_s = 1$.*

Proof All statements in the lemma, except for the last sentence, clearly hold. It is also clear that the elements $f_1, \ldots, f_s$ are pairwise orthogonal idempotents whose sum is equal to 1.

We show first that $f_k \in Z(\Lambda)$ for all $k = 1, \ldots, s$. In fact, for $x \in \Lambda$ we have

$$xf_k = 1xf_k = \sum_{i\in I}\sum_{j\in I_k} e_i x e_j = \sum_{i,j\in I_k} e_i x e_j\,,$$

since $e_i x e_j \in e_i \Lambda e_j = 0$ whenever $i \in I \smallsetminus I_k$ and $j \in I_k$. Similarly, one has $f_k x = \sum_{i,j\in I_k} e_i x e_j$ and therefore $xf_k = f_k x$, and $f_k \in Z(\Lambda)$.

Next we show that, for each $k = 1, \ldots, s$, the central idempotent f_k is primitive in $Z(\Lambda)$. Assume that $f_k = g + h$ is an orthogonal decomposition with central idempotents g and h, and assume that $g \neq 0$. Then $0 \neq g = gf_k$ implies that $ge_i \neq 0$ for some $i \in I_k$ and therefore $ge_i = e_i$. But then, for each $j \in I$ with $e_i\Lambda e_j \neq 0$ one has $0 \neq e_i \Lambda e_j = e_i g \Lambda e_j = e_i \Lambda g e_j$. This implies that $ge_j \neq 0$ and therefore $ge_j = e_j$.

Similarly, also $e_j \Lambda e_i \neq 0$ implies that $ge_j = e_j$. Thus, we obtain $ge_j = e_j$ for all $j \in I_k$. This implies $g = gf_k = \sum_{j\in I_k} ge_j = \sum_{j\in I_k} e_j = f_k$ and $h = 0$. Thus, f_k is a primitive central idempotent.

It is now clear from the orthogonal decomposition $1 = f_1 + \cdots + f_s$ that each primitive central idempotent of Λ must be equal to f_k for some $k \in \{1, \ldots, s\}$. $\square$

5.5 Theorem *Let R be an integral domain with field of fractions K such that $|G|$, $|\mathrm{Out}(U)|$ and $\varphi(|\mathrm{Out}(U)|)$, for $U \leqslant G$, are invertible in K. Then $RB^{\lhd}(G,G)$ has no central idempotent different from 0 and 1. In particular, the ring $B^{\lhd}(G,G)$ is connected.*

Proof Let e be a non-zero central idempotent of $RB^{\lhd}(G,G)$. We will show that $e = 1$. We will use the commutative diagram of canonical embeddings

$$\begin{array}{ccc} RB^{\Delta}(G,G) & \subseteq & RB^{\lhd}(G,G) \\ \downarrow & & \downarrow \\ KB^{\Delta}(G,G) & \subseteq & KB^{\lhd}(G,G) \end{array}$$

Since e is a central idempotent of $RB^{\lhd}(G,G)$ it is also a central idempotent of $KB^{\lhd}(G,G)$. By Corollary 5.3, we obtain that e is a central idempotent of $KB^{\Delta}(G,G)$. From Remark 3.9(a) we obtain that

$$e = \sum_{(U,\chi)\in\mathcal{E}} e_{(U,\chi)}$$

for a subset $\mathcal{E}$ of $\mathcal{E}_G$. By Lemma 5.4 applied to the ring $KB^{\lhd}(G,G)$ and the idempotents $e_{(U,\chi)}$, $(U,\chi) \in \mathcal{E}_G$, we know that the subset $\mathcal{E}$ has the property that if $(U,\chi) \in \mathcal{E}$ and $(U',\chi') \in \mathcal{E}_G$ satisfy $e_{(U,\chi)} \cdot_G KB^{\lhd}(G,G) \cdot_G e_{(U',\chi')} \neq \{0\}$ or $e_{(U',\chi')} \cdot_G KB^{\lhd}(G,G) \cdot_G e_{(U,\chi)} \neq \{0\}$ then also $(U',\chi') \in \mathcal{E}$. However, Corollary 5.8 to Lemma 5.6 below implies:

$$\text{If } (U,1) \in \mathcal{E} \text{ for some } U \in \hat{\Sigma}_G \text{ then } (U',1) \in \mathcal{E} \text{ for all } U' \in \hat{\Sigma}_G. \tag{11}$$

Since $KB^{\Delta}(G,G) \cap RB^{\lhd}(G,G) = RB^{\Delta}(G,G)$, we also obtain that e is a central idempotent of $RB^{\Delta}(G,G)$. Now Theorem 3.8 and Proposition 3.7 imply:

$$\text{If } (U,\chi) \in \mathcal{E} \text{ then } (U,\chi') \in \mathcal{E} \text{ for all } \chi' \in \mathrm{Irr}_K(\mathrm{Out}(U)). \tag{12}$$

Since $e \neq 0$, there exists at least one element $(U,\chi) \in \mathcal{E}$. But then (11) and (12) together imply that $\mathcal{E} = \mathcal{E}_G$, or in other words that $e = 1$. $\square$

For a subset X of a group G, a field K, and a character χ of a KG-module, we set $X^+ := \sum_{x\in X} x \in KG$ and $\chi(X^+) := \sum_{x\in X} \chi(x)$. By χ^* we denote the contragredient character of χ. For a subgroup V of G, we denote by $\mathrm{Aut}_G(V)$ the image of the map $N_G(V) \to \mathrm{Aut}(V)$, $g \mapsto c_g$, and by $\mathrm{Out}_G(V)$ the image of $\mathrm{Aut}_G(V)$ under the canonical epimorphism $\mathrm{Aut}(V) \to \mathrm{Out}(V)$. If χ is a K-character of $\mathrm{Aut}(U)$ for some finite group U and if V is another group that is isomorphic to U then χ_V denotes the character of $\mathrm{Aut}(V)$ defined by $\chi_V(\omega) := \chi(\lambda^{-1} \circ \omega \circ \lambda)$ for any isomorphism $\lambda\colon U \xrightarrow{\cong} V$. The character χ_V is independent of the choice of λ. Similarly, if χ is a K-character of $\mathrm{Out}(U)$ and V is isomorphic to U one defines the character χ_V of $\mathrm{Out}(V)$. Recall the definition of $\tilde{\Sigma}_G(U)$ for $U \in \hat{\Sigma}_G$ from Remark 3.9.

5.6 Lemma *Let K be a field such that $|G|$, $|\mathrm{Out}(U)|$ and $\varphi(|\mathrm{Out}(U)|)$, for $U \leqslant G$, are invertible in K. Then, for any $(U,\chi),(U',\chi') \in \mathcal{E}_G$, one has $e_{(U,\chi)} \cdot_G KB^{\lhd}(G,G) \cdot_G e_{(U',\chi')} \neq \{0\}$ if and only if there exist $V \in \tilde{\Sigma}(U)$, $V' \in \tilde{\Sigma}(U')$, an epimorphism $\alpha\colon V' \to V$ and an element $(\overline{\omega},\overline{\omega'}) \in \mathrm{Out}(V) \times \mathrm{Out}(V')$ such that*

$$(\chi_V^* \times \chi'_{V'})\big([(\mathrm{Out}_G(V) \times \mathrm{Out}_G(V')) \cdot (\overline{\omega},\overline{\omega'}) \cdot \overline{L}_\alpha]^+\big) \neq 0\,. \tag{13}$$

Here, $L_\alpha := \mathrm{stab}_{\mathrm{Aut}(V)\times\mathrm{Aut}(V')}(\alpha)$ under the action $(\omega,\omega')\cdot\alpha := \omega\circ\alpha\circ(\omega')^{-1}$ and $\overline{L}_\alpha \leqslant \mathrm{Out}(V)\times\mathrm{Out}(V')$ denotes the image of L_α under the canonical epimorphism $\mathrm{Aut}(V)\times\mathrm{Aut}(V') \to \mathrm{Out}(V)\times\mathrm{Out}(V')$.

The proof of Lemma 5.6 is very technical and will be given in Section 6.

5.7 Corollary *Let K be as in Lemma 5.6. Then the primitive central idempotents of $KB^{\lhd}(G,G)$ are parametrized by the equivalence classes of $\mathcal{E}_G$ under the equivalence relation $\approx$ defined as the transitive closure of the relation $\sim$ which is defined by*

$$(U,\chi) \sim (U',\chi') :\iff e_{(U,\chi)} \cdot_G KB^{\lhd}(G,G) \cdot_G e_{(U',\chi')} \neq 0 \quad \text{or} \quad e_{(U',\chi')} \cdot_G KB^{\lhd}(G,G) \cdot_G e_{(U,\chi)} \neq 0\,.$$

If $\mathcal{E} \subseteq \mathcal{E}_G$ is such an equivalence class then $\sum_{(U,\chi)\in\mathcal{E}} e_{(U,\chi)}$ is the corresponding primitive central idempotent.

Proof This follows immediately from Lemma 5.4 applied to the idempotents $e_{(U,\chi)}$ of $KB^{\lhd}(G,G)$. They satisfy the hypothesis of the Lemma, since they are the primitive central idempotents of $KB^{\Delta}(G,G)$ (see Remark 3.9) and since each central idempotent of $KB^{\lhd}(G,G)$ is contained in $KB^{\Delta}(G,G)$ (see Corollary 5.3). ∎

5.8 Corollary *Let K be a field as in Lemma 5.6. Then, for any $(U,\chi) \in \mathcal{E}_G$, one has:*

$$e_{(1,1)} \cdot_G KB^{\lhd}(G,G) \cdot_G e_{(U,\chi)} \neq \{0\} \iff \chi = 1\,.$$

Proof By Lemma 5.6, the condition $e_{(1,1)}KB^{\lhd}(G,G)e_{(U,\chi)} \neq \{0\}$ is equivalent to the existence of $V \in \tilde{\Sigma}_G(U)$ and $\omega \in \mathrm{Aut}(V)$ such that

$$(1 \times \chi_V)\big([(\mathrm{Out}_G(1) \times \mathrm{Out}_G(V)) \cdot (1,\overline{\omega}) \cdot \overline{L}_\alpha]^+\big) \neq 0\,.$$

Note that here $\alpha\colon V \to 1$ is the trivial homomorphism and that consequently $L_\alpha = \mathrm{Aut}(1) \times \mathrm{Aut}(V)$. Identifying $\mathrm{Aut}(1) \times \mathrm{Aut}(V)$ with $\mathrm{Aut}(V)$ and $\mathrm{Out}(1) \times \mathrm{Out}(V)$ with $\mathrm{Out}(V)$ via the second projection, we obtain

$$\begin{aligned} & e_{(1,1)} \cdot_G KB^{\lhd}(G,G) \cdot_G e_{(U,\chi)} \neq \{0\} \iff \chi_V\big((\mathrm{Out}_G(V) \cdot \overline{\omega} \cdot \mathrm{Out}(V))^+\big) \neq 0 \\ \iff & \chi_V\big(\mathrm{Out}(V)^+\big) \neq 0 \iff (\chi_V, 1) \neq 0 \iff \chi_V = 1 \iff \chi = 1\,, \end{aligned}$$

and the proof is complete. ∎

5.9 Remark Assume that $\alpha\colon V' \to V$ is a surjective group homomorphism. We want to get a better understanding of the subgroup L_α of $\mathrm{Aut}(V) \times \mathrm{Aut}(V')$ in Lemma 5.6. Set $\mathrm{Aut}(V', \ker(\alpha)) := \{\omega' \in \mathrm{Aut}(V') \mid \omega'(\ker(\alpha)) = \ker(\alpha)\}$. Then α induces a group homomorphism

$$\alpha_*\colon \mathrm{Aut}(V', \ker(\alpha)) \to \mathrm{Aut}(V)$$

where $(\alpha_*(\omega'))(\alpha(v')) := \alpha(\omega'(v'))$ for $\omega' \in \mathrm{Aut}(V', \ker(\alpha))$ and $v' \in V'$. It is now straightforward to verify that $p_1(L_\alpha) = \mathrm{im}(\alpha_*)$, $k_1(L_\alpha) = \{1\}$, $p_2(L_\alpha) = \mathrm{Aut}(V', \ker(\alpha))$, $k_2(L_\alpha) := \ker(\alpha_*)$ and that the isomorphism $p_2(L_\alpha)/k_2(L_\alpha) \to p_1(L_\alpha)/k_1(L_\alpha)$ determined by L_α is equal to the isomorphism $\overline{\alpha}_*\colon \mathrm{Aut}(K, \ker(\alpha))/\ker(\alpha_*) \to \mathrm{im}(\alpha_*)$. Note that $p_1(L_\alpha)$ consists of all automorphisms ω of V which can be "lifted" (via α) to an automorphism ω' of V', i.e., $\omega\alpha = \alpha\omega'\colon V' \to V$.

Moreover, $\mathrm{Inn}(V') \leqslant \mathrm{Aut}(V', \ker(\alpha))$, since $\ker(\alpha)$ is normal in V', and $\alpha_*(c_{v'}) = c_{\alpha(v')}$ for $v' \in V'$, so that $\alpha_*(\mathrm{Inn}(V')) = \mathrm{Inn}(V)$. This implies that the subgroup $\overline{L}_\alpha = L_\alpha{\cdot}(\mathrm{Inn}(V)\times\mathrm{Inn}(V'))/\mathrm{Inn}(V)\times\mathrm{Inn}(V')$ of $\mathrm{Out}(V) \times \mathrm{Out}(V')$ satisfies $p_1(\overline{L}_\alpha) = \mathrm{im}(\alpha_*)/\mathrm{Inn}(V)$, $k_1(\overline{L}_\alpha) = 1$, $p_2(\overline{L}_\alpha) = \mathrm{Aut}(V, \ker(\alpha))/\mathrm{Inn}(V')$, $k_2(\overline{L}_\alpha) = \ker(\alpha_*){\cdot}\mathrm{Inn}(V')/\mathrm{Inn}(V')$ and the isomorphism $p_2(\overline{L}_\alpha)/k_2(\overline{L}_\alpha) \to p_1(\overline{L}_\alpha)/k_1(\overline{L}_\alpha)$ corresponding to $\overline{L}_\alpha$ is induced by α_* and again denoted by $\overline{\alpha}_*$.

We refer the reader to [Bc2, Chapter 2] for the definitions of deflation maps $\mathrm{def}^G_{G/N}\colon R_K(G) \to R_K(G/N)$ and inflation maps $\mathrm{inf}^G_{G/N}\colon R_K(G/N) \to R_K(G)$ when $N \trianglelefteq G$, and the isomorphism maps $\mathrm{iso}_\alpha\colon R_K(G_2) \to R_K(G_1)$ when $\alpha\colon G_1 \xrightarrow{\sim} G_2$ is an isomorphism. Here K is a field such that the orders of G, G_1 and G_2 are invertible in K. More generally one also defines inflation and deflation maps for arbitrary epimorphisms by combining the above definitions with an isomorphism map.

5.10 Lemma *Let K be a field as in Lemma 5.6 and let $\alpha\colon V' \to V$ be a surjective group homomorphism between subgroups of G, and let $\chi \in \mathrm{Irr}_K(\mathrm{Out}(V))$ and $\chi' \in \mathrm{Irr}_K(\mathrm{Out}(V'))$. Asumme further that $\mathrm{Out}_G(V)$ and $\mathrm{Out}_G(V')$ are trivial. Then the following are equivalent:*

(i) *There exists $(\omega, \omega') \in \mathrm{Aut}(V) \times \mathrm{Aut}(V')$ such that (13) holds.*

(ii) *One has $((\chi^* \times \chi')|_{\overline{L}_\alpha}, 1) \neq 0$.*

(iii) *The irreducible character χ is a constituent of the image of χ' under the composition of the following sequence of maps: $\mathrm{res}^{\mathrm{Out}(V')}_{\mathrm{Out}(V',\ker(\alpha))}$, $\mathrm{def}^{\mathrm{Out}(V',\ker(\alpha))}_{\mathrm{Out}(V',\ker(\alpha))/k_2(\overline{L}_\alpha)}$, $\mathrm{iso}_{\overline{\alpha}_*}$, $\mathrm{ind}^{\mathrm{Out}(V)}_{\mathrm{im}(\alpha_*)/\mathrm{Inn}(V)}$.*

(iv) *The irreducible character χ' is a constituent of the image of χ under the composition of the following sequence of maps: $\mathrm{res}^{\mathrm{Out}(V)}_{\mathrm{im}(\alpha_*)/\mathrm{Inn}(V)}$, $\mathrm{iso}^{-1}_{\overline{\alpha}_*}$, $\mathrm{inf}^{\mathrm{Out}(V',\ker(\alpha))}_{\mathrm{Out}(V',\ker(\alpha))/k_2(\overline{L}_\alpha)}$, $\mathrm{ind}^{\mathrm{Out}(V')}_{\mathrm{Out}(V',\ker(\alpha))}$.*

Proof By [BD2, Lemma 7.3], the condition in (13) is equivalent to the condition $(\chi^* \times \chi')(\overline{L}^+_\alpha) \neq 0$ which in turn is equivalent to the condition that $\chi^* \times \chi'$ is a constituent of $\mathrm{ind}^{\mathrm{Out}(V)\times\mathrm{Out}(V')}_{\overline{L}_\alpha}(1)$. Since the latter is a permutation character, this is equivalent to $\chi \times {\chi'}^*$ being a constituent of the same character. The equivalence between (ii) and (iii) now follows from the following general consideration: If A and B are finite groups such that $|A \times B|$ is invertible in K and if L is a subgroup of $A \times B$ then the permutation character of $A \times B/L$ is equal to the sum of the characters $I(\psi) \times \psi^*$, where ψ runs through the irreducible characters of B and $I\colon R_K(B) \to R_K(A)$ is the map induced by tensoring with the (KG, KH)-bimodule $K[A \times B/L]$ over KB. Now the result follows from the decomposition of the transitive biset $A \times B/L$ as in [Bc2, Lemma 2.3.26]. Finally, the equivalence between (ii) and (iv) follows from the last equivalence applied to the dual subgroup $L^\circ := \{(b, a) \in B \times A \mid (a, b) \in L\}$ of L. □

5.11 Corollary *Let K be a field as in Lemma 5.6. Then $(G,\chi)\in\mathcal{E}_G$ for each $\chi\in\mathrm{Irr}_K(\mathrm{Out}(G))$. Moreover, for any $\chi,\chi'\in\mathrm{Irr}_K(\mathrm{Out}(G))$ one has*

$$e_{(G,\chi)}\cdot_G KB^{\lhd}(G,G)\cdot_G e_{(G,\chi')}\neq\{0\}\iff\chi=\chi'\,.$$

Proof Since $K\overline{\mathrm{Inj}}(G,G)\cong K\mathrm{Out}(G)$ as left $K\mathrm{Out}(G)$-modules, each $\chi\in\mathrm{Irr}_K(\mathrm{Out}(G))$ is a constituent of the character of $K\overline{\mathrm{Inj}}(G.G)$. Thus $(G,\chi)\in\mathcal{E}_G$. Now let $\chi,\chi'\in\mathrm{Irr}_K(\mathrm{Out}(G))$. Since $\mathrm{Out}_G(G)$ is trivial we can use Lemma 5.10. First note that if $\alpha\in\mathrm{Aut}(G)$ then $L_\alpha={}^{(\alpha,1)}\Delta(\mathrm{Aut}(G))$ and $\overline{L}_\alpha\leqslant\mathrm{Out}(G)\times\mathrm{Out}(G)$ is conjugate to $\Delta(\mathrm{Out}(G))$. Thus, the composition of the sequence of the maps in (iv) in Lemma 5.10 is the identity map, and the result follows. □

6 Proof of Lemma 5.6

The goal of this section is the proof of Lemma 5.6. Throughout this section, G denotes a finite group and K denotes a field such that $|G|$, $|\mathrm{Out}(U)|$ and $\varphi(|\mathrm{Out}(U)|)$, for $U\leqslant G$, are invertible in K. We first need to recall constructions from [BD1].

6.1 Remark (a) Recall from [BD1, Section 4] that there is an isomorphism

$$\rho_G\colon KB^{\lhd}(G,G)\to KA(G,G)^{G\times G}\,, \tag{14}$$

where $A(G,G)$ is the free abelian group on the set of triples (U,α,V), where U and V are subgroups of G and $\alpha\colon V\to U$ is an epimorphism. Moreover $KA(G,G)$ denotes the K-vector space with the same triples as basis. The group $G\times G$ acts on these triples by ${}^{(g,h)}(U,\alpha,V):=({}^gU,c_g\alpha c_h^{-1},{}^hV)$, and $KA(G,G)^{G\times G}$ denotes the fixed points under the extended action on $KA(G,G)$. The $G\times G$-conjugacy class of (U,α,V) is denoted by $[U,\alpha,V]_{G\times G}$ and the class sums $[U,\alpha,V]^+_{G\times G}$ form a K-basis of $KA(G,G)^{G\times G}$. By [BD1, Theorem 4.7], the multiplication in $KB^{\lhd}(G,G)$ is translated under the isomorphism ρ_G into the multiplication on $KA(G,G)^{G\times G}$ which is the restriction of the following multiplication on $KA(G,G)$:

$$(U,\alpha,V)\cdot_G(V',\beta,W):=\begin{cases}\frac{|C_G(V)|}{|G|}(U,\alpha\circ\beta,W) & \text{if } V=V',\\ 0 & \text{if } V\neq V'.\end{cases}$$

(b) Under the isomorphism ρ_G in (a), the subspace $KB^{\Delta}(G,G)$ is mapped isomorphically onto the $G\times G$-fixed points of the K-span $KA^{\Delta}(G,G)$ of triples of the form (U,α,V), where $\alpha\colon V\to U$ is an isomorphism. The isomorphism

$$\rho_G\circ\sigma_G^{-1}\colon\bigoplus_{U\in\hat{\Sigma}_G}\mathrm{End}_{K\mathrm{Out}(U)}(K\overline{\mathrm{Inj}}(U,G))\xrightarrow{\sim}KA^{\Delta}(G,G)^{G\times G}$$

is given explicitly as follows: For $U\in\hat{\Sigma}_G$ and a $K\mathrm{Out}(U)$-module endomorphism $f\colon[\mu]\mapsto\sum_{[\lambda]}a_{[\lambda],[\mu]}[\lambda]$ of $K\overline{\mathrm{Inj}}(U,G)$, the image of f under $\rho_G\circ\sigma_G^{-1}$ is given by

$$\sum_{\lambda\times_{\mathrm{Aut}(U)}\mu\in\mathrm{Inj}(U,G)\times_{\mathrm{Aut}(U)}\mathrm{Inj}(U,G)}a_{[\lambda],[\mu]}\cdot(\lambda(U),\lambda\circ\mu^{-1},\mu(U))\,,$$

cf. the proof of Theorem 5.5(d) in [BD1]. Here, $\mathrm{Aut}(U)$ acts on $\mathrm{Inj}(U,G)$ from the right via composition and from the left by using the right action and inversion of group elements.

For some of the notation in the following proposition we refer the reader back to Remark 3.9.

6.2 Proposition *For $(U,\chi) \in \mathcal{E}_G$ and $V \in \tilde{\Sigma}(U)$ with $\chi = s(\psi_1 + \cdots + \psi_r)$ a decomposition into absolutely irreducible characters over some extension field of K, one has*

$$\rho_G(e_{(U,\chi,V)}) = \frac{\chi(1)}{s^2 r|\mathrm{Out}(U)|} \sum_{\substack{g,h\in G/N_G(V)\\ \omega\in\mathrm{Aut}(V)}} \chi_V\left((\overline{\omega}^{-1}\cdot \mathrm{Out}_G(V))^+\right)\cdot {}^{(g,h)}(V,\omega,V)\,. \tag{15}$$

Moreover, $e_{(U,\chi,V)} \neq 0$ if and only if $(\chi_V|_{\mathrm{Out}_G(V)},1)\neq 0$.

Proof We set $c := \chi(1)/(s^2 r|\mathrm{Aut}(U)|)$. Using the explicit formula (7) for e_χ in Remark 3.9, the U-component f of $\sigma_G(e_{(U,\chi,V)})$ is given by

$$[\mu] \mapsto c \sum_{\omega\in\mathrm{Aut}(U)} \chi(\omega^{-1})[\mu\omega]\,,$$

for $\mu\in\mathrm{Inj}(U,G)_V$, if we denote the inflation of χ to $\mathrm{Aut}(U)$ again by χ. Thus, the matrix coefficients $a_{[\lambda],[\mu]}$ of f with respect to the basis $\overline{\mathrm{Inj}}(U,G)_V$ are given by

$$a_{[\lambda],[\mu]} = \begin{cases} c\displaystyle\sum_{\substack{\omega\in\mathrm{Aut}(U)\\ [\lambda]=[\mu\omega]}} \chi(\omega^{-1}) & \text{if } \mu\in\mathrm{Inj}(U,G)_V,\\ 0 & \text{otherwise.}\end{cases}$$

Using the explicit description of $\rho_G\circ\sigma_G^{-1}$ in Remark 6.1(b), we obtain

$$\rho_G(e_{(U,\chi,V)}) = \sum_{\lambda\times_{\mathrm{Aut}(U)}\mu\in\mathrm{Inj}(U,G)_V\times_{\mathrm{Aut}(U)}\mathrm{Inj}(U,G)_V}\ \sum_{\substack{\omega\in\mathrm{Aut}(U)\\ [\lambda]=[\mu\omega]}} c\chi(\omega^{-1})\cdot(\lambda(U),\lambda\mu^{-1},\mu(U))\,.$$

Note that $\mathrm{Aut}(U)$ acts freely on $\mathrm{Inj}(U,G)_V\times\mathrm{Inj}(U,G)_V$ so that replacing the summation over $\mathrm{Inj}(U,G)_V\times_{\mathrm{Aut}(U)}\mathrm{Inj}(U,G)_V$ yields

$$\sum_{(\lambda,\mu)\in\mathrm{Inj}(U,G)_V\times\mathrm{Inj}(U,G)_V}\ \sum_{\substack{\omega\in\mathrm{Aut}(U)\\ [\lambda]=[\mu\omega]}} c'\chi(\omega^{-1})\cdot(\lambda(U),\lambda\mu^{-1},\mu(U))\,,$$

where $c' := c/|\mathrm{Aut}(U)|$. Let $\lambda_0\colon U\to V$ be a fixed isomorphism. Note that if g runs through a set of representatives of $G/N_G(V)$ and α runs through $\mathrm{Aut}(U)$ then $c_g\lambda_0\alpha$ runs through $\mathrm{Inj}(U,G)_V$ without repetition. Thus we can rewrite the last expression as

$$\sum_{g,h\in G/N_G(V)}\ \sum_{\substack{\alpha,\beta,\omega\in\mathrm{Aut}(U)\\ [c_g\lambda_0\alpha]=[c_h\lambda_0\beta\omega]}} c'\chi(\omega^{-1})\cdot {}^{(g,h)}(V,\lambda_0\alpha\beta^{-1}\lambda_0^{-1},V)\,.$$

It is straightforward to verify that $[c_g\lambda_0\alpha] = [c_h\lambda_0\beta\omega]$ if and only if $\omega \in \beta^{-1}\lambda_0^{-1}\mathrm{Aut}_G(V)\lambda_0\alpha$. Thus, with $\omega = \beta^{-1}\lambda_0^{-1}\gamma\lambda_0\alpha$ for $\gamma \in \mathrm{Aut}_G(V)$, the last expression can be rewritten as

$$\sum_{g,h\in G/N_G(V)}\ \sum_{\alpha,\beta\in\mathrm{Aut}(U)}\ \sum_{\gamma\in\mathrm{Aut}_G(V)} c'\chi(\alpha^{-1}\lambda_0^{-1}\gamma^{-1}\lambda_0\beta)\cdot\ {}^{(g,h)}(V,\lambda_0\alpha\beta^{-1}\lambda_0^{-1},V)\,.$$

Using $\chi(\alpha^{-1}\lambda_0^{-1}\gamma^{-1}\lambda_0\beta) = \chi(\beta\alpha^{-1}\lambda_0^{-1}\gamma^{-1}\lambda_0) = \chi_V(\lambda_0\beta\alpha^{-1}\lambda_0^{-1}\gamma^{-1})$ and rewriting $\lambda_0\alpha\beta^{-1}\lambda_0^{-1}$ as $\omega \in \mathrm{Aut}(V)$ we obtain

$$\begin{aligned}&\sum_{g,h\in G/N_G(V)}\ \sum_{\omega\in\mathrm{Aut}(V)}\ \sum_{\gamma\in\mathrm{Aut}_G(V)} c''\chi_V(\omega^{-1}\gamma^{-1})\cdot\ {}^{(g,h)}(V,\omega,V)\\ =&\sum_{\substack{g,h\in G/N_G(V)\\ \omega\in\mathrm{Aut}(V)}} c''\cdot\chi_V((\omega^{-1}\mathrm{Aut}_G(V))^+)\cdot\ {}^{(g,h)}(V,\omega,V)\end{aligned}$$

with $c'' = c'\cdot|\mathrm{Aut}(U)| = c$. Since $\chi_V((\omega^{-1}\mathrm{Aut}_G(V))^+) = |\mathrm{Inn}(V)|\cdot\chi_V((\overline{\omega}^{-1}\cdot\mathrm{Out}_G(V))^+)$, we obtain Equation (15).

Finally, note that different choices of triples (g,h,ω) in the sum of Equation (15) lead to different basis elements ${}^{(g,h)}(V,\omega,V)$ of $KA(G,G)$. Thus, $\rho(e_{(U,\chi,V)}) \neq 0$ if and only if there exists $\omega \in \mathrm{Aut}(V)$ such that $\chi_V((\overline{\omega}\cdot\mathrm{Out}_G(V))^+) \neq 0$. By [BD2, Lemma 7.3], this is equivalent to $\chi_V(\mathrm{Out}_G(V)^+) \neq 0$. But this in turn is equivalent to $(\chi_V|_{\mathrm{Out}_G(V)},1) \neq 0$. □

Proof of Lemma 5.6. Let $(U,\chi),(U',\chi') \in \mathcal{E}_G$. Then the orthogonal decomposition (9) implies that

$$e_{(U,\chi)}\cdot_G KB^{\triangleleft}(G,G)\cdot_G e_{(U',\chi')} \neq 0 \tag{16}$$

if and only if there exist $V \in \tilde{\Sigma}(U)$ and $V' \in \tilde{\Sigma}(U')$ satisfying

$$e_{(U,\chi,V)}\cdot_G KB^{\triangleleft}(G,G)\cdot_G e_{(U',\chi',V')} \neq 0\,, \tag{17}$$

Applying the isomorphism ρ_G on both sides, the last condition is equivalent to the existence of a basis element (W,α,W') of $KA(G,G)$ such that the $G\times G$-orbit sum $[W,\alpha,W']^+_{G\times G}$ satisfies $\rho_G(e_{(U,\chi,V)})\cdot_G [W,\alpha,W']^+_{G\times G}\cdot_G\rho_G(e_{(U',\chi',V')}) \neq 0$. Since the last expression is equal to 0 if W is not G-conjugate to V or W' is not G-conjugate to V' (by the explicit formula in (15)), and since we may replace (W,α,W') by any $G\times G$-conjugate, the condition in (17) is equivalent to the existence of an epimorphism $\alpha\colon V'\to V$ such that

$$\rho_G(e_{(U,\chi,V)})\cdot_G [V,\alpha,V']^+_{G\times G}\cdot_G\rho_G(e_{(U',\chi',V')}) \neq 0\,. \tag{18}$$

Recall from [BD1, Proposition 1.7] that one has

$$[V,\alpha,V']^+_{G\times G} = \sum_{(x,y)\in\mathcal{A}\times\mathcal{B}} {}^{(x,y)}(V,\alpha,V')\,,$$

where $\mathcal{A}\subseteq G$ is a set of representatives of the cosets $G/k_1(N_{G\times G}(\Delta(V,\alpha,V')))$ and $\mathcal{B}\subseteq G$ is a set of representatives of the cosets $G/p_2(N_{G\times G}(\Delta(V,\alpha,V')))$. Using the explicit formula (15), the condition in (18) is equivalent to

$$\begin{aligned}\sum_{(x,y)\in\mathcal{A}\times\mathcal{B}}\ \sum_{\substack{g,h\in G/N_G(V)\\ \omega\in\mathrm{Aut}(V)}}\ \sum_{\substack{g',h'\in G/N_G(V')\\ \omega'\in\mathrm{Aut}(V')}} &\chi_V((\omega^{-1}\mathrm{Aut}_G(V))^+)\chi_{V'}((\omega'^{-1}\mathrm{Aut}_G(V'))^+)\cdot\\ &\cdot\ {}^{(g,h)}(V,\omega,V)\cdot_G {}^{(x,y)}(V,\alpha,V')\cdot_G {}^{(g',h')}(V',\omega',V') \neq 0\,.\end{aligned}$$

Since $k_1(N_{G\times G}(\Delta(V,\alpha,V'))) = C_G(V) \leqslant N_G(V)$ and $p_2(N_{G\times G}(\Delta(V,\alpha,V'))) \leqslant N_G(V')$, cf. [BD1, Proposition 1.7], each element $x \in \mathcal{A}$ (resp. $y \in \mathcal{B}$) determines a unique element $h \in G/N_G(V)$ (resp. $g' \in G/N_G(V')$) such that the multiplication $\cdot_G$ is non-zero, and for given $x \in \mathcal{A}$ (resp. $y \in \mathcal{B}$), we may adjust the representatives h of $G/N_G(V)$ (resp. g' of $G/N_G(V')$) such that x (resp. y) occurs as a representative. Thus, the above condition is equivalent to

$$\sum_{(x,y)\in\mathcal{A}\times\mathcal{B}} \sum_{\substack{g\in G/N_G(V)\\ h'\in G/N_G(V')}} \sum_{\substack{\omega\in\mathrm{Aut}(V)\\ \omega'\in\mathrm{Aut}(V')}} \chi_V((\omega^{-1}\mathrm{Aut}_G(V))^+)\chi'_{V'}(({\omega'}^{-1}\mathrm{Aut}_G(V'))^+)\cdot {}^{(g,h')}(V,\omega\alpha\omega',V') \neq 0\,.$$

Since x and y don't occur in the argument of the sum and since $|\mathcal{A}\times\mathcal{B}|$ is invertible in K, we may drop the first sum in the above condition. Moreover, since for the various choices of g and h' in the above sum, the sets $\{{}^{(g,h')}(V,\omega\alpha\omega',V') \mid (\omega,\omega') \in \mathrm{Aut}(V)\times\mathrm{Aut}(V')\}$ of basis elements are pairwise disjoint, the above condition is also equivalent to

$$\sum_{(\omega,\omega')\in\mathrm{Aut}(V)\times\mathrm{Aut}(V')} \chi_V((\omega^{-1}\mathrm{Aut}_G(V))^+)\chi'_{V'}(({\omega'}^{-1}\mathrm{Aut}_G(V'))^+)\cdot(V,\omega\alpha\omega',V') \neq 0\,.$$

Next we fix an element $(\omega_0,\omega_0') \in \mathrm{Aut}(V)\times\mathrm{Aut}(V')$ and determine the coefficient of $(V,\omega_0\alpha\omega_0',V')$ in the above sum. Let $L_\alpha := \mathrm{stab}_{\mathrm{Aut}(V)\times\mathrm{Aut}(V')}(\alpha)$. Then, for any $(\omega,\omega') \in \mathrm{Aut}(V)\times\mathrm{Aut}(V')$, we have

$$\omega\alpha\omega' = \omega_0\alpha\omega_0' \iff (\omega^{-1},\omega') \in L_\alpha(\omega_0^{-1},\omega_0')\,.$$

Thus, writing $(\omega^{-1},\omega') = (\theta,\theta')(\omega_0^{-1},\omega_0')$, for $(\theta,\theta') \in L_\alpha$, the last condition is equivalent to requiring that there exists an element $(\omega_0,\omega_0') \in \mathrm{Aut}(V)\times\mathrm{Aut}(V')$ such that

$$\sum_{(\theta,\theta')\in L_\alpha} \chi_V((\theta\omega_0^{-1}\mathrm{Aut}_G(V))^+)\chi'_{V'}(({\omega_0'}^{-1}{\theta_0'}^{-1}\mathrm{Aut}_G(V'))^+) \neq 0\,.$$

Since $\chi_V((\theta\omega_0^{-1}\mathrm{Aut}_G(V))^+) = \chi_V^*((\mathrm{Aut}_G(V)\omega_0\theta^{-1})^+)$ and $\chi'_{V'}(({\omega_0'}^{-1}{\theta_0'}^{-1}\mathrm{Aut}_G(V'))^+) = \chi'_{V'}((\mathrm{Aut}_G(V'){\omega_0'}^{-1}{\theta'}^{-1})^+)$, the sum in the above equation is equal to

$$\begin{aligned}&\sum_{(\theta,\theta')\in L_\alpha} \chi_V^*((\mathrm{Aut}_G(V)\omega_0\theta^{-1})^+)\chi'_{V'}((\mathrm{Aut}_G(V'){\omega_0'}^{-1}{\theta'}^{-1})^+)\\ &= c\cdot(\chi_V^*\times\chi'_{V'})\big([(\mathrm{Aut}_G(V)\times\mathrm{Aut}_G(V'))\cdot(\omega_0,{\omega_0'}^{-1})\cdot L_\alpha]^+\big)\end{aligned}$$

with $c = |(\mathrm{Aut}_G(V)\times\mathrm{Aut}_G(V'))\cap {}^{(\omega_0,{\omega_0'}^{-1})}L_\alpha|$ which is invertible in K. Finally,

$$\begin{aligned}&(\chi_V^*\times\chi'_{V'})\big([(\mathrm{Aut}_G(V)\times\mathrm{Aut}_G(V'))\cdot(\omega_0,{\omega_0'}^{-1})\cdot L_\alpha]^+\big)\\ &= |\mathrm{Inn}(V)\times\mathrm{Inn}(V')|\cdot(\chi_V^*\times\chi'_{V'})\big([(\mathrm{Out}_G(V)\times\mathrm{Out}_G(V'))\cdot(\overline{\omega}_0,\overline{\omega_0'}^{-1})\cdot\overline{L}_\alpha]^+\big)\,.\end{aligned}$$

In fact, $\mathrm{Inn}(V)\times\mathrm{Inn}(V')$ is contained in $\mathrm{Aut}_G(V)\times\mathrm{Aut}_G(V')$ and the canonical epimorphism $\mathrm{Aut}(V)\times\mathrm{Aut}(V')\to\mathrm{Out}(V)\times\mathrm{Out}(V')$ maps the set $(\mathrm{Aut}_G(V)\times\mathrm{Aut}_G(V'))\cdot(\omega_0,{\omega_0'}^{-1})\cdot L_\alpha$ onto the set $(\mathrm{Out}_G(V)\times\mathrm{Out}_G(V'))\cdot(\overline{\omega}_0,\overline{\omega_0'}^{-1})\cdot\overline{L}_\alpha$ with fibers of cardinality $|\mathrm{Inn}(V)\times\mathrm{Inn}(V')|$, since $\mathrm{Inn}(V)\times\mathrm{Inn}(V')$ acts freely by left multiplication on the first set. Now the proof of Lemma 5.6 is complete. □

7 Examples and Questions

7.1 Example (a) Let U be a cyclic group of order k, let U' be a cyclic group of order k', and let $\alpha\colon U' \to U$ be a surjective homomorphism. Then k divides k'. We want to determine the subgroup L_α of $\mathrm{Aut}(U) \times \mathrm{Aut}(U')$ using Remark 5.9. First note that $\mathrm{Aut}(U', \ker(\alpha)) = \mathrm{Aut}(U')$, since $\ker(\alpha)$ is the only subgroup of order k'/k in U'. Also note that one has a canonical isomorphism $(\mathbb{Z}/k'\mathbb{Z})^\times \to \mathrm{Aut}(U')$ mapping the residue class of an integer i which is coprime to k' to the automorphism which raises each element to its i-th power. Note that if $\omega' \in \mathrm{Aut}(U')$ corresponds to i then $\alpha_*(\omega') \in \mathrm{Aut}(U)$ also corresponds to i. Recall that the canonical map $(\mathbb{Z}/k'\mathbb{Z})^\times \to (\mathbb{Z}/k\mathbb{Z})^\times$ is surjective. Thus, $\alpha_*\colon \mathrm{Aut}(U') \to \mathrm{Aut}(U)$ is the canonical surjective map $p_{k,k'}$ which sends the automorphism $u' \mapsto {u'}^i$ of U' to the automorphism $u \mapsto u^i$ of U, for any integer i which is coprime to k'. In particular, α_* does not depend on α.

(b) Now let G be a cyclic group of order n and let K be a field of characteristic not dividing $n \cdot \varphi(n)$. Assume that U and U' as in (a) are subgroups of G and let $\alpha\colon U' \to U$ be a surjective group homomorphism. Moreover, let $\chi \in \mathrm{Irr}_K(\mathrm{Aut}(U))$ and $\chi' \in \mathrm{Irr}_K(\mathrm{Aut}(U'))$. Note that $(U,\chi) \in \mathcal{E}_G$, since $\overline{\mathrm{Inj}}(U,G) = \mathrm{Out}(U) = \mathrm{Aut}(U)$. Note that $\mathrm{Out}_G(U)$ and $\mathrm{Out}_G(U')$ are trivial so that we can use Lemma 5.10 which implies that $e_{(U,\chi)} \cdot_G KB^\lhd(G,G) \cdot_G e_{(U',\chi')} \neq 0$ if and only if $\chi' = \chi \circ p_{k,k'}$, that is, if and only if χ' is the inflation of χ with respect to $p_{k,k'}$.

(c) We define a partial order $\leqslant$ on the set $\mathcal{E}_G$ by setting $(U,\chi) \leqslant (U',\chi')$ if and only if $|U|$ divides $|U'|$ and $\chi' = \chi p_{|U|,|U'|}$. Note that the symmetric closure of $\leqslant$ is the same relation as $\sim$ in Corollary 5.7. Thus, the connected components (i.e., the equivalence classes of the symmetric and transitive closure of $\leqslant$) are the equivalence classes of $\mathcal{E}_G$ describing the primitive central idempotents of $KB^\lhd(G,G)$. We call a pair $(U,\chi) \in \mathcal{E}_G$ *primitive* if it is minimal with respect to $\leqslant$. It is well-known that for each element $(U',\chi') \in \mathcal{E}_G$ there exists a unique primitive element (U,χ) with $(U,\chi) \leqslant (U',\chi')$. It is now straightforward to see that the equivalence classes of $\mathcal{E}_G$ are represented by the elements (G,ϑ), $\vartheta \in \mathrm{Irr}_K(\mathrm{Aut}(G))$, or also by the set of primitive pairs of $\mathcal{E}_G$.

We summarize the results developed above in the following Theorem.

7.2 Theorem *Let G be a cyclic group of order n and let K be a field such that n and $\varphi(n)$ are invertible in K. Then each pair (U,χ) with $U \leqslant G$ and $\chi \in \mathrm{Irr}_K(\mathrm{Aut}(U))$ is contained in $\mathcal{E}_G$. The set of primitive central idempotents of $KB^\lhd(G,G)$ is parametrized by $\mathrm{Irr}_K(\mathrm{Aut}(G))$. For $\vartheta \in \mathrm{Irr}_K(\mathrm{Aut}(G))$ the corresponding primitive idempotent is the sum of the idempotents $e_{(U,\chi)}$ with $(U,\chi) \in \mathcal{E}_G$ satisfying $\vartheta = \chi p_{|U|,n}$.*

7.3 Remark Let G and K be as in the above theorem. From Theorem 8.11 and Remark 8.12(a) in [BD2] one can see that the primitive central idempotents of the full double Burnside algebra $KB(G,G)$ are indexed by the pairs $(U,\chi) \in \mathcal{E}_G$. Thus, the primitive central idempotents of $KB^\lhd(G,G)$ must split in $KB(G,G)$. Note also that the primitive central idempotents of $KB^\Delta(G,G)$ were also indexed by $\mathcal{E}_G$.

7.4 Example Let $G = \langle x_1, \ldots, x_n\rangle$ be an elementary abelian p-group of rank n for a prime p and let K be an algebraically closed field of characteristic 0.

(a) For $i = 0, \ldots, n$ set $U_i := \langle x_1, \ldots, x_i\rangle$ and $V_i := \langle x_{i+1}, \ldots, x_n\rangle$, thus $G = U_i \oplus V_i$ as $\mathbb{F}_p$-vector space. We can choose $\hat{\Sigma}_G$ as $\{U_0, \ldots, U_n\}$. For $0 \leqslant i \leqslant j \leqslant n$, let $\pi_{i,j}\colon U_j \to U_i$ denote the canonical projection which is the identity on U_i and has kernel $\langle x_{i+1}, \ldots, x_j\rangle =: V_{i,j}$. For $i = 0, \ldots, n$ we identify

$\mathrm{Aut}(U_i)$ with $G_i := \mathrm{GL}_i(\mathbb{F}_p)$ using the basis $(x_1,\ldots,x_i)$ of U_i (with $G_0 = \{1\}$). For $0 \leqslant i \leqslant j \leqslant n$, the projections $\pi_{i,j}\colon U_j \to U_i$ induce surjections $(\pi_{i,j})_*\colon P_{i,j} := \mathrm{Aut}(U_j, V_{i,j}) \to G_i$, where $P_{i,j}$ is the parabolic subgroup of G_j of shape $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$ with upper left corner of size $i\times i$ and lower right corner of size $(i-j)\times(i-j)$. Moreover, we set $Q_{i,j} := \ker((\pi_{i,j})_*)$, which is the subgroup of all elements in $P_{i,j}$ having top left corner equal to the identity matrix of size i.

(b) For all $i = 0,\ldots,n$ one has $\overline{\mathrm{Inj}}(U_i, G) = \mathrm{Inj}(U_i, G)$ which is a disjoint union of free right $\mathrm{Aut}(U_i)$-sets. Thus, for each $\chi_i \in \mathrm{Irr}_K(\mathrm{Aut}(U_i))$, one has $(U_i,\chi_i) \in \mathcal{E}_G$. We will show that for $(U_i,\chi_i), (U_j,\chi_j) \in \mathcal{E}_G$ one has

$$e_{(U_i,\chi_i)} \cdot_G KB^{\triangleleft}(G,G) \cdot_G e_{(U_j,\chi_j)} \neq \{0\} \iff i \leqslant j \text{ and } \chi_j \mid \mathrm{ind}_{P_{i,j}}^{G_j}(\mathrm{inf}_{(\pi_{i,j})_*}(\chi_i)). \tag{19}$$

Here, for characters χ and χ' of a finite group, we write $\chi \mid \chi'$ if χ is a summand of χ'. In fact, if $\tilde{U}_j$ (resp. $\tilde{U}_i$) is a subgroup of G isomorphic to U_j (resp. U_i) and if $\alpha\colon \tilde{U}_j \to \tilde{U}_i$ is an epimorphism then there exist isomorphisms $\lambda_j\colon U_j \xrightarrow{\sim} \tilde{U}_j$ and $\lambda_i\colon U_i \xrightarrow{\sim} \tilde{U}_i$ such that $\lambda_i^{-1}\alpha\lambda_j = \pi_{i,j}$. Applying Lemma 5.10, we see that there exists $(\omega_i,\omega_j) \in \mathrm{Aut}(\tilde{U}_i)\times\mathrm{Aut}(\tilde{U}_j)$ such that the condition in (13) holds if and only if the trivial character is a constituent of $\mathrm{res}_{L_\alpha}^{\mathrm{Aut}(\tilde{U}_i)\times\mathrm{Aut}(\tilde{U}_j)}((\chi_i)^*_{\tilde{U}_i} \times (\chi_j)_{\tilde{U}_j})$. Using the isomorphisms λ_i and λ_j this condition is equivalent to 1 being a constituent of $\mathrm{res}_{L_{\lambda_i^{-1}\alpha\lambda_j}}^{\mathrm{Aut}(U_i)\times\mathrm{Aut}(U_j)}(\chi_i^* \times \chi_j)$, and the claim follows again with Lemma 5.10.

(c) This implies that each element $(U_i,\chi_i) \in \mathcal{E}_G$ is equivalent (with respect to the equivalence relation $\approx$ in Corollary 5.7) to an element of the form (G,χ) for some $\chi \in \mathrm{Irr}_K(\mathrm{Aut}(G))$, namely for any χ occurring as a constituent in $\mathrm{ind}_{P_{i,n}}^{G_n}(\mathrm{inf}_{(\pi_{i,n})_*}(\chi_i))$. But we cannot, in general, determine which of the elements (G,χ), $\chi \in \mathrm{Irr}_K(\mathrm{Aut}(G))$, are equivalent. For $n=2$ one computes easily that $(G,\chi) \approx (G,\chi')$ if and only if $\chi = \chi'$ or if $\chi,\chi' \in \{1, \mathrm{St}\}$, where St denotes the Steinberg character.

(d) In this part we show that the set

$$\{(U_i,\chi_i) \in \mathcal{E}_G \mid i \in \{0,\ldots,n\},\, \chi_i \in \mathrm{Irr}_K(G_i) \text{ unipotent}\} \tag{20}$$

is an equivalence class under the equivalence relation $\approx$ in Corollary 5.7. To see this assume that (U_i,χ_i) and (U_j,χ_j) satisfy (19). We will first show that χ_i is unipotent if and only if χ_j is unipotent.

If χ_i is unipotent, i.e., $\chi_i \mid \mathrm{ind}_{B_i}^{G_i}(1)$, where B_i denotes the subgroup of lower triangular matrices in G_i, then

$$\chi_j \mid \mathrm{ind}_{P_{i,j}}^{G_j}\mathrm{inf}_{(\pi_{i,j})_*}(\chi_i) \mid \mathrm{ind}_{P_{i,j}}^{G_j}\mathrm{inf}_{(\pi_{i,j})_*}\mathrm{ind}_{B_i}^{G_i}(1)\,.$$

But

$$\begin{aligned}\mathrm{inf}_{(\pi_{i,j})_*}\mathrm{ind}_{B_i}^{G_i}(1) &= \mathrm{ind}_{(\pi_{i,j})_*^{-1}(B_i)}^{P_{i,j}}\mathrm{inf}_{(\pi_{i,j})_*\colon (\pi_{i,j})_*^{-1}(B_i)\to B_i}(1)\\ &= \mathrm{ind}_{(\pi_{i,j})_*^{-1}(B_i)}^{P_{i,j}}(1) \mid \mathrm{ind}_{(\pi_{i,j})_*^{-1}(B_i)}^{P_{i,j}}\mathrm{ind}_{B_j}^{(\pi_{i,j})_*^{-1}(B_i)}(1) = \mathrm{ind}_{B_j}^{P_{i,j}}(1)\,,\end{aligned} \tag{21}$$

since $B_j \leqslant (\pi_{i,j})_*^{-1}(B_i)$. Altogether we obtain $\chi_j \mid \mathrm{ind}_{P_{i,j}}^{G_j}\mathrm{ind}_{B_j}^{P_{i,j}}(1) = \mathrm{ind}_{B_j}^{G_j}(1)$ so that also χ_j is unipotent.

Conversely, if χ_j is unipotent, then $\chi_j \mid \mathrm{ind}_{B_j}^{G_j}(1)$ and $\chi_i \mid \mathrm{def}_{(\pi_{i,j})_*}\mathrm{res}_{P_{i,j}}^{G_j}(\chi_j)$ by (19) and the obvious adjunctions. This implies

$$\chi_i \mid \mathrm{def}_{(\pi_{i,j})_*}\mathrm{res}_{P_{i,j}}^{G_j}\mathrm{ind}_{B_j}^{G_j}(1)\,.$$

But, Mackey's decomposition formula yields

$$\mathrm{res}^{G_j}_{P_{i,j}}\mathrm{ind}^{G_j}_{B_j}(1) = \sum_g \mathrm{ind}^{P_{i,j}}_{P_{i,j}\cap {}^gB_j}(1)$$

where g runs over some subset of G. Thus, there exists $g \in G$ such that

$$\chi_i \mid \mathrm{def}_{(\pi_{i,j})_*}\mathrm{ind}^{P_{i,j}}_{P_{i,j}\cap {}^gB_j}(1) = \mathrm{iso}_{\overline{(\pi_{i,j})_*}}\mathrm{ind}^{P_{i,j}/Q_{i,j}}_{(P_{i,j}\cap {}^gB_j)Q_{i,j}/Q_{i,j}}(1)\,,$$

where $\overline{(\pi_{i,j})_*}\colon P_{i,j}/Q_{i,j} \xrightarrow{\sim} G_i$ denotes the isomorphism induced by $(\pi_{i,j})_*$ and $\mathrm{iso}_{\overline{(\pi_{i,j})_*}}$ denotes the corresponding isomorphism $R_K(P_{i,j}/Q_{i,j}) \xrightarrow{\sim} R_K(G_i)$. By Lemma 7.5 below, the subgroup of G_i corresponding to $(P_{i,j}\cap {}^gB_j)Q_{i,j}/Q_{i,j}$ under $\mathrm{iso}_{\overline{(\pi_{i,j})_*}}$ is conjugate to B_i. Thus, we obtain $\chi_i \mid \mathrm{ind}^{G_i}_{B_i}(1)$ and χ_i is unipotent.

Finally, we will show that each element in the set (20) is equivalent to $(U_1, 1)$. Note that, for $i = 1, \ldots, n-1$, one has $(\pi_{i,i+1})_*^{-1}(B_i) = B_{i+1}$, so that (21) becomes $\mathrm{inf}_{(\pi_{i,i+1})_*}\mathrm{ind}^{G_i}_{B_i}(1) = \mathrm{ind}^{P_{i,i+1}}_{B_{i+1}}(1)$. An easy induction argument now shows that

$$(\mathrm{ind}^{G_{i+1}}_{P_{i,i+1}}\mathrm{inf}_{(\pi_{i,i+1})_*})\circ(\mathrm{ind}^{G_i}_{P_{i-1,i}}\mathrm{inf}_{(\pi_{i-1,i})_*})\circ\cdots\circ(\mathrm{ind}^{G_2}_{P_{1,2}}\mathrm{inf}_{(\pi_{1,2})_*})(1) = \mathrm{ind}^{G_{i+1}}_{B_{i+1}}(1)\,.$$

But this implies that, for each $i = 1, \ldots, n-1$ and each unipotent character $\chi_{i+1} \in \mathrm{Irr}_K(G_{i+1})$, there exists a chain of unipotent characters $\chi_j \in \mathrm{Irr}_K(G_j)$, $j = 2, \ldots, i$, such that $(U_1, 1) \sim (U_2, \chi_2) \sim \cdots \sim (U_i, \chi_i) \sim (U_{i+1}, \chi_{i+1})$. Now the proof of the claim is complete.

7.5 Lemma *Let $\mathcal{F}\colon \{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$ be a chain of subspaces in a vector space V over a field K with $\dim_K V_i = i$ for $i = 0, \ldots, n$. Moreover, let U be a subspace of V and let $\bar{\cdot}\colon V \to V/U$, $v \mapsto \overline{v}$, denote the canonical epimorphism. Let P denote the stabilizer of U in $\mathrm{Aut}(V)$ and let $\pi\colon P \to \mathrm{Aut}(\overline{V})$ denote the epimorphism given by $(\pi(f))(\overline{v}) = \overline{f(v)}$ for $f \in P$ and $v \in V$. Then π maps $\mathrm{stab}_P(\mathcal{F})$ onto $\mathrm{stab}_{\mathrm{Aut}(\overline{V})}(\overline{\mathcal{F}})$, where $\overline{\mathcal{F}}$ denotes the chain $0 = \overline{V}_0 \subseteq \overline{V}_1 \subseteq \cdots \subseteq \overline{V}_n = \overline{V}$ of subspaces of $\overline{V}$.*

Proof It is straightforward to check that if $f \in \mathrm{stab}_P(\mathcal{F})$ then $\pi(f) \in \mathrm{stab}_{\mathrm{Aut}(\overline{V})}(\overline{\mathcal{F}})$.

Conversely, assume that $g \in \mathrm{Aut}(\overline{V})$ stabilizes $\overline{\mathcal{F}}$. By induction on i we will construct a sequence $f_i \in \mathrm{Aut}(V_i)$, $i = 0, \ldots, n$, such that

$$\begin{aligned} f_i|_{V_{i-1}} &= f_{i-1} \text{ for } i = 1, \ldots, n, \\ f_i(V_i \cap U) &= f_i(V_i \cap U) \text{ for } i = 0, \ldots, n, \text{ and} \\ f_i(v) + U &= g(v + U) \text{ for } v \in V_i \text{ and } i = 0, \ldots, n. \end{aligned} \tag{22}$$

To this end, let v_i be an element of V_i not contained in V_{i-1}, for $i = 1, \ldots, n$. Then $v_1, \ldots, v_i$ is a basis of V_i for $i = 0, \ldots, n$. We start with defining f_0 as the zero-map and assume we have already defined f_i satisfying the above properties. Then we define f_{i+1} as the unique extension of f_i with the property that $f_{i+1}(v_{i+1}) = \alpha v_{i+1} + w$ for elements $0 \neq \alpha \in K$ and $w \in V_i$ that will be determined by distinction of two cases. First note that the equations

$$\dim(V_i \cap U) + \dim(V_i + U) = \dim V_i + \dim U$$
$$\dim(V_{i+1} \cap U) + \dim(V_{i+1} + U) = \dim V_{i+1} + \dim U$$

imply that either $V_i \cap U = V_{i+1} \cap U$ or $V_i + U = V_{i+1} + U$. In the first case one has $V_{i+1} + U = Kv_{i+1} \oplus (V_i + U)$ and in the second case one has $V_{i+1} \cap U = K(v_{i+1} - v) \oplus (V_i \cap U)$, where $v \in V_i$ is any element satisfying $v_{i+1} + U = v + U$. In the first case we have $g(\overline{v}_{i+1}) = \alpha\overline{v}_{i+1} + \overline{v}$ for some $0 \neq \alpha \in K$ and $v \in V_i$, since g stabilizes $\overline{\mathcal{F}}$. In this case we define $f_{i+1}(v_{i+1}) := \alpha v_{i+1} + v$. In the second case let $v \in V_i$ be such that $v_{i+1} + U = v + U$ and set $f_{i+1}(v_{i+1}) := v_{i+1} - v + f_i(v)$. It is now straightforward to show that also f_{i+1} satisfies the requirements in (7) for the parameter i replaced with $i + 1$. Finally, the automorphism f_n of V has the property that $f \in P$, that f stabilizes V_i for all $i = 0, \ldots, n$, and that $\pi(f) = g$. □

7.6 Questions The key condition (13) for the relation $\sim$ (cf. Corollary 5.7) that leads to the blocks of $\mathbb{C}B^{\triangleleft}(G, G)$ is of the form $\chi((AxB)^+) \neq 0$ for a finite group X, subgroups A and B of X, an element $x \in X$, and an irreducible character $\chi \in \mathrm{Irr}(X)$ (we assume that $K = \mathbb{C}$ for simplicity).

(a) In all examples that we checked, we saw that

$$\textit{if } \chi((AxB)^+) \neq 0 \textit{ for some } x \in X \textit{ then also } \chi((AB)^+) \neq 0. \tag{23}$$

We are not able to prove this and would like to know if (23) holds for all finite groups X, subgroups A and B of X, and $\chi \in \mathrm{Irr}(X)$. This would simplify the condition in (13).

Note that $(AxB)^+ \in \mathbb{C}X$ is a canonical basis element of $e_A\mathbb{C}Xe_B \cong \mathrm{Hom}_{\mathbb{C}X}(\mathbb{C}Xe_A, \mathbb{C}Xe_B)$, where $e_A := |A|^{-1}\sum_{a\in A} a$. Thus, the statement (23) can be rephrased as: $\chi(e_A\mathbb{C}Xe_B) \neq \{0\}$ implies $\chi(e_Ae_B) \neq 0$. This implies that (23) holds for instance whenever $A = B$.

(b) Moreover, it is straightforward to see that $\chi((AB)^+)$ is always a real number. In fact,

$$\sum_{(a,b)\in A\times B} \chi(ab) = \sum_{(a,b)\in A\times B} \chi(ba) = \sum_{(a,b)\in A\times B} \chi(b^{-1}a^{-1}) = \sum_{(a,b)\in A\times B} \chi((ab)^{-1}) = \sum_{(a,b)\in A\times B} \overline{\chi(ab)}\,.$$

Surprisingly, we also observed in all examples that we checked that

$$\chi((AB)^+) \textit{ is a non-negative real number.} \tag{24}$$

We have no explanation for this, and would like to know if this is true in general for any finite group X, subgroups A and B of X, and $\chi \in \mathrm{Irr}(X)$.

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