The Depth of Subgroups of PSL$(2, q)$

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Abstract

Let $G := \mathrm{PSL}(2, q)$ with q a prime power. The depth of each subgroup H of G, which is the depth of the inclusion of group algebras $kH \subseteq kG$ for a field k of characteristic 0, is determined. It turns out that for $q \geq 13$ almost every proper nontrivial subgroup has depth 3 in G. The exceptions are the dihedral subgroups of order $2(q+1)$ for q even and the semidirect products with a Sylow p-subgroup of G as normal subgroup for q odd respectively which have depth 5 in G. If $q \leq 11$ the values of the depths range between 2 and 5.

1 Introduction

The notion depth was introduced in connection with von Neumann-algebras [6]. It was then developed in the direction to what it means for Hopf-algebras respectively Frobenius-algebras having depth 2 [12, 13, 11]. This led to the definition of higher depths of Frobenius extensions [10] and afterwards to the definition for semisimple algebras as well as the depth of a subgroup H in a finite group G [3]. That is, simplified, the question for the smallest $n \in \mathbb{N}$ for which an integer a exists, such that $\mathbb{C}G \otimes_{\mathbb{C}H} \cdots \otimes_{\mathbb{C}H} \mathbb{C}G$ ($n+1$ times $\mathbb{C}G$) is a direct summand of $\bigoplus_{i=1}^{a} \mathbb{C}G \otimes_{\mathbb{C}H} \cdots \otimes_{\mathbb{C}H} \mathbb{C}G$ (n times $\mathbb{C}G$) as canonical $\mathbb{C}H$-$\mathbb{C}H$-bimodule respectively as canonical $\mathbb{C}G$-$\mathbb{C}H$-bimodule.
The depth of the inclusion of finite groups $H < G$ is defined as the depth of the inclusion of group algebras $\mathbb{C}H \subseteq \mathbb{C}G$ which can be obtained by its inclusion matrix M. Let $\chi_1, \ldots, \chi_s$ be the irreducible characters of G and $\psi_1, \ldots, \psi_r$ be the ones of H, then M is an $r \times s$ matrix with entries $m_{ij} = \langle \psi_i^G, \chi_j \rangle$ (which is the same as $\langle \psi_i, \chi_j\big|_H \rangle$ by Frobenius reciprocity). The powers of M are given by $M^{2l} := M^{2l-1}M^t$ and $M^{2l+1} := M^{2l}M$ for $l \geq 1$. The depth $d(H, G)$ of H in G is the smallest integer n, such that the inequality $M^{n+1} \leq aM^{n-1}$ holds for some $a \in \mathbb{N}$. If $M^{k+1} \leq aM^{k-1}$ for some $a, k \in \mathbb{N}$, then also $M^{k+2} \leq \tilde{a}M^k$ for some $\tilde{a} \in \mathbb{N}$. Here, for real matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size, the relation $A \leq B$ (resp. $A < B$) is defined by $a_{ij} \leq b_{ij}$ (resp. $a_{ij} < b_{ij}$) for all i, j.
For two irreducible characters $\alpha, \beta \in \mathrm{Irr}(H)$ of H the relation $\alpha \sim \beta$ means that there is a $\chi \in \mathrm{Irr}(G)$ with both, α and β, as constituents. Using this relation, a distance $d(\alpha, \beta)$ between two irreducible characters α, β of H can be defined. If $\alpha \neq \beta$ one sets $d(\alpha, \beta) = m$ in case m is the smallest integer, such that there exists a chain $\alpha = \psi_0 \sim \psi_1 \sim \cdots \sim \psi_m = \beta$ of irreducible characters of H. Further $d(\alpha, \beta) := -\infty$ if there is no such chain and $d(\alpha, \beta) := 0$ if $\alpha = \beta$. Moreover, given an irreducible character $\chi \in \mathrm{Irr}(G)$, the subset $\mathcal{X} \subseteq \mathrm{Irr}(H)$ is the set of all irreducible constituents of $\chi\big|_H$. The maximal distance between an irreducible character of H and the set $\mathcal{X}$ is given by

$$m(\chi) = \max_{\alpha \in \mathrm{Irr}(H)} \min_{\psi \in \mathcal{X}} d(\alpha, \psi).$$

The information about the distances is nearly sufficient to determine the depth of H in G [3]:

Theorem 1.1. *(i) Let $m \geq 1$. Then H has depth $\leq 2m+1$ in G if and only if the distance between any two irreducible characters of H is at most m.*

(ii) Let $m \geq 2$. Then H has depth $\leq 2m$ in G if and only if $m(\chi) \leq m-1$ for all $\chi \in \mathrm{Irr}(G)$.

In Section 2 we will apply this theorem several times. That is why we need some character tables there whose derivations can be found in [4, 8, 9] and [14] respectively.

Besides the description by the inclusion matrix, the depth of H in G can be obtained by purely group theoretic properties [3]: ($\mathrm{Core}_G(H)$ denotes the maximal normal subgroup of H in G.)

Theorem 1.2. *Suppose H is a subgroup of a finite group G and $N := \mathrm{Core}_G(H)$ is the intersection of m conjugates of H. Then H has depth $\leq 2m$ in G. If N is in the center of G, H has depth $\leq 2m-1$ in G.*

Moreover, H has depth ≤ 2 in G if and only if H is normal in G [11] and the depth of H in G is 1 if and only if $G = HC_G(x)$ for all $x \in H$ where $C_G(x)$ is the centralizer of x in G [1]. Furthermore, the depth of H in G does not change if $\mathbb{C}$ is replaced by an arbitrary field of characteristic 0 [2].

Up to now, there are only a few inclusions of groups $H < G$ for which the depth is known [3]. In this paper, we determine the depths of the subgroups of the finite projective special linear groups $\mathrm{PSL}(2,q)$ of degree 2.

Let $q = p^f$ with a prime p and some $f \in \mathbb{N}$. The subgroups of $\mathrm{PSL}(2,q)$ are known by a theorem of Dickson. A complete list of all subgroups of $\mathrm{PSL}(2,q)$ is the following [7]:

1. Elementary-abelian p-groups.
2. Cyclic groups of order z, where z divides $(q \pm 1)/k$ with $k = \gcd(q-1, 2)$.
3. Dihedral subgroups of order $2z$ with z as in 2.
4. Alternating groups $\mathfrak{A}_4$ if $p > 2$ or $p = 2$ and $f \equiv 0 \pmod 2$.
5. Symmetric groups $\mathfrak{S}_4$ if $q^2 - 1 \equiv 0 \pmod{16}$.
6. Alternating groups $\mathfrak{A}_5$ if $p = 5$ or $q^2 - 1 \equiv 0 \pmod 5$.
7. Semidirect products $\mathfrak{C}_p^m \rtimes \mathfrak{C}_t$ of elementary-abelian groups of order p^m with cyclic groups of order t, where t divides $p^m - 1$ as well as $(q-1)/k$.
8. Groups $\mathrm{PSL}(2, p^m)$ if $m \mid f$ and $\mathrm{PGL}(2, p^m)$ if $2m \mid f$.

In Section 2 we go through this list step by step. There, we need some more facts about $\mathrm{PSL}(2,q)$ which are also proven in [7]. Let E be the identity of $\mathrm{PSL}(2,q)$. The group $\mathrm{PSL}(2,q)$ acts 2-transitive on the projective line $\mathbb{P}^1(\mathbb{F}_q)$. With the help of this action, the following results can be obtained:

Theorem 1.3. *1. Let $\mathcal{P}$ be a Sylow p-subgroup of $\mathrm{PSL}(2,q)$. Then all the elements of $\mathcal{P}$ have a common fixed point and any element of $\mathcal{P} \setminus E$ has only this fixed point. The Sylow p-subgroups of $\mathrm{PSL}(2,q)$ are TI-subgroups.*

2. Let $\mathcal{U} < \mathrm{PSL}(2,q)$ be cyclic of order $(q-1)/k$, $k = \gcd(q-1,2)$. Then $\mathcal{U}$ has two fixed points and any nontrivial element of $\mathcal{U}$ has no further ones. Moreover, there is no element in $\mathrm{PSL}(2,q) \setminus \mathcal{U}$ that fixes both of these points, so $\mathcal{U}$ is a TI-subgroup. The normalizer of any nontrivial subgroup of $\mathcal{U}$ is a dihedral group of order $2(q-1)/k$.

3. Let $\mathcal{S} < \mathrm{PSL}(2,q)$ be cyclic of order $(q+1)/k$. Then none of the elements of $\mathcal{S} \setminus E$ has a fixed point. The normalizer of any nontrivial subgroup of $\mathcal{S}$ is a dihedral subgroup of order $2(q+1)/k$. Furthermore, $\mathcal{S}$ is a TI-subgroup.

4. Let $\mathcal{V} \cong \mathfrak{C}_2 \times \mathfrak{C}_2$ be a subgroup of $\mathrm{PSL}(2,q)$ with q odd. If 16 does not divide q^2-1, then $N(\mathcal{V}) \cong \mathfrak{A}_4$ and all subgroups of $\mathrm{PSL}(2,q)$ that are isomorphic to $\mathfrak{A}_4$ are conjugate. However, if 16 divides q^2-1, then $N(\mathcal{V}) \cong \mathfrak{S}_4$. In this case, $\mathrm{PSL}(2,q)$ contains two conjugacy classes of subgroups isomorphic to $\mathfrak{S}_4$.

As already mentioned, we need to know the character table of PSL$(2,q)$ in several cases. For $q \equiv 1 \pmod 4$ it looks as follows:

	1	P_1	P_2	A^l	A^d	B^m
$\mathbb{1}$	1	1	1	1	1	1
ζ_1	$\frac{q+1}{2}$	$\frac{1+\sqrt{q}}{2}$	$\frac{1-\sqrt{q}}{2}$	$(-1)^l$	$(-1)^d$	0
ζ_2	$\frac{q+1}{2}$	$\frac{1-\sqrt{q}}{2}$	$\frac{1+\sqrt{q}}{2}$	$(-1)^l$	$(-1)^d$	0
θ_j	$q-1$	-1	-1	0	0	$-\sigma^{jm}-\sigma^{-jm}$
ψ	q	0	0	1	1	-1
δ_i	$q+1$	1	1	$\rho^{il}+\rho^{-il}$	$\rho^{id}+\rho^{-id}$	0

Table 1: Character table of $PSL(2,q)$ for $q \equiv 1 \pmod 4$

Thereby, $P_1, P_2, A, B \in$ PSL$(2,q)$, such that $o(P_1) = o(P_2) = p$, $o(A) = (q-1)/2$ and $o(B) = (q+1)/2$, where $o(X)$ denotes the order of an element X. Moreover, ρ and σ are primitive roots of unity of order $(q-1)/2$ and $(q+1)/2$ respectively. In addition $d = (q-1)/4$, i.e. $o(A^d) = 2$. The indices i, l respectively j, m run from 1 to $(q-5)/4$ and $(q-1)/4$ respectively. Write $\hat{G}$ for the conjugacy class of $G \in$ PSL$(2,q)$, then $|\hat{P}_1| = |\hat{P}_2| = (q^2-1)/2$, $|\hat{A}^l| = q(q+1)$, $|\hat{B}^m| = q(q-1)$ and $|\hat{A}^d| = q(q+1)/2$.
If $q \equiv 3 \pmod 4$ the character table of PSL$(2,q)$ looks similar to the preceding table.

	1	P_1	P_2	A^l	B^m	B^e
$\mathbb{1}$	1	1	1	1	1	1
ζ_1	$\frac{q-1}{2}$	$\frac{-1+\sqrt{-q}}{2}$	$\frac{-1-\sqrt{-q}}{2}$	0	$(-1)^{m+1}$	$(-1)^{e+1}$
ζ_2	$\frac{q-1}{2}$	$\frac{-1-\sqrt{-q}}{2}$	$\frac{-1+\sqrt{-q}}{2}$	0	$(-1)^{m+1}$	$(-1)^{e+1}$
θ_j	$q-1$	-1	-1	0	$-\sigma^{jm}-\sigma^{-jm}$	$-\sigma^{je}-\sigma^{-je}$
ψ	q	0	0	1	-1	-1
δ_i	$q+1$	1	1	$\rho^{il}+\rho^{-il}$	0	0

Table 2: Character table of PSL$(2,q)$ for $q \equiv 3 \pmod 4$

Here, P_1, P_2, A, B, ρ and σ as well as the length of the conjugacy classes (except for $\hat{B}^e$) are as above. Moreover $e = (q+1)/4$, i.e. $o(B^e) = 2$. The indices i, j, l, m all run from 1 to $(q-3)/4$ and $|\hat{B}^e| = q(q-1)/2$.
Finally, the character table of PSL$(2,q)$ for q even follows.

	1	P	A^l	B^m
$\mathbb{1}$	1	1	1	1
θ_j	$q-1$	-1	0	$-\sigma^{jm}-\sigma^{-jm}$
ψ	q	0	1	-1
δ_i	$q+1$	1	$\rho^{il}+\rho^{-il}$	0

Table 3: Character table of PSL$(2,q)$ for q even

Here $o(P) = 2$, $o(A) = q-1$ and $o(B) = q+1$. Furthermore, ρ is a primitive $(q-1)$th root of unity and σ a primitive $(q+1)$th one. Now $1 \le i, l \le (q-2)/2$ and $1 \le j, m \le q/2$. Moreover, we have $|\hat{P}| = q^2-1$, $|\hat{A}^l| = q(q+1)$ and $|\hat{B}^m| = q(q-1)$.

2 Calculation of the Depths

The groups $\mathrm{PSL}(2,q)$ are simple for $q>3$. Hence, any proper nontrivial subgroup has depth ≥ 3 in $\mathrm{PSL}(2,q)$ if $q>3$. We therefore often show only the other inequality for the depth of a subgroup in $\mathrm{PSL}(2,q)$.
We introduce some notations used in this paper. As usual, $N_G(X)$ and $C_G(X)$ denote the normalizer and centralizer of a subset X in a group G. Moreover, for $g \in G$ and X as before we write X^g for $g^{-1}Xg$. Finally, we just write $N(X)$ and $C(X)$ if $G = \mathrm{PSL}(2,q)$.

2.1 Abelian p-subgroups and cyclic subgroups

The Sylow p-subgroups of $\mathrm{PSL}(2,q)$ are nonnormal TI-subgroups. Since every p-subgroup $\mathcal{P}$ is contained in a Sylow p-subgroup of $\mathrm{PSL}(2,q)$, there exists a conjugate $\mathcal{Q}$ of $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{Q} = \{E\}$. Hence, $\mathcal{P}$ has depth 3 in $\mathrm{PSL}(2,q)$.

The cyclic subgroups of $\mathrm{PSL}(2,q)$, $q \neq 2$, having order prime to p are nonnormal TI-subgroups as well. Thus, for a cyclic subgroup $\mathcal{C}$, we can find a conjugate $\mathcal{D}$ satisfying $\mathcal{C} \cap \mathcal{D} = \{E\}$. This implies $d(\mathcal{C}, \mathrm{PSL}(2,q)) = 3$ for $q \neq 2$.
The only subgroup of order 3 in $\mathrm{PSL}(2,2)$ is self-centralizing and therefore has depth 2 in $\mathrm{PSL}(2,2)$.

2.2 Dihedral Subgroups

Similar to the abelian subgroups above, most of the dihedral subgroups $\mathcal{D}$ of $\mathrm{PSL}(2,q)$ have a conjugate whose intersection with $\mathcal{D}$ is trivial. In detail we get the following:

Lemma 2.1. *Let $\mathcal{D} < \mathrm{PSL}(2,q)$ be a dihedral group of order dividing $2 \cdot (q-1)/k$, $k = \gcd(q-1, 2)$. Then there exists some $G \in \mathrm{PSL}(2,q)$ such that $\mathcal{D} \cap \mathcal{D}^G = \{E\}$.*

Proof. Suppose $|\mathcal{D}| = 2(q-1)/k$. Then $\mathcal{D}$ is the normalizer of a cyclic subgroup $\mathcal{C}$ with fixed points $x, y \in \mathbb{P}^1(\mathbb{F}_q)$, that means the normalizer of the set $\{x,y\}$. Because of $\mathcal{C} \cap \mathcal{C}^G = \{E\}$ for all $G \in \mathrm{PSL}(2,q) \setminus \mathcal{D}$, the intersection of $\mathcal{D}$ and $\mathcal{D}^G$ can only contain elements of order 2 (besides E) whenever $\mathcal{D} \neq \mathcal{D}^G$. Of course, these elements do not lie in $\mathcal{C}$.
The elements of order 2 in $\mathcal{D} \setminus \mathcal{C}$ permute the points x and y. Now, $\mathrm{PSL}(2,q)$ is 2-transitive and that is why we can choose $G \in \mathrm{PSL}(2,q)$ and $z \in \mathbb{P}^1(\mathbb{F}_q) \setminus \{x,y\}$ such that $Gy = x$ and $Gx = z$. It follows $GHG^{-1}x = GHy = Gx = z \notin \{x,y\}$ for all $H \in \mathcal{D} \setminus \mathcal{C}$ of order 2 and hence $\mathcal{D} \cap \mathcal{D}^G = \{E\}$.
Any dihedral group $\mathcal{D}$ with order dividing $2(q-1)/k$ is contained in a dihedral group of order $2(q-1)/k$. Thus, there is a conjugate of $\mathcal{D}$ whose intersection with $\mathcal{D}$ is trivial. □

Lemma 2.2. *Let $q>3$ be odd and $\mathcal{D} < \mathrm{PSL}(2,q)$ be a dihedral group of order dividing $q+1$. Then there exists some $G \in \mathrm{PSL}(2,q)$ such that $\mathcal{D} \cap \mathcal{D}^G = \{E\}$.*

Proof. Each dihedral group whose order divides $q+1$ lies in a dihedral group of order $q+1$. So, we will only consider the case $|\mathcal{D}| = q+1$.
Suppose $q \equiv 3 \pmod 4$, $q>3$ and $|\mathcal{D}| = q+1$. We know that nontrivial elements which are contained in $\mathcal{D}$ as well as in one of its conjugates have order 2. We therefore pay attention to the set $\{(T,\mathcal{H}) : \mathcal{H} \cong \mathcal{D}, T \in \mathcal{D} \cap \mathcal{H}, T \text{ an involution}\}$. Since $q \equiv 3 \pmod 4$, there is an element X in the center of $\mathcal{D}$ with order 2 and $\mathcal{D} = N(X)$. Thus, we find exactly $(q+1)/2$ subgroups containing X which are isomorphic to $\mathcal{D}$. The center of each of these subgroups is generated by an element of order 2 that also lies in $\mathcal{D}$. So, we only count the elements of the set $\mathcal{S} := \{(T,\mathcal{H}) : \mathcal{H} \cong \mathcal{D}, \mathcal{H} \neq \mathcal{D}, T \in \mathcal{D} \cap \mathcal{H}, T \text{ an involution}, T \neq X\}$. Apart from X, there are $(q+1)/2$ elements of order 2 in $\mathcal{D}$ and each of them lies in $(q+1)/2$ further subgroups isomorphic to $\mathcal{D}$. It follows $|\mathcal{S}| + 1 \leq ((q+1)/2)^2 + 1 = (q^2+2q+5)/4$, whence there are at most $(q^2+2q+5)/4$ conjugates of $\mathcal{D}$ in $\mathrm{PSL}(2,q)$ whose intersection with $\mathcal{D}$ is nontrivial.
Otherwise $\mathrm{PSL}(2,q)$ contains $|G : N(\mathcal{D})| = |G : \mathcal{D}| = (q^2-q)/2$ conjugates of $\mathcal{D}$. The inequality $(q^2-q)/2 > (q^2+2q+5)/4$ is satisfied by all $q \geq 7$ and we conclude, that there exists a conjugate $\tilde{\mathcal{D}}$ of $\mathcal{D}$ such that $\mathcal{D} \cap \tilde{\mathcal{D}} = \{E\}$.

Now, we assume $q \equiv 1 \pmod 4$. Then, the intersection of $\mathcal{D}$ and one of its conjugates ($\neq \mathcal{D}$) has at most two elements. We are interested in the size of the set $\mathcal{S} := \{(T, \mathcal{H}) : \mathcal{H} \cong \mathcal{D}, \mathcal{H} \neq \mathcal{D}, T \in \mathcal{D} \cap \mathcal{H}, T \text{ an involution}\}$ as above. The elements of order 2 in $\mathcal{D}$ are all conjugate. Moreover, we have $|C(X)| = q-1$ and $|C(X) \cap \mathcal{D}| = 2$, hence X lies in $(q-1)/2$ conjugates of $\mathcal{D}$ (including $\mathcal{D}$). Certainly, $\mathcal{D}$ contains $(q+1)/2$ different elements of order 2 and we get $|\mathcal{S}| + 1 = (q+1)/2 \cdot ((q-1)/2 - 1) + 1 = (q^2 - 2q + 1)/4$. The number of conjugates of $\mathcal{D}$ in $\mathrm{PSL}(2,q)$ is equal to $|G : N(\mathcal{D})| = |G : \mathcal{D}| = (q^2 - q)/2$. Since each $q > 1$ satisfies $(q^2 - q)/2 > (q^2 - 2q + 1)/4$, we find some conjugate $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ such that $\widetilde{\mathcal{D}} \cap \mathcal{D} = \{E\}$. □

Lemma 2.3. *Let $q > 2$ be even and $\mathcal{D} < \mathrm{PSL}(2,q)$ be a dihedral group of order dividing $2(q+1)$. Then there exists some $G \in \mathrm{PSL}(2,q)$ such that $\mathcal{D} \cap \mathcal{D}^G = \{E\}$ if and only if $|\mathcal{D}| < 2(q+1)$.*

Proof. We start following the same argumentation as in the proof of the preceding lemma. Suppose $|\mathcal{D}| = 2(q+1)$. Two different conjugates of $\mathcal{D}$ have at most two elements in common. Any element $X \in \mathcal{D}$ of order 2 lies in $q/2$ conjugate subgroups of $\mathcal{D}$ since $|C(X)| = q$. Thus, there are exactly $(q+1)(q/2 - 1) + 1 = (q^2 - q)/2$ conjugates of $\mathcal{D}$ which have a nontrivial intersection with $\mathcal{D}$. But the length of the conjugacy class of $\mathcal{D}$ in $\mathrm{PSL}(2,q)$ is also $(q^2 - q)/2$. Hence $|\mathcal{D} \cap \mathcal{D}^G| = 2$ for all $G \in \mathrm{PSL}(2,q) \setminus \mathcal{D}$.
A proper dihedral subgroup $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ contains less elements of order 2 than $\mathcal{D}$. Consequently, we find some $G \in \mathrm{PSL}(2,q)$ satisfying $\widetilde{\mathcal{D}} \cap \widetilde{\mathcal{D}}^G = \{E\}$. □

Proposition 2.4. *Apart from the following cases a dihedral group $\mathcal{D} < \mathrm{PSL}(2,q)$ has depth 3 in $\mathrm{PSL}(2,q)$: a dihedral group of order 4 has depth 2 in $\mathrm{PSL}(2,3)$ and one of order $2(q+1)$ has depth 5 in $\mathrm{PSL}(2,q)$ for q even.*

Proof. Due to the Lemmas 2.1, 2.2 and 2.3, we know that $\mathcal{D}$ has depth 3 in $\mathrm{PSL}(2,q)$ if $q \neq 3$ or $|\mathcal{D}| < 2(q+1)$ and q is even. Moreover, the dihedral group of order 4 in $\mathrm{PSL}(2,3)$ is the normalizer of any of its nontrivial elements and therefore has depth 2 in $\mathrm{PSL}(2,3)$.
Assume $|\mathcal{D}| = 2(q+1)$ and q even. We use character tables for the calculation of the depth. For the dihedral group $\mathfrak{D}_z$ of order $2z$ with z odd it looks as follows:

	1	X^n	Y
$\tilde{\mathbb{1}}$	1	1	1
λ	1	1	-1
φ_t	2	$\varepsilon^{tn} + \varepsilon^{-tn}$	0

Table 4: Character table of $\mathfrak{D}_z$ for z odd

Thereby $1 \leq n, t \leq (z-1)/2$, ε is a primitive zth root of unity, $o(X) = z$ and $o(Y) = 2$.
The restriction of the irreducible characters of $\mathrm{PSL}(2,q)$ on $\mathcal{D}$ gives (with a suitable choice of X)

	1	X^n	Y
$\mathbb{1}\big\vert_{\mathcal{D}}$	1	1	1
$\theta_j\big\vert_{\mathcal{D}}$	$q-1$	$-(\varepsilon^{jn} + \varepsilon^{-jn})$	-1
$\psi\big\vert_{\mathcal{D}}$	q	-1	0
$\delta_i\big\vert_{\mathcal{D}}$	$q+1$	0	1

This leads to the decompositions

$$\mathbb{1}\big|_{\mathcal{D}} = \tilde{\mathbb{1}}\,, \quad \theta_j\big|_{\mathcal{D}} = \lambda + \sum_{t=1}^{q/2} \varphi_t - \varphi_j\,, \quad \psi\big|_{\mathcal{D}} = \sum_{t=1}^{q/2} \varphi_t \quad \text{and} \quad \delta_i\big|_{\mathcal{D}} = \tilde{\mathbb{1}} + \sum_{t=1}^{q/2} \varphi_t\,,$$

and we see that two irreducible characters of $\mathcal{D}$ have at most the distance 2. Furthermore, the distance of $\tilde{\mathbb{1}}$ and λ is exactly 2, so $m(\mathbb{1}) = 2$ and we deduce that $\mathcal{D}$ has depth 5 in $\mathrm{PSL}(2,q)$. □

2.3 Permutation Subgroups

Remark 2.5. The alternating group $\mathfrak{A}_4$ is isomorphic to the semidirect product $(\mathfrak{C}_2 \times \mathfrak{C}_2) \rtimes \mathfrak{C}_3$. That means we have a subcase of Section 2.4 if q is even. Taking the results from Section 2.4 we get $d(\mathfrak{A}_4, \mathrm{PSL}(2,4)) = 5$ and $d(\mathfrak{A}_4, \mathrm{PSL}(2,2^{2f})) = 3$ if $f > 1$.

Lemma 2.6. *Let G be a finite group and $H \le G$ a subgroup. Assume $x \in H$ belongs to the conjugacy class $\hat{K}$ of G and $\hat{K} \cap H$ has n orbits under conjugation with the elements of $N(H)$. Then*

$$|\{H^g : g \in G,\, \langle x\rangle \in H^g,\, H \neq H^g\}| \le n \cdot \max_{t \in \hat{K} \cap H} \left\{ \frac{|C(x)|}{|C_{N(H)}(t)|} \right\} - 1.$$

Proof. If $x \in H \cap H^a$ for some $a \in G$, there exists a $y \in H$ such that $y^a = x$. Suppose $z \in H$ and y are conjugate in $N(H)$, i. e. there is an $h \in N(H)$ such that $z^h = y$. Of course, we have $H^a = H^{ha}$. Hence, the sets $\{H^g : g \in G,\, y^g = x\}$ and $\{H^{hg} : g \in G,\, z^{hg} = x\}$ are equal or equivalently $\{H^g : g \in G,\, y^g = x\} = \{H^g : g \in G,\, z^g = x\}$.
It is clear that $|\{H^g : g \in G,\, y^g = x\}| = |C(x)|/|C_{N(H)}(y)|$ for any $y \in \hat{K} \cap H$. In conclusion, we have $|\{H^g : g \in G,\, \langle x\rangle \in H^g\}| \le n \cdot \max_{t \in \hat{K} \cap H} \{|C(x)|/|C_{N(H)}(t)|\}$. □

Lemma 2.7. *Let $q \ge 7$ be odd and $\mathcal{A} \cong \mathfrak{A}_4$ be a subgroup of $\mathrm{PSL}(2,q)$. Then there exists some $G \in \mathrm{PSL}(2,q)$ such that $\mathcal{A} \cap \mathcal{A}^G = \{E\}$.*

Proof. Let $X \in \mathcal{A}$ be an element of order 2. Each element of order 2 which lies in an X containing conjugate subgroup of $\mathcal{A}$ commutes with X. Thus, there are only $(q \pm 1)/2$ such elements. From the fact that the normalizer of a subgroup isomorphic to $\mathfrak{C}_2 \times \mathfrak{C}_2$ of $\mathrm{PSL}(2,q)$ is either isomorphic to $\mathfrak{A}_4$ or $\mathfrak{S}_4$, we deduce that there are $(q \pm 1)/4$ subgroups isomorphic to $\mathcal{A}$ and containing X. There are three elements of order 2 in $\mathcal{A}$ and therefore exactly $3(q \pm 1)/4 - 3$ subgroups isomorphic to $\mathcal{A}$ in whose intersection with $\mathcal{A}$ lie exactly two elements.
Now, let $X \in \mathcal{A}$ be an element of order 3. For $q \not\equiv 0 \pmod 3$ the centralizer of X in $N(\mathcal{A})$ has three elements, the centralizer of X in $\mathrm{PSL}(2,q)$ has $(q \pm 1)/2$ elements. If $q^2 \equiv 1 \pmod{16}$ we have $N(\mathcal{A}) \cong \mathfrak{S}_4$ and all elements of order 3 are conjugate in $N(\mathcal{A})$. Bearing in mind that $\mathcal{A}$ has four subgroups of order 3 we deduce that there are at most $4((q \pm 1)/6 - 1)$ conjugates of $\mathcal{A}$ whose intersection with $\mathcal{A}$ contains exactly three elements by Lemma 2.6. However, if $q^2 \not\equiv 1 \pmod{16}$, then $\mathcal{A}$ is self-normalizing, so $N(\mathcal{A})$ has two conjugacy classes of elements of order 3. Lemma 2.6 now yields that there are at most $4(2(q \pm 1)/6 - 1)$ conjugates of $\mathcal{A}$ in whose intersection with $\mathcal{A}$ lie exactly three elements.
Consequently, there are at most $1 + 3(q \pm 1)/4 - 3 + 4(q \pm 1)/3 - 4 \le (25q - 47)/12$ conjugates of $\mathcal{A}$ which intersect $\mathcal{A}$ nontrivially for $q \not\equiv 0 \pmod 3$, $q^2 \not\equiv 1 \pmod{16}$. The conjugacy class of $\mathcal{A}$ has size $|\mathrm{PSL}(2,q)|/12 = (q^3 - q)/24$. The inequality $(q^3 - q)/24 > (25q - 47)/12$ is satisfied for all $q \ge 11$ and thus we can find some conjugate $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ with $\widetilde{\mathcal{A}} \cap \mathcal{A} = \{E\}$ if $q \ge 11$.
For $q^2 \equiv 1 \pmod{16}$ we get $N(\mathcal{A}) \cong \mathfrak{S}_4$, so the conjugacy class of $\mathcal{A}$ is of size $|\mathrm{PSL}(2,q)|/24 = (q^3 - q)/48$. The number of conjugates of $\mathcal{A}$ which intersect $\mathcal{A}$ nontrivially is at most $1 + 3(q \pm 1)/4 - 3 + 4(q \pm 1)/6 - 4 \le (17q - 55)/12$. This leads to the inequality $(q^3 - q)/48 > (17q - 55)/12$ that holds for any $q \ge 13$. Moreover, it is known that $\mathcal{A}$ is a nonnormal TI-subgroup of $\mathrm{PSL}(2,7)$ and that is why we find some $G \in \mathrm{PSL}(2,q)$, such that $\mathcal{A} \cap \mathcal{A}^G = \{E\}$ under the above assumptions.
The case $q \equiv 0 \pmod 3$ can be done in a similar way. If $q \equiv 3 \pmod 4$, i. e. $q^2 \not\equiv 1 \pmod{16}$, an element of order 3 is not conjugate to its inverse and the centralizer of an element of order 2 contains $(q + 1)/2$ elements. We then get the inequality $1 + 3(q + 1)/4 - 3 + 4q/3 - 4 = (25q - 63)/12 < (q^3 - q)/24$ which certainly holds for $q \ge 27$. For $q \equiv 1 \pmod 4$, i. e. $q^2 \equiv 1 \pmod{16}$, all elements of order 3 are conjugate in $N(\mathcal{A})$. This leads to the inequality $1 + 3(q - 1)/4 - 3 + 4q/3 - 4 = (25q - 81)/12 < (q^3 - q)/48$ which is satisfied for all $q \ge 9$ and thus, there is some $G \in \mathrm{PSL}(2,q)$, such that $\mathcal{A} \cap \mathcal{A}^G = \{E\}$ for all $q > 3$ that are divisible by 3. □

Proposition 2.8. *Any subgroup $\mathcal{A} \cong \mathfrak{A}_4$ of $\mathrm{PSL}(2,q)$ has depth 3 for $q \ge 7$ and depth 5 for $q \in \{4,5\}$ in $\mathrm{PSL}(2,q)$.*

Proof. This is an immediate consequence from Lemma 2.7, Remark 2.5 as well as $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$. □

The calculation of the depth of a subgroup $\mathcal{S} \cong \mathfrak{S}_4$ in $\mathrm{PSL}(2,q)$ can be done as for $\mathfrak{A}_4$. With the help of Lemma 2.6, we will show again that for a sufficiently large q there is always a conjugate of $\mathcal{S}$ that has only E with $\mathcal{S}$ in common.

Lemma 2.9. *Suppose $q \geq 17$ with $q^2 \equiv 1 \pmod{16}$ and $\mathcal{S} \cong \mathfrak{S}_4$ is a subgroup of $\mathrm{PSL}(2,q)$. Then there is some $G \in \mathrm{PSL}(2,q)$ such that $\mathcal{S} \cap \mathcal{S}^G = \{E\}$.*

Proof. At first, we consider the case $q \not\equiv 0 \pmod 3$. The eight elements of order 3 lie in a single conjugacy class in $\mathcal{S}$, so, by Lemma 2.6, there are at most $4(q \pm 1)/6 - 4$ conjugates $\neq \mathcal{S}$ of $\mathcal{S}$ whose intersection with $\mathcal{S}$ contains elements of order 3. We argue in the same way for elements of order 4: Since $|C_{\mathcal{S}}(X)| = 4$ for any $X \in \mathcal{S}$ of order 4, there are at most $3(q \pm 1)/8 - 3$ conjugates $\neq \mathcal{S}$ of $\mathcal{S}$ in whose intersection with $\mathcal{S}$ lie elements of order 4.
It is not necessary to consider the nontrivial elements of the normal subgroup $\mathcal{N} \cong \mathfrak{C}_2 \times \mathfrak{C}_2$ of $\mathcal{S}$. The normalizer in $\mathrm{PSL}(2,q)$ of one of these elements contains only two elements of order 4, so if this element lies in $\mathcal{S} \cap \mathcal{S}^G$ for some $G \in \mathrm{PSL}(2,q)$ an element of order 4 also does. The six remaining elements of order 2 form a single conjugacy class in $\mathcal{S}$. However, they are conjugate to the nontrivial elements of $\mathcal{N}$ in $\mathrm{PSL}(2,q)$, so Lemma 2.6 yields that at most $6 \cdot (2(q \pm 1)/4 - 1)$ conjugates $\neq \mathcal{S}$ of $\mathcal{S}$ contain elements of order 2 from $\mathcal{S} \setminus \mathcal{N}$.
Taken together, there are at most $1 + 4(q \pm 1)/6 - 4 + 3(q \pm 1)/8 - 3 + 12(q \pm 1)/4 - 6 < 97q/24$ conjugates of $\mathcal{S}$ which intersect $\mathcal{S}$ nontrivially. The size of the conjugacy class of $\mathcal{S}$ in $\mathrm{PSL}(2,q)$ is $(q^3 - q)/48$, so this leads to the inequality $97q/24 < (q^3 - q)/48$ which holds for $q \geq 17$ and we are done.
The case $q \equiv 0 \pmod 3$ can be treated as the above. The only difference is the number of conjugates of $\mathcal{S}$ that have elements of order 3 with $\mathcal{S}$ in common. □

Proposition 2.10. *A subgroup $\mathcal{S} < \mathrm{PSL}(2,q)$ that is isomorphic to $\mathfrak{S}_4$ has depth 3 in $\mathrm{PSL}(2,q)$ for $q \geq 17$, depth 4 for $q = 9$ and depth 5 for $q = 7$.*

Proof. For $q \geq 17$ this is an easy consequence of the preceding lemma. The depth of $\mathcal{S}$ in $\mathrm{PSL}(2,7)$ respectively $\mathrm{PSL}(2,9)$ will be found with the help of the inclusion matrix M. For $q = 7$ we get

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

It is easy to see $M^3 \not> 0$ and $M^4 > 0$ and therefore $M^n > 0$ for $n \geq 4$. That is why we can find some $a \in \mathbb{N}$ such that $M^6 \leq aM^4$ whereas no $b \in \mathbb{N}$ satisfying $M^5 \leq bM^3$ exists, so $\mathcal{S}$ has depth 5 in $\mathrm{PSL}(2,7)$.
The choice $q = 9$ leads to the inclusion matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

Here we have $M^2 \not> 0$ and $M^3 > 0$, that means $\mathcal{S}$ has depth 4 in $\mathrm{PSL}(2,9)$. □

Remark 2.11. It is well known that $\mathfrak{A}_5 \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$. Thus, for $q \equiv 0 \pmod 2$ and $q \equiv 0 \pmod 5$ the determination of the depth of $\mathfrak{A}_5$ in $\mathrm{PSL}(2,q)$ is a special case of what will be done in Section 2.5. The Propositions 2.17 and 2.18 yield that $\mathfrak{A}_5$ has depth 3 in $\mathrm{PSL}(2,q)$ whenever $q > 5$ and $q \equiv 0 \pmod 2$ respectively $q \equiv 0 \pmod 5$.

Lemma 2.12. *Let $q \geq 29$ be odd such that $q^2 \equiv 1 \pmod 5$ and $\mathcal{A} \cong \mathfrak{A}_5$ be a subgroup of* $\mathrm{PSL}(2,q)$. *Then there is some $G \in \mathrm{PSL}(2,q)$ such that $\mathcal{A} \cap \mathcal{A}^G = \{E\}$.*

Proof. Looking at the list of subgroups of $\mathrm{PSL}(2,q)$, we immediately deduce $N(\mathcal{A}) = \mathcal{A}$. This implies that the length of the conjugacy class of $\mathcal{A}$ in $\mathrm{PSL}(2,q)$ is $(q^3-q)/120$.
We start with the case $q \not\equiv 0 \pmod 3$. The fifteen elements of order 2 of $\mathcal{A}$ form a single conjugacy class as well as the twenty elements of order 3 of $\mathcal{A}$ do. However, there are two conjugacy classes of elements of order 5 in $\mathcal{A}$. Lemma 2.6 therefore yields that at most $1 + 10(q \pm 1)/6 - 10 + 15(q \pm 1)/4 - 15 + 12(q \pm 1)/10 - 6 < 397q/60$ conjugates of $\mathcal{A}$ have a nontrivial intersection with $\mathcal{A}$. The inequality $(q^3-q)/120 > 397q/60$ certainly holds for $q \geq 29$ and so we are done.
The argumentation for $q \equiv 0 \pmod 3$ is analog. □

Proposition 2.13. *A subgroup $\mathcal{A} \cong \mathfrak{A}_5$ of $\mathrm{PSL}(2,q)$ has depth 3 in $\mathrm{PSL}(2,q)$ for $q \geq 16$ and depth 5 for $q \in \{9, 11\}$.*

Proof. Due to Lemma 2.12 and Remark 2.11, there are only the cases $q = 9$, 11 and 19 remaining. Once more we are working with the inclusion matrices and the distance of characters. For $q = 19$ we obtain $m(\delta_3|_{\mathcal{A}}) = 1$ and thus $\mathcal{A}$ has depth 3 in $\mathrm{PSL}(2,19)$. If $q = 9$ we get the inclusion matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

which satisfies $M^4 > 0$ and $M^3 \not> 0$, that means $\mathcal{A}$ has depth 5 in $\mathrm{PSL}(2,9)$.
For $q = 11$ the inclusion matrix is

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and we get $M^4 > 0$ and $M^3 \not> 0$ again, so the proposition is proved. □

2.4 Semidirect Products

Lemma 2.14. *Suppose $\mathcal{G} = \mathcal{Q} \rtimes \mathcal{C}$ is a subgroup of $\mathrm{PSL}(2,p^f)$ where $\mathcal{Q} \cong \mathfrak{C}_p^m$, $1 \leq m < f$, fixes $(1:0) \in \mathbb{P}^1(\mathbb{F}_{p^f})$. Moreover, let $\mathcal{C} = \left\langle \begin{pmatrix} c & b \\ 0 & c^{-1} \end{pmatrix} \right\rangle$ with $b, c \in \mathbb{F}_{p^f}$ and $|\langle c \rangle| > 1$ divides $\gcd(p^m-1, (p^f-1)/k)$ for $k = \gcd(p^f-1, 2)$. Then there exists a conjugate $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ such that $\mathcal{G} \cap \widetilde{\mathcal{G}} = \{E\}$.*

Proof. Let $G \in \mathcal{G} \setminus \mathcal{Q}$ and $x = (x_1 : x_2) \in \mathbb{P}^1(\mathbb{F}_{p^f})$ be a fixed point of G with $x_2 \neq 0$ so w.l.o.g. $x_2 = 1$. Then we can write $G = \begin{pmatrix} a & b+u \\ 0 & a^{-1} \end{pmatrix}$ with $u \in \mathbb{F}_{p^f}$ such that $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in \mathcal{Q}$ and $a \in \mathbb{F}_{p^f}$ with $a^2 \in \mathbb{F}_{p^m}^{\times}$. Because of

$$\begin{pmatrix} a & b+u \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ 1 \end{pmatrix} = \begin{pmatrix} ax_1 + b + u \\ a^{-1} \end{pmatrix},$$

there is some $d \in \mathbb{F}_{p^f}$ such that

$$\begin{pmatrix} x_1 \\ 1 \end{pmatrix} = d \cdot \begin{pmatrix} ax_1 + b + u \\ a^{-1} \end{pmatrix}.$$

Thus, we get $d = a$ implying $a^2x_1 + a(b+u) = x_1$. Now $a^2 = 1$ yields $b + u = 0$ so $G = E \in \mathcal{Q}$ contrary to our assumption. Consequently, we have $a^2 \neq 1$. If $b+u = 0$ we therefore get $x_1 = 0$, and $b+u \neq 0$ leads to $x_1(b+u)^{-1} = a(1-a^2)^{-1}$. It follows $(x_1(b+u)^{-1})^2 = a^2(1-a^2)^{-2} \in \mathbb{F}_{p^m}^\times$ whence any fixed point $x = (x_1 : 1)$ of an element of $\mathcal{G}$ satisfies either $(x_1(b+u)^{-1})^2 \in \mathbb{F}_{p^m}^\times$ or $x_1 = 0$. Each element of $\mathbb{F}_{p^m}^\times$ has at most two square roots in $\mathbb{F}_{p^f}$, the only square root of 0 is clearly 0. Thus, there are at most $2(p^m-1)+1+1 = 2p^m < p^f + 1$ elements in $\mathbb{F}_{p^f}$ which can be fixed points of an element of $\mathcal{G}$ (the last summand comes from $\infty = (1:0)$). This implies that there exists some $y \in \mathbb{P}^1(\mathbb{F}_{p^f})$ which is fixed by none of the elements of $\mathcal{G}$. Due to Sylow's theorem, we find a conjugate $\widetilde{\mathcal{Q}}$ of $\mathcal{Q}$ in $\mathrm{PSL}(2,p^f)$ that fixes y. It follows that there is a conjugate $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ which also fixes y so $\widetilde{\mathcal{G}}$ intersects $\mathcal{G}$ trivially. □

Proposition 2.15. *Suppose $\mathcal{H} \cong \mathfrak{C}_p^m \rtimes \mathfrak{C}_t$ is a subgroup of $\mathrm{PSL}(2,p^f)$, $1 \le m < f$, such that $t > 1$ divides $\gcd(p^m-1,(p^f-1)/k)$ for $k = \gcd(p^f-1,2)$. Then $\mathcal{H}$ has depth 3 in $\mathrm{PSL}(2,p^f)$.*

Proof. According to Sylow's theorem there is some $G \in \mathrm{PSL}(2,p^f)$, such that the Sylow p-subgroup $\mathcal{P}$ of $\mathcal{H}^G$ fixes $(1:0) \in \mathbb{P}^1(\mathbb{F}_{p^f})$. We know that there are no further fixed points of $\mathcal{P}$, so $(1:0)$ is actually the fixed point of $\mathcal{H}^G$. By suitable choices of $\mathcal{Q}$ and b from Lemma 2.14, the group $\mathcal{G}$ of this lemma contains $\mathcal{H}^G$. That is why we can find a conjugate of $\mathcal{H}^G$ whose intersection with $\mathcal{H}^G$ is trivial. Clearly, the same is valid for $\mathcal{H}$ and therefore $d(\mathcal{H}, \mathrm{PSL}(2,p^f)) = 3$. □

It is easy to show that an upper bound of the depth of a subgroup $\mathcal{H} < \mathrm{PSL}(2,p^f)$, which is isomorphic to a subgroup of $\mathfrak{C}_p^f \rtimes \mathfrak{C}_t$, is 5: Let $\mathcal{P}$ be the Sylow p-subgroup of $\mathcal{H}$ and $\mathcal{D}$ be the intersection of three different conjugates of $N(\mathcal{P})$. Then any element of $\mathcal{D}$ has at least three fixed points, i. e. $\mathcal{D} = \{E\}$. Since $\mathcal{H} < N(\mathcal{P})$, the depth of $\mathcal{H}$ in $\mathrm{PSL}(2,p^f)$ is at most 5.
It remains to show that $d(\mathcal{H}, \mathrm{PSL}(2,q)) > 4$ for $q = p^f$. For this, we use the irreducible characters of $\mathcal{H}$. A description how to find them is given in [14]. W.l.o.g. let $\mathcal{H}$ be a subgroup of

$$\mathcal{G} = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{PSL}(2,q) \right\}.$$

Further, let $\mathcal{P}$ be the Sylow p-subgroup of $\mathcal{G}$ and $\mathcal{C}$ a subgroup of $\mathcal{G}$ with order $t > 1$ not divisible by p such that $\mathcal{H} = \mathcal{P} \rtimes \mathcal{C}$. Since $C(X)$ is a p-group for $X \in \mathcal{P}$, we get $C_{\mathcal{G}}(X) = \mathcal{P}$. Thus, $\mathcal{H}$ consists of the following conjugacy classes: $\{E\}$, $(q-1)/t$ classes with elements of order p, and $t-1$ classes each represented by a nontrivial element of $\mathcal{C}$. Hence $\mathcal{H}$ has t different irreducible characters of the form $\mathbb{1}_{\mathcal{P}} \otimes \mu$, where μ is an irreducible character of $\mathcal{C}$. The degree of each of the other $(q-1)/t$ irreducible characters of $\mathcal{H}$ divides t, because $\mathcal{P}$ is a normal abelian subgroup of $\mathcal{H}$. On the other hand $t \cdot 1^2 + (q-1)/t \cdot t^2 = qt = |\mathcal{H}|$ and thus, any nonlinear irreducible character φ of $\mathcal{H}$ has degree t. This also implies $\varphi|_{\mathcal{C}} = 0$. The character table of $\mathcal{H}$ looks as follows ($P_i \in \mathcal{P}$, $T_j \in \mathcal{C}$, $r = (q-1)/t$ and ε a primitive tth root of unity):

	1	P_1	$\cdots$	P_r	T_1	$\cdots$	T_{t-1}
$\tilde{\mathbb{1}}$	1	1	$\cdots$	1	1	$\cdots$	1
μ_1	1	1	$\cdots$	1	ε	$\cdots$	ε^{-1}
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
μ_{t-1}	1	1	$\cdots$	1	ε^{-1}	$\cdots$	ε
φ_1	t	$*$	$\cdots$	$*$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
φ_r	t	$*$	$\cdots$	$*$	0	$\cdots$	0

Further, we obtain from the orthogonality relations $\sum_{l=1}^{r}\varphi_l(P_a) = \sum_{l=1}^{r}\varphi_a(P_l) = -1$ for all $1 \le a \le r$.
The restrictions of the irreducible characters of $\mathrm{PSL}(2,q)$ on $\mathcal{H}$ have (with a suitable choice of the T_j) the values

	1	P_1	$\cdots$	P_r	T_1	T_2	$\cdots$	T_{t-1}
$\mathbb{1}\big\vert_{\mathcal{H}}$	1	1	$\cdots$	1	1	1	$\cdots$	1
$\theta_j\big\vert_{\mathcal{H}}$	$q-1$	-1	$\cdots$	-1	0	0	$\cdots$	0
$\psi\big\vert_{\mathcal{H}}$	q	0	$\cdots$	0	1	1	$\cdots$	1
$\delta_i\big\vert_{\mathcal{H}}$	$q+1$	1	$\cdots$	1	$\varepsilon^i+\varepsilon^{-i}$	$\varepsilon^{2i}+\varepsilon^{-2i}$	$\cdots$	$\varepsilon^{(t-1)i}+\varepsilon^{-(t-1)i}$

and in addition for $q \equiv 1 \pmod 4$

	1	P_1	$\cdots$	P_r	T_1	T_2	$\cdots$	T_{t-1}
$\zeta_{1/2}\big\vert_{\mathcal{H}}$	$\frac{q+1}{2}$	$\frac{1\pm\sqrt{q}}{2}$	$\cdots$	$\frac{1\pm\sqrt{q}}{2}$	$(-1)^{q-1/2t}$	$(-1)^{2(q-1)/2t}$	$\cdots$	$(-1)^{(t-1)(q-1)/2t}$

respectively for $q \equiv 3 \pmod 4$

	1	P_1	$\cdots$	P_r	T_1	T_2	$\cdots$	T_{t-1}
$\zeta_{1/2}\big\vert_{\mathcal{H}}$	$\frac{q-1}{2}$	$\frac{-1\pm\sqrt{-q}}{2}$	$\cdots$	$\frac{-1\pm\sqrt{-q}}{2}$	0	0	$\cdots$	0

It is clear that

$$\mathbb{1}\big|_{\mathcal{H}} = \tilde{\mathbb{1}}\,, \quad \psi\big|_{\mathcal{H}} = \tilde{\mathbb{1}} + \sum_{l=1}^{r}\varphi_l \quad \text{and} \quad \theta_j\big|_{\mathcal{H}} = \sum_{l=1}^{r}\varphi_l\,.$$

Moreover $|C_{\mathcal{H}}(P)| = q$ and $|C_{\mathcal{H}}(T)| = t$ for $P \in \mathcal{P}$ and $T \in \mathcal{C}$ which leads to

$$\langle \delta_i\big|_{\mathcal{H}}, \tilde{\mathbb{1}}\rangle = \frac{1}{qt}\left[(q+1) + \frac{q-1}{t}t + q\sum_{s=1}^{t-1}(\varepsilon^{si}+\varepsilon^{-si})\right] = \frac{1}{qt}\left[2q + q\cdot\begin{cases}2t-2, & t \mid i\\ -2, & t \nmid i\end{cases}\right] = \begin{cases}2, & t \mid i\\ 0, & t \nmid i\end{cases}$$

as well as

$$\langle \delta_i\big|_{\mathcal{H}}, \mu_u\rangle = \frac{1}{qt}\left[(q+1) + \frac{q-1}{t}t + q\sum_{s=1}^{t-1}(\varepsilon^{si}+\varepsilon^{-si})\varepsilon^{su}\right] = \begin{cases}2, & t \mid (u+i) \text{ and } t \mid (u-i)\\ 1, & t \text{ divides either } (u+i) \text{ or } (u-i)\\ 0, & t \nmid (u+i) \text{ and } t \nmid (u-i)\end{cases}$$

for $1 \le u \le t-1$ and

$$\langle \delta_i\big|_{\mathcal{H}}, \varphi_a\rangle = \frac{1}{qt}\left[(q+1)\cdot t + t\cdot\sum_{l=1}^{r}\varphi_a(P_l)\right] = \frac{qt+t-t}{qt} = 1\,.$$

If q is odd, then the elements of order p lie in two conjugacy classes of the same size of $\mathcal{H}$ since $\mathcal{H}$ contains a Sylow p-subgroup of $\mathrm{PSL}(2,q)$ in which the Sylow p-subgroups are TI-subgroups. In the case $q \equiv 1 \pmod 4$ the value of the restriction of ζ_1 and ζ_2 on $\mathcal{H}$ is therefore $(1+\sqrt{q})/2$ on exactly half of the conjugacy classes of elements of order p and $(1-\sqrt{q})/2$ on the other ones. This implies

$$\begin{aligned}\langle \zeta_{1/2}\big|_{\mathcal{H}}, \tilde{\mathbb{1}}\rangle &= \frac{1}{qt}\left[\frac{q+1}{2} + \frac{q-1}{2t}\cdot t\cdot\frac{1+\sqrt{q}}{2} + \frac{q-1}{2t}\cdot t\cdot\frac{1-\sqrt{q}}{2} + q\cdot\sum_{s=1}^{t-1}(-1)^{s(q-1)/2t}\right]\\ &= \frac{1}{qt}\left[q + q\cdot\begin{cases}-1, & \frac{q-1}{2t} \text{ odd}\\ t-1, & \frac{q-1}{2t} \text{ even}\end{cases}\right] = \begin{cases}0, & \frac{q-1}{2t} \text{ odd}\\ 1, & \frac{q-1}{2t} \text{ even}\end{cases}\end{aligned}$$

for $q \equiv 1 \pmod 4$. If $q \equiv 3 \pmod 4$ we analogously get

$$\langle \zeta_{1/2}|_{\mathcal{H}}, \tilde{\mathbb{1}} \rangle = \frac{1}{qt}\left[\frac{q-1}{2} + \frac{q-1}{2t}\cdot t \cdot \frac{-1+\sqrt{-q}}{2} + \frac{q-1}{2t}\cdot t\cdot \frac{-1-\sqrt{-q}}{2}\right] = 0\,.$$

However, if $(q-1)/2t$ is even (implying $q \equiv 1 \pmod 4$) then

$$\begin{aligned}\langle \zeta_{1/2}|_{\mathcal{H}}, \mu_u \rangle &= \frac{1}{qt}\left[\frac{q+1}{2} + \frac{q-1}{2t}\cdot t\cdot\left(\frac{1+\sqrt{q}}{2} + \frac{1-\sqrt{q}}{2}\right) + q\cdot \sum_{s=1}^{t-1} (-1)^{s(q-1)/2t}\varepsilon^{su}\right]\\ &= \frac{1}{qt}\left[q + q\cdot\sum_{s=1}^{t-1}\varepsilon^{su}\right] = 0\,.\end{aligned}$$

Now $t \geq 2$, whence none of the irreducible characters of $\mathrm{PSL}(2,q)$ has (restricted on $\mathcal{H}$) $\tilde{\mathbb{1}}$ and μ_1 as constituents. From the decompositions of $\delta_1|_{\mathcal{H}}$ and $\psi|_{\mathcal{H}}$ we obtain $\mu_1 \sim \varphi_1 \sim \tilde{\mathbb{1}}$, so the distance of $\tilde{\mathbb{1}}$ and μ_1 is 2. This yields $m(\mathbb{1}) = 2$ and thus $d(\mathcal{H}, \mathrm{PSL}(2,q)) > 4$. This completes the proof of

Proposition 2.16. *The depth of* $\mathfrak{C}_p^f \rtimes \mathfrak{C}_t$ *in* $\mathrm{PSL}(2,p^f)$ *is* 5.

2.5 Projective linear Subgroups

The only remaining part is the calculation of the depth of subgroups $\mathcal{H} \cong \mathrm{PSL}(2,p^m)$ and $\mathcal{H} \cong \mathrm{PGL}(2,p^m)$ respectively in $\mathrm{PSL}(2,p^f)$. In the case $f > 2m$, it is possible to find conjugates of $\mathcal{H}$ which intersect $\mathcal{H}$ trivially. However, if $f = 2m$, such conjugates do not exist. That is why we only make use of the distance of characters in the following.

Proposition 2.17. *Let* $1 \leq m < f$ *and* $\mathcal{H} \cong \mathrm{PSL}(2,2^m)$ *be a subgroup of* $\mathrm{PSL}(2,2^f)$. *Then* $\mathcal{H}$ *has depth* 3 *in* $\mathrm{PSL}(2,2^f)$.

Proof. We will prove the proposition by showing $m(\psi) = 1$ for the irreducible character ψ of $PSL(2,2^f)$. The following inner products are meant as inner products of characters of $\mathcal{H}$. We denote a character χ of $\mathcal{H}$ by "$\tilde{\chi}$" in order to distinguish between characters of $\mathcal{H}$ and $\mathrm{PSL}(2,2^f)$. Further, let $\tilde{\rho}$ be a primitive (2^m-1)th root of unity and $\tilde{\sigma}$ a primitive (2^m+1)th root of unity. Suppose f/m odd whence $(2^m+1) \mid (2^f+1)$. We then get

$$\begin{aligned}\langle \psi|_{\mathcal{H}}, \tilde{\mathbb{1}}\rangle &= \frac{1}{2^m(2^{2m}-1)}\left[2^f + \frac{2^m-2}{2}\cdot 2^m(2^m+1) - \frac{2^m}{2}\cdot 2^m(2^m-1)\right] = \frac{2^{f-m}-1}{2^{2m}-1}\,,\\ \left\langle \psi|_{\mathcal{H}}, \tilde{\psi}\right\rangle &= \frac{1}{2^m(2^{2m}-1)}\left[2^f 2^m + \frac{2^m-2}{2}\cdot 2^m(2^m+1) + \frac{2^m}{2}\cdot 2^m(2^m-1)\right]\\ &= \frac{2^f + 2^{2m} - 2^m - 1}{2^{2m}-1}\,,\\ \left\langle \psi|_{\mathcal{H}}, \tilde{\theta}_j\right\rangle &= \frac{1}{2^m(2^{2m}-1)}\left[2^f(2^m-1) + 2^m(2^m-1)\sum_{r=1}^{\frac{2^m}{2}}\left(\tilde{\sigma}^{jr} + \tilde{\sigma}^{-jr}\right)\right]\\ &= \frac{1}{2^m(2^{2m}-1)}\left[2^f(2^m-1) + 2^m(2^m-1)\sum_{r=1}^{2^m}\tilde{\sigma}^{jr}\right] = \frac{2^{f-m}-1}{2^m+1}\end{aligned}$$

and

$$\begin{aligned}\left\langle \psi|_{\mathcal{H}}, \tilde{\delta}_i\right\rangle &= \frac{1}{2^m(2^{2m}-1)}\left[2^f(2^m+1) + 2^m(2^m+1)\sum_{l=1}^{\frac{2^m-2}{2}}\left(\tilde{\rho}^{il} + \tilde{\rho}^{-il}\right)\right]\\ &= \frac{1}{2^m(2^{2m}-1)}\left[2^f(2^m+1) + 2^m(2^m+1)\sum_{l=1}^{2^m-2}\tilde{\rho}^{il}\right] = \frac{2^{f-m}-1}{2^m-1}\,.\end{aligned}$$

If f/m is even, then $(2^m + 1) \mid (2^f - 1)$, leading to

$$\langle \psi|_{\mathcal{H}}, \tilde{\mathbb{1}} \rangle = \frac{2^{f-m} + 2^{2m} - 2^m - 1}{2^{2m} - 1}\,, \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\psi} \right\rangle = \frac{2^f - 1}{2^{2m} - 1}\,, \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\theta_j} \right\rangle = \frac{2^{f-m} + 1}{2^m + 1} \quad \text{and}$$

$$\left\langle \psi|_{\mathcal{H}}, \tilde{\delta_i} \right\rangle = \frac{2^{f-m} - 1}{2^m - 1}\,.$$

The assumption $f > m$ guarantees that all these inner products are positive. □

Similar calculations can be done to determine $d(\mathrm{PSL}(2, p^m), \mathrm{PSL}(2, p^f))$ for p^f odd. Thereby, not only the parity of f/m but also the cases $p^f \equiv 1 \pmod 4$ and $p^f \equiv 3 \pmod 4$ have to be distinguished.

Proposition 2.18. *Let p be odd and $1 \leq m < f$. Suppose $\mathcal{H} \cong \mathrm{PSL}(2, p^m)$ is a subgroup of $\mathrm{PSL}(2, p^f)$. Then $\mathcal{H}$ has depth 3 in $\mathrm{PSL}(2, p^f)$.*

Proof. We use the "~" as in the proof of Proposition 2.17. For $p^f \equiv 1 \pmod 4$ and f/m odd we get

$$\langle \psi|_{\mathcal{H}}, \tilde{\mathbb{1}} \rangle = 2\frac{p^{f-m} - 1}{p^{2m} - 1}\,, \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\zeta}_{1/2} \right\rangle = \frac{p^{f-m} - 1}{p^m - 1}\,, \qquad \left\langle \psi|_{\mathcal{H}}, \tilde{\theta_j} \right\rangle = 2\frac{p^{f-m} - 1}{p^m + 1}\,,$$

$$\left\langle \psi|_{\mathcal{H}}, \tilde{\psi} \right\rangle = \frac{2p^f + p^{2m} - 2p^m - 1}{p^{2m} - 1} \qquad \text{as well as} \qquad \left\langle \psi|_{\mathcal{H}}, \tilde{\delta_i} \right\rangle = 2\frac{p^{f-m} - 1}{p^m - 1}\,,$$

if $p^f \equiv 1 \pmod 4$ and f/m even, then

$$\langle \psi|_{\mathcal{H}}, \tilde{\mathbb{1}} \rangle = \frac{2p^{f-m} + p^{2m} - 2p^m - 1}{p^{2m} - 1}\,, \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\zeta}_{1/2} \right\rangle = \frac{p^{f-m} - 1}{p^m - 1}\,,$$

$$\left\langle \psi|_{\mathcal{H}}, \tilde{\theta_j} \right\rangle = 2\frac{p^{f-m} + 1}{p^m + 1}\,, \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\psi} \right\rangle = 2\frac{p^f - 1}{p^{2m} - 1} \quad \text{and} \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\delta_i} \right\rangle = 2\frac{p^{f-m} - 1}{p^m - 1}\,,$$

the case $p^f \equiv 3 \pmod 4$ and f/m odd leads to

$$\langle \psi|_{\mathcal{H}}, \tilde{\mathbb{1}} \rangle = 2\frac{p^{f-m} - 1}{p^{2m} - 1}\,, \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\zeta}_{1/2} \right\rangle = \frac{p^{f-m} - 1}{p^m + 1}\,, \qquad \left\langle \psi|_{\mathcal{H}}, \tilde{\theta_j} \right\rangle = 2\frac{p^{f-m} - 1}{p^m + 1}\,,$$

$$\left\langle \psi|_{\mathcal{H}}, \tilde{\psi} \right\rangle = \frac{2p^f + p^{2m} - 2p^m - 1}{p^{2m} - 1} \qquad \text{as well as} \qquad \left\langle \psi|_{\mathcal{H}}, \tilde{\delta_i} \right\rangle = 2\frac{p^{f-m} - 1}{p^m - 1}$$

and finally, $p^f \equiv 3 \pmod 4$ and f/m even, gives

$$\langle \psi|_{\mathcal{H}}, \tilde{\mathbb{1}} \rangle = \frac{2p^{f-m} + p^{2m} - 2p^m - 1}{p^{2m} - 1}\,, \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\zeta}_{1/2} \right\rangle = \frac{p^{f-m} + 1}{p^m + 1}\,,$$

$$\left\langle \psi|_{\mathcal{H}}, \tilde{\theta_j} \right\rangle = 2\frac{p^{f-m} + 1}{p^m + 1}\,, \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\psi} \right\rangle = 2\frac{p^f - 1}{p^{2m} - 1} \quad \text{and} \quad \left\langle \psi|_{\mathcal{H}}, \tilde{\delta_i} \right\rangle = 2\frac{p^{f-m} - 1}{p^m - 1}\,.$$

Therefore, any irreducible character of $\mathcal{H}$ is a constituent of $\psi|_{\mathcal{H}}$ and the proposition is established. □

At last, we determine the depth of projective general linear subgroups $\mathrm{PGL}(2, p^m)$ of $\mathrm{PSL}(2, p^f)$ for p odd. The character table of $\mathrm{PGL}(2, q)$ is similar to the one of $\mathrm{PSL}(2, q)$. Here, $P, A, B \in \mathrm{PGL}(2, q)$ are elements of order p, $q - 1$ and $q + 1$ respectively, ρ is a primitive $(q-1)$th root of unity, σ a primitive $(q+1)$th root of unity and $d = (q-1)/2$, $e = (q+1)/2$, i. e. A^d and B^e are elements of order 2. The indices i, l as well as j, m run from 1 to $(q-3)/2$ and $(q-1)/2$ respectively. The conjugacy classes are of the following lengths: $|\hat{P}| = q^2 - 1$, $|\hat{A}^l| = q(q+1)$, $|\hat{A}^d| = q(q+1)/2$, $|\hat{B}^m| = q(q-1)$ and $|\hat{B}^e| = q(q-1)/2$.

	1	P	A^l	A^d	B^m	B^e
$\mathbb{1}$	1	1	1	1	1	1
λ	1	1	$(-1)^l$	$(-1)^d$	$(-1)^m$	$(-1)^e$
θ_j	$q-1$	-1	0	0	$-\sigma^{jm}-\sigma^{-jm}$	$-\sigma^{je}-\sigma^{-je}$
ψ_1	q	0	1	1	-1	-1
ψ_2	q	0	$(-1)^l$	$(-1)^d$	$(-1)^{m+1}$	$(-1)^{e+1}$
δ_i	$q+1$	1	$\rho^{il}+\rho^{-il}$	$\rho^{id}+\rho^{-id}$	0	0

Table 5: Character table of $\mathrm{PGL}(2,q)$ for q odd

Assume p is odd and $\mathcal{H} \cong \mathrm{PGL}(2,p^m)$ is a subgroup of $\mathrm{PSL}(2,p^f)$. As above, we make use of the "~" to distinguish between characters of $\mathcal{H}$ and $\mathrm{PSL}(2,p^f)$ and make clear of which order the roots of unity are. Since f/m is even, each element of order prime to p has an order that divides $(p^f-1)/2$. Thus $\theta_j\big|_{\mathcal{H}}(1) = p^f-1$, $\theta_j\big|_{\mathcal{H}}(P) = -1$ and $\theta_j\big|_{\mathcal{H}}(H) = 0$ for all the other $H \in \mathcal{H}$ and every j. This yields

$$\left\langle \theta_j\big|_{\mathcal{H}}, \tilde{\mathbb{1}} \right\rangle = p^m \frac{p^{f-2m}-1}{p^{2m}-1}, \quad \left\langle \theta_j\big|_{\mathcal{H}}, \tilde{\lambda} \right\rangle = p^m \frac{p^{f-2m}-1}{p^{2m}-1}, \quad \left\langle \theta_j\big|_{\mathcal{H}}, \tilde{\theta_j} \right\rangle = \frac{p^{f-m}+1}{p^m+1},$$

$$\left\langle \theta_j\big|_{\mathcal{H}}, \tilde{\psi}_{1/2} \right\rangle = \frac{p^f-1}{p^{2m}-1} \quad \text{and} \quad \left\langle \theta_j\big|_{\mathcal{H}}, \tilde{\delta}_i \right\rangle = \frac{p^{f-m}-1}{p^m-1}.$$

That means, any irreducible character of $\mathcal{H}$ is a constituent of $\theta_j\big|_{\mathcal{H}}$ if $f > 2m$. This implies

Proposition 2.19. *For $1 \leq m < f/2$ each subgroup of $\mathrm{PSL}(2,p^f)$ isomorphic to $\mathrm{PGL}(2,p^m)$ has depth 3 in $\mathrm{PSL}(2,p^f)$.*

The calculations for the depth of $\mathrm{PGL}(2,p^r)$ in $\mathrm{PSL}(2,p^{2r})$ are longer than the above mentioned. They are summarized in

Lemma 2.20. *Let $p^r > 3$ be odd and $\mathcal{H} \cong \mathrm{PGL}(2,p^r)$ be a subgroup of $\mathrm{PSL}(2,p^{2r})$. Then the distance of two irreducible characters of $\mathcal{H}$ (w.r.t. $\mathrm{PSL}(2,p^{2r})$) is 1.*

Proof. The results above show that $\theta_j\big|_{\mathcal{H}}$ is the sum of all nonlinear irreducible characters of $\mathcal{H}$. Moreover, it is clear that $\psi\big|_{\mathcal{H}} = \tilde{\mathbb{1}} + \theta_j\big|_{\mathcal{H}}$.
The decomposition of the characters $\zeta_1\big|_{\mathcal{H}}$ and $\zeta_2\big|_{\mathcal{H}}$ requires a bit more work. Since $(p^{2r}-1)/4$ is divisible by p^r-1 if and only if $p \equiv 3 \pmod 4$ and divisible by p^r+1 if and only if $p^r \equiv 1 \pmod 4$ respectively, the values on the respective conjugacy classes are (w.l.o.g.)

	1	P	A^l	A^d	B^m	B^e
$\zeta_1\big|_{\mathcal{H}}$	$\frac{p^{2r}+1}{2}$	$\frac{1+p^r}{2}$	1	1	$(-1)^m$	1
$\zeta_2\big|_{\mathcal{H}}$	$\frac{p^{2r}+1}{2}$	$\frac{1-p^r}{2}$	1	1	$(-1)^m$	1

for $p^r \equiv 3 \pmod 4$ and

	1	P	A^l	A^d	B^m	B^e
$\zeta_1\big|_{\mathcal{H}}$	$\frac{p^{2r}+1}{2}$	$\frac{1+p^r}{2}$	$(-1)^l$	1	1	1
$\zeta_2\big|_{\mathcal{H}}$	$\frac{p^{2r}+1}{2}$	$\frac{1-p^r}{2}$	$(-1)^l$	1	1	1

for $p^r \equiv 1 \pmod 4$.

Then, we get as inner products

$$\left\langle \zeta_1\big|_{\mathcal{H}}, \tilde{\mathbb{1}} \right\rangle = \frac{1}{p^r(p^{2r}-1)} \left[\frac{p^{2r}+1}{2} + (p^{2r}-1)\frac{1+p^r}{2} + \frac{p^r(p^r+1)}{2} + \frac{p^r(p^r-1)}{2} + \right.$$
$$\left. + \begin{cases} \frac{p^r-3}{2} p^r(p^r+1) - p^r(p^r-1) \\ \frac{p^r-1}{2} p^r(p^r-1) - p^r(p^r+1) \end{cases} \right]$$
$$= \frac{1}{p^r(p^{2r}-1)} \left[\frac{p^{2r}+1+p^{3r}+p^{2r}-p^r-1}{2} + \frac{p^{3r}-2p^{2r}-p^r}{2} \right] = 1 \,,$$

$$\left\langle \zeta_2\big|_{\mathcal{H}}, \tilde{\mathbb{1}} \right\rangle = \frac{1}{p^r(p^{2r}-1)} \left[\frac{p^{2r}+1}{2} + (p^{2r}-1)\frac{1-p^r}{2} + \frac{p^r(p^r+1)}{2} + \frac{p^r(p^r-1)}{2} + \right.$$
$$\left. + \begin{cases} \frac{p^r-3}{2} p^r(p^r+1) - p^r(p^r-1) \\ \frac{p^r-1}{2} p^r(p^r-1) - p^r(p^r+1) \end{cases} \right]$$
$$= \frac{1}{p^r(p^{2r}-1)} \left[\frac{p^{2r}+1-p^{3r}+p^{2r}+p^r-1}{2} + \frac{p^{3r}-2p^{2r}-p^r}{2} \right] = 0 \,.$$

With the further calculations done analogously, we obtain

$$\left\langle \zeta_1\big|_{\mathcal{H}}, \tilde{\lambda} \right\rangle = 1 \,, \ \left\langle \zeta_2\big|_{\mathcal{H}}, \tilde{\lambda} \right\rangle = 0 \,, \ \left\langle \zeta_1\big|_{\mathcal{H}}, \tilde{\psi_1} \right\rangle = \left\langle \zeta_2\big|_{\mathcal{H}}, \tilde{\psi_1} \right\rangle = \begin{cases} 0 \,, & p^r \equiv 1 \pmod 4 \\ 1 \,, & p^r \equiv 3 \pmod 4 \end{cases} \,,$$
$$\left\langle \zeta_1\big|_{\mathcal{H}}, \tilde{\psi_2} \right\rangle = \left\langle \zeta_2\big|_{\mathcal{H}}, \tilde{\psi_2} \right\rangle = \begin{cases} 1 \,, & p^r \equiv 1 \pmod 4 \\ 0 \,, & p^r \equiv 3 \pmod 4 \end{cases} \,,$$
$$\left\langle \zeta_1\big|_{\mathcal{H}}, \tilde{\theta}_j \right\rangle = 0 \,, \quad \left\langle \zeta_2\big|_{\mathcal{H}}, \tilde{\theta}_j \right\rangle = 1 \,, \quad \left\langle \zeta_1\big|_{\mathcal{H}}, \tilde{\delta}_i \right\rangle = 1 \,, \quad \text{and} \quad \left\langle \zeta_2\big|_{\mathcal{H}}, \tilde{\delta}_i \right\rangle = 0 \,.$$

Combining the previous results we deduce that the distance between two irreducible characters of $\mathcal{H}$ is 1, if for each $1 \le j \le (p^r-1)/4$ there is a character $\delta_i\big|_{\mathcal{H}}$ which has $\tilde{\lambda}$ as well as $\tilde{\theta}_j$ as constituents, and a similar assertion is valid for $\tilde{\lambda}$ as well as $\tilde{\psi_1}$ and $\tilde{\psi_2}$ respectively. The values of δ_i on the respective conjugacy classes of $\mathcal{H}$ are:

	1	P	A^l	A^d	B^m	B^e
$\delta_i\big|_{\mathcal{H}}$	$p^{2r}+1$	1	$\tilde{\rho}^{il}+\tilde{\rho}^{-il}$	$\tilde{\rho}^{id}+\tilde{\rho}^{-id}$	$\tilde{\sigma}^{im}+\tilde{\sigma}^{-im}$	$\tilde{\sigma}^{ie}+\tilde{\sigma}^{-ie}$

With the help of the following calculations the results of the inner products below can be easily verified.

$$\sum_{l=1}^{\frac{p^r-3}{2}} \left(\tilde{\rho}^{il} + \tilde{\rho}^{-il} \right) + \frac{\tilde{\rho}^{id}+\tilde{\rho}^{-id}}{2} = \sum_{l=1}^{p^r-2} \tilde{\rho}^{il} = \begin{cases} p^r-2, & (p^r-1) \mid i \\ -1, & \text{else} \end{cases} \,,$$

$$\sum_{l=1}^{\frac{p^r-3}{2}} (-1)^l \left(\tilde{\rho}^{il} + \tilde{\rho}^{-il} \right) + (-1)^d \frac{\tilde{\rho}^{id}+\tilde{\rho}^{-id}}{2}$$
$$= -\sum_{l=1}^{\frac{p^r-1}{4}} \left(\tilde{\rho}^{i(2l-1)} + \tilde{\rho}^{-i(2l-1)} \right) + \sum_{l=1}^{\frac{p^r-5}{4}} \left(\tilde{\rho}^{2il} + \tilde{\rho}^{-2il} \right) + \tilde{\rho}^{2id/2}$$
$$= -\sum_{l=1}^{\frac{p^r-1}{4}} \left(\tilde{\rho}^{i(2l-1)} + \tilde{\rho}^{-i(2l-1)} \right) - \sum_{l=1}^{\frac{p^r-5}{4}} \left(\tilde{\rho}^{2il} + \tilde{\rho}^{-2il} \right) - \tilde{\rho}^{2id/2} + 2 \left(\sum_{l=1}^{\frac{p^r-5}{4}} \left(\tilde{\rho}^{2il} + \tilde{\rho}^{-2il} \right) + \tilde{\rho}^{2id/2} \right)$$
$$= -\sum_{l=1}^{p^r-2} \tilde{\rho}^{il} + 2 \sum_{l=1}^{\frac{p^r-3}{2}} \tilde{\rho}^{2il} = \begin{cases} -p^r+2, & (p^r-1) \mid i \\ 1, & \text{else} \end{cases} \quad + \quad 2 \cdot \begin{cases} \frac{p^r-3}{2}, & \frac{p^r-1}{2} \mid i \\ -1, & \text{else} \end{cases}$$
$$= \begin{cases} p^r-2, & i = (2n+1)\frac{p^r-1}{2}, \ n \in \mathbb{Z} \\ -1, & \text{else} \end{cases}$$

if $p^r \equiv 1 \pmod 4$ and, after an analog calculation, the same result for $p \equiv 3 \pmod 4$.

Similarly we get

$$\sum_{m=1}^{\frac{p^r-1}{2}} \left(\tilde{\sigma}^{im} + \tilde{\sigma}^{-im}\right) + \frac{\tilde{\sigma}^{ie} + \tilde{\sigma}^{-ie}}{2} = \sum_{m=1}^{p^r} \tilde{\sigma}^{im} = \begin{cases} p^r, & (p^r+1) \mid i \\ -1, & \text{else} \end{cases} \quad \text{and}$$

$$\sum_{m=1}^{\frac{p^r-1}{2}} (-1)^m \left(\tilde{\sigma}^{im} + \tilde{\sigma}^{-im}\right) + (-1)^e \frac{\tilde{\sigma}^{ie} + \tilde{\sigma}^{-ie}}{2} = \begin{cases} p^r, & i = (2n+1)\frac{p^r+1}{2},\ n \in \mathbb{Z} \\ -1, & \text{else} \end{cases}.$$

Finally it is

$$\begin{aligned}
&\sum_{m=1}^{\frac{p^r-1}{2}} \left(\tilde{\sigma}^{im} + \tilde{\sigma}^{-im}\right)\left(\tilde{\sigma}^{jm} + \tilde{\sigma}^{-jm}\right) + \frac{(\tilde{\sigma}^{ie} + \tilde{\sigma}^{-ie})(\tilde{\sigma}^{je} + \tilde{\sigma}^{-je})}{2} \\
&\quad = \sum_{m=1}^{\frac{p^r-1}{2}} \left(\tilde{\sigma}^{(i+j)m} + \tilde{\sigma}^{-(i+j)m}\right) + \frac{\tilde{\sigma}^{(i+j)e} + \tilde{\sigma}^{-(i+j)e}}{2} + \\
&\quad\quad + \sum_{m=1}^{\frac{p^r-1}{2}} \left(\tilde{\sigma}^{(i-j)m} + \tilde{\sigma}^{-(i-j)m}\right) + \frac{\tilde{\sigma}^{(i-j)e} + \tilde{\sigma}^{-(i-j)e}}{2} \\
&\quad = \begin{cases} p^r, & i+j \equiv 0 \pmod{p^r+1} \\ -1, & \text{else} \end{cases} + \begin{cases} p^r, & i-j \equiv 0 \pmod{p^r+1} \\ -1, & \text{else} \end{cases} \\
&\quad = \begin{cases} p^r - 1, & i \equiv \pm j \pmod{p^r+1} \\ -2, & \text{else} \end{cases}.
\end{aligned}$$

Setting $i = (p^r-1)/2$ then yields $\left\langle \delta_i\big|_{\mathcal{H}}, \tilde{\lambda} \right\rangle = 1$, $\left\langle \delta_i\big|_{\mathcal{H}}, \tilde{\psi}_1 \right\rangle = 1$ and $\left\langle \delta_i\big|_{\mathcal{H}}, \tilde{\psi}_2 \right\rangle = 2$.
Further, if $i = (p^r+1)/2$, we get $\left\langle \delta_i\big|_{\mathcal{H}}, \tilde{\lambda} \right\rangle = 1$ and $\left\langle \delta_i\big|_{\mathcal{H}}, \tilde{\theta}_j \right\rangle = 1$ for all $1 \leq j \leq (p^r-1)/2$.
The assumption $p^r > 3$ implies $(p^r+1)/2 < (p^{2r}-5)/4$, whence for these values of i irreducible characters δ_i of $\mathrm{PSL}(2, p^{2r})$ really exist. ☐

Proposition 2.21. *Let $\mathcal{H} \cong \mathrm{PGL}(2, p^r)$ be a subgroup of $\mathrm{PSL}(2, p^{2r})$. Then $\mathcal{H}$ has depth 3 in $\mathrm{PSL}(2, p^{2r})$ if $p^r \neq 3$ and depth 4 in $\mathrm{PSL}(2, p^{2r})$ if $p^r = 3$.*

Proof. Since $\mathrm{PGL}(2, 2^r) \cong \mathrm{PSL}(2, 2^r)$ and $\mathrm{PGL}(2, 3) \cong \mathfrak{S}_4$, the proposition is immediately obtained from Lemma 2.20, Proposition 2.17 and Proposition 2.10. ☐

2.6 Conclusion

The results of the previous sections lead to the following

Theorem 2.22. *Almost all proper nontrivial subgroups of $\mathrm{PSL}(2, q)$ have depth 3 in $\mathrm{PSL}(2, q)$. The exceptions are as follows.*

1. *The depth of $\mathfrak{C}_3$ in $\mathrm{PSL}(2, 2)$ and the depth of $\mathfrak{D}_2$ in $\mathrm{PSL}(2, 3)$ is 2.*
2. *The depth of $\mathfrak{S}_4$ in $\mathrm{PSL}(2, 9)$ is 4.*
3. *The depth of $\mathfrak{A}_4$ in $\mathrm{PSL}(2, 4) \cong \mathrm{PSL}(2, 5)$, $\mathfrak{S}_4$ in $\mathrm{PSL}(2, 7)$, $\mathfrak{A}_5$ in $\mathrm{PSL}(2, 9)$ and $\mathrm{PSL}(2, 11)$ respectively, $\mathfrak{D}_{2^f+1}$ in $\mathrm{PSL}(2, 2^f)$ as well as $\mathfrak{C}_p^f \rtimes \mathfrak{C}_t$ with $t > 1$ in $\mathrm{PSL}(2, p^f)$ is 5.*

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References

[1] R. Boltje, B. Külshammer, *Group algebra extensions of depth one*, Algebra Number Theory **5** (2011), 63–73.

[2] R. Boltje, S. Danz, B. Külshammer, *On the depth of subgroups and group algebra extensions*, J. Algebra **335** (2011), 258–281.

[3] S. Burciu, L. Kadison, B. Külshammer, *On subgroup depth*, Int. Electron. J. Algebra **9** (2011), 133–166.

[4] L. Dornhoff, *Group representation theory*, Part A, Marcel Dekker, New York 1971.

[5] T. Fritzsche, *Die Tiefe von Untergruppen von* $\mathrm{PSL}(2, q)$, Diploma thesis, Friedrich Schiller University Jena, unpublished.

[6] F. Goodman, P. de la Harpe, V. F. R. Jones, *Coxeter graphs and towers of algebras*, Springer, New York, 1989.

[7] B. Huppert, *Endliche Gruppen I*, Springer, Berlin 1967.

[8] I. M. Isaacs, *Character theory of finite groups*, Dover, New York 1976.

[9] G. James, M. Liebeck, *Representations and characters of groups*, Cambridge University Press, Cambridge 2001.

[10] L. Kadison, *Finite depth and Jacobson-Bourbaki correspondence*, J. Pure Appl. Algebra **212** (2008), 1822–1839.

[11] L. Kadison, B. Külshammer, *Depth two, normality and a trace ideal condition for Frobenius extensions*, Commun. Alg. **34** (2006), 3103–3122.

[12] L. Kadison, D. Nikshych, *Hopf algebra actions on strongly separable extensions of depth two*, Adv. in Math. **163** (2001), 258–286.

[13] L. Kadison, K. Szlachányi, *Bialgebroid actions on depth two extensions and duality*, Adv. in Math. **179** (2003), 75–121.

[14] J. P. Serre, *Lineare Darstellungen endlicher Gruppen*, Akademie-Verlag, Berlin 1972.