A NOTE ON BLOCKS WITH ABELIAN DEFECT GROUPS

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Dedicated to Michel Broué on the occasion of his 60th birthday

Abstract. A recent result by H. Meyer shows that, for a field F of characteristic $p > 0$ and a finite group G with an abelian Sylow p-subgroup, the F-subspace $\mathrm{Z}_{p'}FG$ of the group algebra FG spanned by all p-regular class sums in G is multiplicatively closed, i.e. a subalgebra of the center $\mathrm{Z}FG$ of FG. Here we generalize this result to blocks. More precisely, we show that, for a block A of a group algebra FG with an abelian defect group, the F-subspace $\mathrm{Z}_{p'}A := A \cap \mathrm{Z}_{p'}FG$ is multiplicatively closed, i.e. a subalgebra of the center $\mathrm{Z}A$ of A. We also show that this subalgebra is invariant under perfect isometries and hence under derived equivalences.

1. Introduction

In a recent paper [8], H. Meyer proved that, for a field F of characteristic $p > 0$ and a finite group G with an abelian Sylow p-subgroup P, the F-subspace $\mathrm{Z}_{p'}FG$ of the group algebra FG spanned by all p-regular class sums in G is multiplicatively closed, i.e. a subalgebra of the center $\mathrm{Z}FG$ of FG. His result was motivated by an earlier paper by J. Murray [9] who showed that $\mathrm{Z}_{p'}FG$ is multiplicatively closed for G a symmetric or alternating group. And Murray's proof made use of results in [4] and [5]. On the other hand, H. Meyer gives several examples of finite groups G such that $\mathrm{Z}_{p'}FG$ is not multiplicatively closed.

In this paper, we will consider the F-subspace $\mathrm{Z}_{p'}A := A \cap \mathrm{Z}_{p'}FG$ of the center $\mathrm{Z}A$ of A, for a block A of a group algebra FG. As is well-known, the dimension of $\mathrm{Z}_{p'}A$ coincides with the number of simple A-modules. Our main result will be a generalization of Meyer's theorem to blocks. In §2 we show that, for a block A with an abelian defect group D (in a finite group with an arbitrary Sylow p-subgroup P), the F-subspace $\mathrm{Z}_{p'}A$ is multiplicatively closed, i.e. a subalgebra of $\mathrm{Z}A$.

This generalization is not quite straightforward since Meyer's proof uses a transfer argument which is not directly applicable to blocks. Our way around this problem is to use results by Watanabe [10], Fan [3] and Külshammer-Okuyama-Watanabe [7] instead.

Our main result has applications to perfect isometries, as defined by M. Broué [1]. In §3 we show that, for blocks A and B with abelian defect groups in finite groups G and H, respectively, the isomorphism of F-algebras $\mathrm{Z}B \longrightarrow \mathrm{Z}A$ induced by a perfect isometry between A and B maps $\mathrm{Z}_{p'}B$ onto $\mathrm{Z}_{p'}A$. In particular, this applies to all situations where Broué's Abelian Defect Group Conjecture is known to hold. Hence the F-algebra $\mathrm{Z}_{p'}A$ provides a new invariant of perfect isometries and hence of derived equivalences, at least for blocks with abelian defect groups.

2. Abelian defect groups

In the following, we fix a prime number p and a p-modular system $(K, \mathcal{O}, F)$, that is, $\mathcal{O}$ is a complete discrete valuation ring with field of fractions K of characteristic 0 and residue field F of characteristic p. The $\mathcal{O}$-algebras we consider will always be free of finite rank as $\mathcal{O}$-modules. It will be important, at a key step for our main result, to work over $\mathcal{O}$, and not just over F, though the following first two results hold not only for $\mathcal{O}$. We start with a somewhat technical result on abstract algebras whose relevance will become clear later.

2.1 Proposition. *Let A be an $\mathcal{O}$-algebra, and let L be a local unitary subalgebra of the center $\mathrm{Z}A$ of A. Moreover, let e be an idempotent in A such that $AeA = A$, and suppose that C is a unitary subalgebra of eAe such that the multiplication map*

$$\mu_C : L \otimes_{\mathcal{O}} C \longrightarrow eAe, \quad z \otimes c \longmapsto zc,$$

is an isomorphism of $\mathcal{O}$-algebras. Then there exists a unitary subalgebra B of A such that $C = eBe$ and such that the multiplication map

$$\mu_B : L \otimes_{\mathcal{O}} B \longrightarrow A, \quad z \otimes b \longmapsto zb,$$

is an isomorphism of $\mathcal{O}$-algebras.

Proof. We write $e = \sum_{r=1}^{l} \sum_{s=1}^{m_r} f_{rs}$ with pairwise orthogonal primitive idempotents f_{rs} in C where f_{rs} and $f_{r's'}$ are conjugate in C if and only if $r = r'$. Since L is local and since μ_C is an isomorphism, every f_{rs} is also primitive in eAe and thus in A. Moreover, f_{rs} and $f_{r's'}$ are conjugate in A if and only if they are conjugate in eAe if and only if they are conjugate in C.

Next, note that every primitive idempotent of A is conjugate to one in eAe since $AeA = A$; so we combine our decomposition of e with a similar decomposition of $1 - e$

in order to obtain a decomposition $1 = \sum_{r=1}^{l} \sum_{s=1}^{n_r} f_{rs}$ into pairwise orthogonal primitive idempotents f_{rs} in A where f_{rs} and $f_{r's'}$ are conjugate in A if and only if $r = r'$, and $n_r \geq m_r$ for $r = 1, \ldots, l$. We conclude that

$$A = \bigoplus_{r,r'=1}^{l} \bigoplus_{s=1}^{n_r} \bigoplus_{s'=1}^{n_{r'}} f_{rs} A f_{r's'},$$

and

$$eAe = \bigoplus_{r,r'=1}^{l} \bigoplus_{s=1}^{m_r} \bigoplus_{s'=1}^{m_{r'}} f_{rs} A f_{r's'}, \qquad C = \bigoplus_{r,r'=1}^{l} \bigoplus_{s=1}^{m_r} \bigoplus_{s'=1}^{m_{r'}} f_{rs} C f_{r's'}.$$

For $r = 1, \ldots, l$ and $s = 1, \ldots, n_r$, we can choose elements $x_{rs} \in f_{rs} A f_{r1}$ and $y_{rs} \in f_{r1} A f_{rs}$ such that $f_{rs} = x_{rs} y_{rs}$ and $f_{r1} = y_{rs} x_{rs}$; here we may and will assume that $x_{rs} \in f_{rs} C f_{r1}$ and $y_{rs} \in f_{r1} C f_{rs}$ whenever $s \leq m_r$. Then

$$f_{rs} A f_{r's'} = x_{rs} y_{rs} A x_{r's'} y_{r's'} \subseteq x_{rs} A y_{r's'} \subseteq f_{rs} A f_{r's'},$$

and

$$x_{rs} A y_{r's'} = x_{rs} f_{r1} A f_{r'1} y_{r's'} \subseteq x_{rs}(eAe) y_{r's'} \subseteq x_{rs} A y_{r's'};$$

therefore, $f_{rs} A f_{r's'} = x_{rs} A y_{r's'} = x_{rs}(eAe) y_{r's'}$ for $r, r' = 1, \ldots, l$, $s = 1, \ldots, n_r$ and $s' = 1, \ldots, n_{r'}$. And similarly, $f_{rs} C f_{r's'} = x_{rs} C y_{r's'}$ whenever $s \leq m_r$ and $s' \leq m_{r'}$. This implies that, for $r, r' = 1, \ldots, l$, $s = 1, \ldots, n_r$, $s' = 1, \ldots, n_{r'}$, the maps

$$x_{rs} A y_{r's'} \longrightarrow x_{r1} A y_{r'1}, \quad a \longmapsto y_{rs} a x_{r's'},$$

and

$$x_{r1} A y_{r'1} \longrightarrow x_{rs} A y_{r's'}, \quad a \longmapsto x_{rs} a y_{r's'},$$

are mutually inverse isomorphisms of L-modules, which induce $\mathcal{O}$-module isomorphisms $x_{rs} C y_{r's'} \cong x_{r1} C y_{r'1}$.

We claim that

$$B := \bigoplus_{r,r'=1}^{l} \bigoplus_{s=1}^{n_r} \bigoplus_{s'=1}^{n_{r'}} x_{rs} C y_{r's'}$$

is a unitary subalgebra of A satisfying the conclusion of the proposition. In fact, we have

$$1 = \sum_{r=1}^{l} \sum_{s=1}^{n_r} f_{rs} = \sum_{r=1}^{l} \sum_{s=1}^{n_r} x_{rs} f_{r1} y_{rs} \in \sum_{r=1}^{l} \sum_{s=1}^{n_r} x_{rs} C y_{rs} \subseteq B.$$

Moreover, we see that $x_{rs} C y_{r's'} \cdot x_{r''s''} C y_{r'''s'''} = 0$ unless $r' = r''$ and $s' = s''$, and that

$$x_{rs} C y_{r's'} \cdot x_{r's'} C y_{r''s''} = x_{rs} C f_{r'1} C y_{r''s''} \subseteq x_{rs} C y_{r''s''} \subseteq B$$

for all possible $r, r', r'', r''', s, s', s'', s'''$. Thus B is a unitary subalgebra of A containing C; hence $C = eCe \subseteq eBe$. On the other hand, we have $e x_{rs} C y_{r's'} e = 0$ whenever $s > m_r$

or $s' > m_{r'}$, and $ex_{rs}Cy_{r's'}e = x_{rs}Cy_{r's'}$ whenever $s \leq m_r$ and $s' \leq m_{r'}$. This shows that $eBe \subseteq C$, and we have proved that $eBe = C$. At last, since the multiplication map μ_B is certainly a homomorphism of $\mathcal{O}$-algebras, it is enough to show that the following restriction of μ_B is an $\mathcal{O}$-isomorphism

$$\mu_B|_{L\otimes_{\mathcal{O}}(x_{rs}Cy_{r's'})} : L \otimes_{\mathcal{O}} (x_{rs}Cy_{r's'}) \to x_{rs}(eAe)y_{r's'},\ z \otimes x_{rs}cy_{r's'} \mapsto x_{rs}zcy_{r's'}.$$

The hypothesis that the multiplication map μ_C is surjective implies that the above map is surjective; and it is injective because the following isomorphism implies that the $\mathcal{O}$-ranks of both sides are equal:

$$L \otimes_{\mathcal{O}} (x_{rs}Cy_{r's'}) \cong L \otimes_{\mathcal{O}} (x_{r1}Cy_{r'1}) \overset{\mu_C}{\cong} x_{r1}(eAe)y_{r'1} = f_{r1}Af_{r'1} \cong f_{rs}Af_{r's'}.$$

We can use Proposition 2.1 in order to prove the following structure theorem for certain blocks with abelian defect groups.

2.2 Theorem. *Let G be a finite group, and let A be a block of the group algebra $\mathcal{O}G$ with maximal A-subpair (D, b_D). Suppose that D is abelian and that $Q := \mathrm{C}_D(\mathrm{N}_G(D, b_D)) \subseteq \mathrm{Z}(G)$. Then there exists a unitary subalgebra B of A such that the multiplication map*

$$\mu : \mathcal{O}Q \otimes_{\mathcal{O}} B \longrightarrow A, \quad x \otimes y \longmapsto xy,$$

is an isomorphism of $\mathcal{O}$-algebras.

Proof. We fix an idempotent i in A such that iAi is a source algebra of A. Results of Fan [3] (which is proved without restrictions on the size of $\mathcal{O}$) and Külshammer-Okuyama-Watanabe [7] imply that there exists a unitary subalgebra C of iAi such that the multiplication map

$$\mathcal{O}Q \otimes_{\mathcal{O}} C \longrightarrow iAi, \quad x \otimes y \longmapsto xy,$$

is an isomorphism of $\mathcal{O}$-algebras. Since $AiA = A$, the result follows from Proposition 2.1.

In fact, the theorem above (for sufficiently large $\mathcal{O}$) was already contained in a preliminary version of [7]; however, it was not incorporated into the final version. We also note that the $\mathcal{O}$-algebra B above is clearly isomorphic to the image $\widetilde{A}$ of A in $\mathcal{O}[G/Q]$, which is a block with defect group D/Q when $\mathcal{O}$ is large enough.

In the following, we fix a finite group G and a block $A = \mathcal{O}Ge$ of the group algebra $\mathcal{O}G$, with block idempotent e and defect group D. We denote by $G_{p'}$ the set of p-regular elements in G, and by $\mathcal{O}G_{p'}$ the $\mathcal{O}$-sublattice of $\mathcal{O}G$ spanned by $G_{p'}$. In general, neither $\mathcal{O}G_{p'}$ nor $\mathrm{Z}_{p'}\mathcal{O}G := \mathrm{Z}\mathcal{O}G \cap \mathcal{O}G_{p'}$ are multiplicatively closed (not even when the Sylow p-subgroups of G are abelian), as easy examples show. Note that the p-regular class sums form an $\mathcal{O}$-basis of $\mathrm{Z}_{p'}\mathcal{O}G$. We set

$$\mathrm{Z}_{p'}A := A \cap \mathrm{Z}_{p'}\mathcal{O}G = (\mathrm{Z}_{p'}\mathcal{O}G)e;$$

the last equality follows from a result of Iizuka (cf. [6]).

2.3 Corollary. *In the situation of Theorem 2.2, we have* $\mathrm{Z}_{p'}A \subseteq \mathrm{Z}B$.

Proof. Let C be a p-regular conjugacy class of G, with class sum $C^+ := \sum_{c\in C} c$ in $\mathcal{O}G$. We need to show that $C^+e \in \mathrm{Z}B$. Since the multiplication map $\mu : \mathcal{O}Q \otimes_{\mathcal{O}} B \longrightarrow A$ is an isomorphism of $\mathcal{O}$-algebras, we have decompositions

$$A = \bigoplus_{u\in Q} uB \quad \text{and} \quad \mathrm{Z}A = \mu(\mathrm{Z}(\mathcal{O}Q \otimes_{\mathcal{O}} B)) = \mu(\mathcal{O}Q \otimes_{\mathcal{O}} \mathrm{Z}B) = \bigoplus_{u\in Q} u\mathrm{Z}B.$$

First we assume that the group algebra $\mathcal{O}\langle c\rangle$ for $c \in C$ is split semisimple, i.e. is isomorphic to a direct product of copies of $\mathcal{O}$; in particular, c is an $\mathcal{O}$-linear combination of idempotents in $\mathcal{O}\langle c\rangle$. Then we can write $C^+e = \sum_{m=1}^{n} \alpha_m j_m$ with idempotents $j_m \in A$ (not necessarily orthogonal) and coefficients $\alpha_m \in \mathcal{O}$. Since $A/J(A) \cong B/J(B)$, every idempotent of A is conjugate to one in B; so we may choose units u_m in A such that $i_m := u_m j_m u_m^{-1} \in B$, for $m = 1, \ldots, n$. Then

$$C^+e = \sum_{m=1}^{n} \alpha_m j_m \equiv \sum_{m=1}^{n} \alpha_m i_m =: b \pmod{[A, A]},$$

where $b \in B$, and $[A, A]$ is the $\mathcal{O}$-sublattice of A spanned by all commutators $xy - yx$ $(x, y \in A)$. And note that

$$[A, A] = \mu([\mathcal{O}Q \otimes_{\mathcal{O}} B, \mathcal{O}Q \otimes_{\mathcal{O}} B]) = \mu(\mathcal{O}Q \otimes_{\mathcal{O}} [B, B]) = \bigoplus_{u\in Q} u[B, B].$$

On the other hand, we may write $C^+e = \sum_{u\in Q} uz_u$ with uniquely determined elements $z_u \in \mathrm{Z}B$. Then

$$\sum_{u\in Q} uz_u - b = C^+e - b \in [A, A] = \bigoplus_{u\in Q} u[B, B],$$

and we conclude that $z_u \in [B, B]$ whenever $1 \neq u \in Q$. Hence $z_u \in \mathrm{Z}B \cap [B, B] = 0$ (since we are working in characteristic 0) whenever $1 \neq u \in Q$, so we obtain $C^+e = z_1 \in \mathrm{Z}B$.

Next assume that $\mathcal{O}\langle c\rangle$ for $c \in C$ is not split. Since c is p-regular, there is a finite extension $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ such that $\widetilde{\mathcal{O}}\langle c\rangle$ is split semisimple; hence, in

$$\widetilde{A} := \widetilde{\mathcal{O}} \otimes_{\mathcal{O}} A = \bigoplus_{u\in Q} u(\widetilde{\mathcal{O}} \otimes_{\mathcal{O}} B) = \bigoplus_{u\in Q} u\widetilde{B}$$

where $\widetilde{B} := \widetilde{\mathcal{O}} \otimes_{\mathcal{O}} B$, we have $C^+e \in \mathrm{Z}\widetilde{B}$ (the arguments of the previous paragraph are in fact based on the structure $A = \bigoplus_{u\in Q} uB$ and independent of the two-sided indecomposability of A). So

$$C^+e \in \mathrm{Z}A \cap \mathrm{Z}\widetilde{B} = \Big(\bigoplus_{u\in Q} u\mathrm{Z}B\Big) \cap \mathrm{Z}\widetilde{B} = \mathrm{Z}B.$$

Now we turn our attention to characteristic p. In the following, we denote the canonical maps $\mathcal{O} \longrightarrow F$ and $\mathcal{O}G \longrightarrow FG$ by $x \longmapsto \overline{x}$. Moreover, we denote by $FG_{p'}$ the F-subspace of the group algebra FG spanned by $G_{p'}$. Then $\mathrm{Z}_{p'}FG := \mathrm{Z}FG \cap FG_{p'}$ is the F-subspace of $\mathrm{Z}FG$ spanned by the p-regular class sums of G.

Since A is a block of $\mathcal{O}G$ with defect group D, its image $\overline{A}$ is a block of FG with defect group D. We are interested in the F-subspace $\mathrm{Z}_{p'}\overline{A} := \overline{A} \cap \mathrm{Z}_{p'}FG = (\mathrm{Z}_{p'}FG)\overline{e}$ of $\mathrm{Z}\overline{A}$. Note that $\mathrm{Z}_{p'}FG = \overline{\mathrm{Z}_{p'}\mathcal{O}G}$ and $\mathrm{Z}_{p'}\overline{A} = \overline{\mathrm{Z}_{p'}A}$. We call A *p'-closed* if $\mathrm{Z}_{p'}\overline{A}$ is multiplicatively closed, i.e. a subalgebra of $\mathrm{Z}\overline{A}$. Similarly, we call $\mathcal{O}G$ *p'-closed* if $\mathrm{Z}_{p'}FG$ is a subalgebra of $\mathrm{Z}FG$. By H. Meyer's result [8], group algebras of groups with abelian Sylow p-subgroups are p'-closed. We prove the following generalization.

2.4 Theorem. *Blocks with abelian defect groups are p'-closed.*

Proof. First we assume that $\mathcal{O}$ is large enough. Let $A = \mathcal{O}Ge$ be a minimal counterexample, where e denotes the block idempotent of A. Let (D, b_D) be a maximal A-subpair, so that D is a defect group of A. Since D is abelian, we have $D = Q \times R$ where $Q := \mathrm{C}_D(\mathrm{N}_G(D, b_D))$ and $R := [D, \mathrm{N}_G(D, b_D)]$. Assume that $\overline{z}_1, \overline{z}_2 \in \mathrm{Z}_{p'}\overline{A}$ but $\overline{z}_1\overline{z}_2 \notin \mathrm{Z}_{p'}\overline{A}$, and let

$$\overline{z}_1\overline{z}_2 = \sum_{g \in G} \overline{z}_g g, \qquad \overline{z}_g \in F.$$

Then there is a p-singular element $g \in G$ such that $\overline{z}_g \neq 0$. Writing $g = su = us$ with a p-element $u \in G$ and a p-regular element $s \in C_G(u)$, and applying the Brauer homomorphism $\mathrm{Br}_{\langle u \rangle}$, we obtain

$$\mathrm{Br}_{\langle u \rangle}(\overline{z}_1),\ \mathrm{Br}_{\langle u \rangle}(\overline{z}_2) \in \mathrm{Z}_{p'}F\mathrm{C}_G(u)\mathrm{Br}_{\langle u \rangle}(\overline{e}),$$

but

$$\mathrm{Br}_{\langle u \rangle}(\overline{z}_1)\mathrm{Br}_{\langle u \rangle}(\overline{z}_2) \notin \mathrm{Z}_{p'}F\mathrm{C}_G(u)\mathrm{Br}_{\langle u \rangle}(\overline{e}).$$

Since the defect groups of the blocks appearing in $F\mathrm{C}_G(u)\mathrm{Br}_{\langle u \rangle}(\overline{e})$ are still abelian, one of these blocks is still a counterexample to the theorem. By the minimality of the counterexample $A = \mathcal{O}Ge$, it must be the case that $\mathrm{C}_G(u) = G$. So u belongs to D and is centralized by $\mathrm{N}_G(D, b_D)$, hence belongs to Q. We conclude that the p-factor of any $g \in G$ such that $\overline{z}_g \neq 0$ belongs to Q.

Let b_Q be the unique block of $\mathcal{O}\mathrm{C}_G(Q)$ such that $(Q, b_Q) \leq (D, b_D)$. Then a result by Watanabe [10] implies that the map

$$\beta : \mathrm{Z}\overline{A} \longrightarrow \mathrm{Z}\overline{b}_Q, \quad \overline{y} \longmapsto \mathrm{Br}_Q(\overline{y})1_{\overline{b}_Q},$$

is an isomorphism of F-algebras; this isomorphism maps $\mathrm{Z}_{p'}\overline{A}$ onto $\mathrm{Z}_{p'}\overline{b}_Q$, so that $\overline{b}_Q$ is also a counterexample to the theorem. Thus it must be the case that $\mathrm{C}_G(Q) = G$, in other words, Q is a central p-subgroup of G. Now, on one hand we may write

$$\overline{z}_1\overline{z}_2 = \sum_{u \in Q} u\overline{s}_u = \sum_{u \in Q} u\overline{s}_u\overline{e}$$

with $\overline{s}_u \in \mathrm{Z}_{p'}FG$, hence $\overline{s}_u\overline{e} \in \mathrm{Z}_{p'}\overline{A}$ for $u \in Q$. On the other hand, by Theorem 2.2, we have a unitary subalgebra B of A such that

$$\overline{A} = \bigoplus_{u\in Q} u\overline{B}, \qquad \text{and} \qquad \mathrm{Z}\overline{A} = \bigoplus_{u\in Q} u\mathrm{Z}\overline{B},$$

where $\overline{B}$ denotes the image of B in $\overline{A}$. Then, by Corollary 2.3, we have $\mathrm{Z}_{p'}A \subseteq \mathrm{Z}B$ and therefore $\mathrm{Z}_{p'}\overline{A} \subseteq \mathrm{Z}\overline{B}$. In particular, $\mathrm{Z}\overline{B}$ contains $\overline{z}_1$, $\overline{z}_2$ and $\overline{s}_u\overline{e}$ for $u \in Q$. But then also $\overline{z}_1\overline{z}_2 \in \mathrm{Z}\overline{B}$. Since $\mathrm{Z}\overline{A} = \bigoplus_{u\in Q} u\mathrm{Z}\overline{B}$ we conclude that $\overline{s}_u\overline{e} = 0$ whenever $1 \neq u \in Q$. This shows that $\overline{z}_1\overline{z}_2 = \overline{s}_1\overline{e} \in \mathrm{Z}_{p'}\overline{A}$, and we have reached a contradiction.

If $\mathcal{O}$ is not large enough, then we can extend it to a large enough one $\widetilde{\mathcal{O}}$, and extend A to $\widetilde{A} = \widetilde{\mathcal{O}} \otimes_{\mathcal{O}} A$. It is easy to see that

$$\mathrm{Z}\overline{A} \cap \mathrm{Z}_{p'}\overline{\widetilde{A}} = \mathrm{Z}_{p'}\overline{A}.$$

In $\widetilde{A}$, decompose e as a sum of primitive central idempotents: $e = e_1 + \cdots + e_n$; then $e_1, \ldots, e_n$ are conjugate under a suitable Galois group, and $\widetilde{A}_t = \widetilde{A}e_t$ for $t = 1, \ldots, n$ are blocks with defect group D. For $\overline{z}_1, \overline{z}_2 \in \mathrm{Z}_{p'}\overline{A}$, by the conclusion proved above,

$$\overline{z}_1\overline{z}_2\overline{e}_t \in \mathrm{Z}_{p'}\overline{\widetilde{A}_t}, \qquad t = 1, \ldots, n;$$

so

$$\overline{z}_1\overline{z}_2 = \sum_{t=1}^{n} \overline{z}_1\overline{z}_2\overline{e}_t \in \bigoplus_{t=1}^{n} \mathrm{Z}_{p'}\overline{\widetilde{A}_t} = \mathrm{Z}_{p'}\overline{\widetilde{A}};$$

in conclusion, we get

$$\overline{z}_1\overline{z}_2 \in \mathrm{Z}\overline{A} \cap \mathrm{Z}_{p'}\overline{\widetilde{A}} = \mathrm{Z}_{p'}\overline{A}.$$

3. Perfect isometries

In this section we assume that K and F are splitting fields for all the algebras we will consider, and we point out a connection between Theorem 2.4 and perfect isometries, as defined by M. Broué [1]. We start by recalling the definition of a perfect isometry and some of its consequences.

Suppose that $A = \mathcal{O}Ge$ and $B = \mathcal{O}Hf$ are blocks of finite groups G and H, respectively; here e and f are the corresponding block idempotents. We denote the set of irreducible characters of G in A by $\mathrm{Irr}(A)$, the group of virtual characters of G in A by $\mathbb{Z}\mathrm{Irr}(A)$, and the set of class functions $G \longrightarrow K$ in A by $\mathrm{CF}(A, K)$.

An isometry $I : \mathbb{Z}\mathrm{Irr}(B) \longrightarrow \mathbb{Z}\mathrm{Irr}(A)$ (with respect to the canonical inner products) is called *perfect* if the virtual character μ of $G \times H$ defined by

$$\mu(g, h) := \sum_{\beta\in\mathrm{Irr}(B)} (I\beta)(g) \cdot \beta(h^{-1}), \qquad (g \in G,\ h \in H),$$

satisfies the following two conditions, for $g \in G$ and $h \in H$:

(Int) $\mu(g,h)/|\mathrm{C}_G(g)| \in \mathcal{O}$ and $\mu(g,h)/|\mathrm{C}_H(h)| \in \mathcal{O}$.

(Sep) If $\mu(g,h) \neq 0$ then g and h are either both p-regular or both p-singular.

Since I is an isometry, each $\beta \in \mathrm{Irr}(B)$ defines a sign $\epsilon_\beta \in \{1,-1\}$ and an irreducible character $\alpha_\beta \in \mathrm{Irr}(A)$ such that $I\beta = \epsilon_\beta \alpha_\beta$. The map $\beta \longmapsto \alpha_\beta$ is a defect-preserving bijection between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(A)$, and I extends to a K-linear bijection, also denoted by I, between $\mathrm{CF}(B,K)$ and $\mathrm{CF}(A,K)$. Note that

$$(I\psi)(g) = |H|^{-1} \sum_{h \in H} \mu(g,h)\psi(h), \qquad (\psi \in \mathrm{CF}(B,K),\ g \in G).$$

Recall that the group algebra KG is a symmetric K-algebra with respect to the canonical bilinear form $(.|.)$ mapping a pair of elements $x, y \in KG$ to the coefficient $z_1 \in K$ in the product $xy = z = \sum_{g \in G} z_g g$. The restriction of $(.|.)$ turns $KA := K \otimes_{\mathcal{O}} A$ into a symmetric K-algebra. Hence the vector spaces $\mathrm{Z}KA$ and $\mathrm{CF}(A,K)$ are isomorphic via the map associating to each class function $\phi \in \mathrm{CF}(A,K)$ the element $\phi^\circ \in \mathrm{Z}KA$ satisfying $(\phi^\circ|a) = \phi(a)$ for $a \in KA$. Explicitly, we have

$$\phi^\circ = \sum_{g \in G} \phi(g^{-1})g.$$

In a similar way, $\mathrm{Z}KB$ and $\mathrm{CF}(B,K)$ are in canonical K-linear bijection. Thus the K-linear bijection $I : \mathrm{CF}(B,K) \longrightarrow \mathrm{CF}(A,K)$ yields a K-linear bijection $I^\circ : \mathrm{Z}KB \longrightarrow \mathrm{Z}KA$ such that $I^\circ \psi^\circ = (I\psi)^\circ$ for $\psi \in \mathrm{CF}(B,K)$.

For $\beta \in \mathrm{Irr}(B)$, let $f_\beta = (\beta(1)/|H|) \sum_{h \in H} \beta(h^{-1})h = (\beta(1)/|H|)\beta^\circ$ denote the primitive idempotent in $\mathrm{Z}KB$ corresponding to β. Similarly, for $\alpha \in \mathrm{Irr}(A)$, let

$$e_\alpha = (\alpha(1)/|G|) \sum_{g \in G} \alpha(g^{-1})g = (\alpha(1)/|G|)\alpha^\circ$$

denote the primitive idempotent of $\mathrm{Z}KA$ corresponding to α. Then

$$I^\circ f_\beta = (\beta(1)/|H|)I^\circ\beta^\circ = (\beta(1)/|H|)\epsilon_\beta\alpha_\beta^\circ = \epsilon_\beta(\beta(1)/|H|)(|G|/\alpha_\beta(1))e_{\alpha_\beta}$$

for $\beta \in \mathrm{Irr}(B)$. Since $f = \sum_{\beta \in \mathrm{Irr}(B)} f_\beta$ we conclude

$$I^\circ f = \sum_{\beta \in \mathrm{Irr}(B)} \epsilon_\beta(\beta(1)/|H|)(|G|/\alpha_\beta(1))e_{\alpha_\beta}.$$

In general, I° is K-linear but not necessarily a homomorphism of K-algebras. It is easy to see that

$$I^\circ z = \sum_{g \in G} \big[|H|^{-1} \sum_{h \in H} \mu(g^{-1},h)z_{h^{-1}}\big]g \quad \text{for} \quad z = \sum_{h \in H} z_h h \in \mathrm{Z}KB.$$

Thus the condition (Int) implies that I° restricts to an $\mathcal{O}$-linear map $\mathrm{Z}B \longrightarrow \mathrm{Z}A$ which we will also denote by I°.

3.1 Lemma. *In the situation above, we have* $I^\circ(\mathrm{Z}_{p'}B) = \mathrm{Z}_{p'}A$.

Proof. Let $z = \sum_{h\in H} z_h h \in \mathrm{Z}_{p'}B$. Then $z_h = 0$ whenever $h \notin H_{p'}$, and $I^\circ z = \sum_{g\in G} y_g g$ with $y_g = |H|^{-1}\sum_{h\in H} \mu(g^{-1},h)z_{h^{-1}}$ for $g \in G$. If $y_g \neq 0$ then $0 \neq \mu(g^{-1},h)z_{h^{-1}}$ for some $h \in H$, so $h \in H_{p'}$ since $z_{h^{-1}} \neq 0$. But now (Sep) implies that $g \in G_{p'}$, and we have proved that $I^\circ(\mathrm{Z}_{p'}B) \subseteq \mathrm{Z}_{p'}A$. Thus the lemma follows by symmetry.

The $\mathcal{O}$-linear bijection $I^\circ : \mathrm{Z}B \longrightarrow \mathrm{Z}A$ induces an F-linear bijection $\overline{I}^\circ : \mathrm{Z}\overline{B} \longrightarrow \mathrm{Z}\overline{A}$ such that $\overline{I}^\circ(\mathrm{Z}_{p'}\overline{B}) = \mathrm{Z}_{p'}\overline{A}$. In general, $\overline{I}^\circ$ is not a homomorphism of F-algebras, as easy examples show. On the other hand, the perfect isometry I between A and B induces an isomorphism of K-algebras

$$\iota : \mathrm{Z}KB \longrightarrow \mathrm{Z}KA$$

in such a way that $\iota(f_\beta) = e_{\alpha_\beta}$ for $\beta \in \mathrm{Irr}(B)$. The maps ι and I° are related by the formula (see [1])

$$\iota(z) = I^\circ(zR^\circ(e)) \quad \text{for} \quad z \in \mathrm{Z}KB;$$

here $R = I^{-1}$ denotes the perfect isometry inverse to I. Hence ι restricts to an isomorphism of $\mathcal{O}$-algebras $\mathrm{Z}B \longrightarrow \mathrm{Z}A$ also denoted by ι. Note that $e \in \mathrm{Z}_{p'}A$ by a result of Osima (cf. [6]). Thus $R^\circ(e) \in \mathrm{Z}_{p'}B$ by Lemma 3.1. We denote by $\overline{\iota} : \mathrm{Z}\overline{B} \longrightarrow \mathrm{Z}\overline{A}$ the isomorphism of F-algebras induced by $\iota : \mathrm{Z}B \longrightarrow \mathrm{Z}A$. Then $\overline{\iota}(\overline{z}) = \overline{I}^\circ(\overline{z}\overline{R}^\circ(\overline{e}))$ for $\overline{z} \in \mathrm{Z}\overline{B}$.

3.2 Proposition. *Let I be a perfect isometry between the blocks $A = \mathcal{O}Ge$ and $B = \mathcal{O}Hf$, and suppose that B is p'-closed. Then A is also p'-closed, and the isomorphism of F-algebras $\overline{\iota} : \mathrm{Z}\overline{B} \longrightarrow \mathrm{Z}\overline{A}$ defined by I satisfies $\overline{\iota}(\mathrm{Z}_{p'}\overline{B}) = \mathrm{Z}_{p'}\overline{A}$.*

Proof. Let $\overline{z} \in \mathrm{Z}_{p'}\overline{B}$. Since $\overline{R}^\circ(\overline{e}) \in \mathrm{Z}_{p'}\overline{B}$ and since B is p'-closed, we conclude that $\overline{z}\overline{R}^\circ(\overline{e}) \in \mathrm{Z}_{p'}\overline{B}$. Thus Lemma 3.1 implies that $\overline{\iota}(\overline{z}) = \overline{I}^\circ(\overline{z}\overline{R}^\circ(\overline{e})) \in \mathrm{Z}_{p'}\overline{A}$. So we have proved that $\overline{\iota}(\mathrm{Z}_{p'}\overline{B}) \subseteq \mathrm{Z}_{p'}\overline{A}$. Since $\dim \mathrm{Z}_{p'}\overline{B} = \dim \mathrm{Z}_{p'}\overline{A}$, we obtain $\overline{\iota}(\mathrm{Z}_{p'}\overline{B}) = \mathrm{Z}_{p'}\overline{A}$, and the second assertion is proved. Since $\mathrm{Z}_{p'}\overline{A}$ is the image of the subalgebra $\mathrm{Z}_{p'}\overline{B}$ under the isomorphism of F-algebras $\overline{\iota}$, it is certainly a subalgebra of $\mathrm{Z}\overline{A}$, and the result follows.

We obtain the following consequence.

3.3 Corollary. *Let I be a perfect isometry between blocks $A = \mathcal{O}Ge$ and $B = \mathcal{O}Hf$ of finite groups G and H, respectively, and suppose that the defect groups of B are abelian. Then A and B are p'-closed, and the isomorphism of F-algebras $\overline{\iota} : \mathrm{Z}\overline{B} \longrightarrow \mathrm{Z}\overline{A}$ defined by I maps $\mathrm{Z}_{p'}\overline{B}$ onto $\mathrm{Z}_{p'}\overline{A}$.*

Proof. By Theorem 2.4, the block B is p'-closed. Hence the result follows from Proposition 3.2.

The result above implies that the F-algebra $\mathrm{Z}_{p'}\overline{A}$ is an invariant of perfect isometries and hence an invariant of derived equivalences, at least for blocks with abelian defect groups. (We recall that the dimension of $\mathrm{Z}_{p'}\overline{A}$ is just the number of simple $\overline{A}$-modules.) In the situation of Corollary 3.3, it is not true, however, in general, that $\iota(\mathrm{Z}_{p'}B) = \mathrm{Z}_{p'}A$, as elementary examples show.

Of course, one would expect that, in the situation of Corollary 3.3, the defect groups of A are also abelian. (This would follow, for example, from Brauer's Height Zero Conjecture.) One would perhaps even expect that A and B have isomorphic defect groups.

The result above motivates the following problem.

3.4 Question. Let $\bar{\iota} : \mathrm{Z}\overline{B} \longrightarrow \mathrm{Z}\overline{A}$ be the isomorphism of F-algebras defined by a perfect isometry I between blocks $A = \mathcal{O}Ge$ and $B = \mathcal{O}Hf$ of finite groups G and H, respectively. Is $\bar{\iota}(\mathrm{Z}_{p'}\overline{B}) = \mathrm{Z}_{p'}\overline{A}$?

J. Murray [9] has proved that blocks of finite symmetric and alternating groups are p'-closed. Thus, by Proposition 3.2, Question 3.4 has a positive answer whenever H is a symmetric or alternating group. (We also note that M. Enguehard has proved that p-blocks of the same weight in finite symmetric groups are always perfectly isometric [2]).

One may view Question 3.4 as a problem concerning the p-sections of 1 in G and H, respectively. Thus one may ask whether similar properties hold for other p-sections as well, at least in the presence of an isotypy (cf. [1]). However, this does not seem to be the case, as easy examples show, not even for blocks with abelian defect groups in the situation of Broué's Abelian Defect Group Conjecture.

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References

1. M. Broué, Isométries parfaites, types de blocs, catégories dérivées, *Astérisque* **181-182** (1990), 61-92
2. M. Enguehard, Isométries parfaites entre blocs de groupes symétriques, *Astérisque* **181-182** (1990), 157-171
3. Y. Fan, Relative local control and the block source algebras, *Sci. China* Ser. A **40** (1997), 785-798
4. B. Külshammer, Bemerkungen über die Gruppenalgebra als symmetrische Algebra II, *J. Algebra* **75** (1982), 59-69

5. B. Külshammer, Group-theoretical descriptions of ring-theoretical invariants of group algebras, *Progr. Math.* **95** (1991), 425-442
6. B. Külshammer, Central idempotents in p-adic group rings, *J. Austral. Math. Soc.* Ser. A **56** (1994), 278-289
7. B. Külshammer, T. Okuyama and A. Watanabe, A lifting theorem with applications to blocks and source algebras, *J. Algebra* **232** (2000), 299-309
8. H. Meyer, On a subalgebra of the centre of a group ring, *J. Algebra* **295** (2006), 293-302
9. J. Murray, Squares in the centre of the group algebra of a symmetric group, *Bull. London Math. Soc.* **34** (2002), 155-164
10. A. Watanabe, Note on a p-block of a finite group with abelian defect group, *Osaka J. Math.* **26** (1989), 829-836