

The vertices of a class of Specht modules and simple modules for symmetric groups in characteristic 2

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Abstract

We study Specht modules $S^{(n-2,2)}$ and simple modules $D^{(n-2,2)}$ for symmetric groups \mathfrak{S}_n of degree n over a field of characteristic 2. In particular, we determine the vertices of these modules, and also provide some information on their sources.

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1 Introduction and results

Let G be a finite group and F a field of characteristic $p > 0$. Any finite-dimensional indecomposable FG -module M has a *vertex*, which is a group theoretic invariant. A vertex is a subgroup P of G , minimal with respect to the property that M is relatively P -projective, that is, the canonical map $FG \otimes_{FP} M \rightarrow M$ splits. Such a vertex P of M is a p -subgroup of G , and it is unique up to conjugation in G . Moreover, there is an indecomposable FP -module L , unique up to isomorphism and conjugation with elements in $N_G(P)$ such that M is isomorphic to a direct summand of $\text{Ind}_P^G(L)$. The FP -module L has also vertex P , and is called a *source* of M .

One would like to know the vertices for distinguished classes of modules, in particular for simple modules. For symmetric groups, there are further classes of distinguished modules, such as Specht modules, and Young modules. Vertices of Young modules have been determined by Grabmeier a while ago (see [8]). On the other hand, vertices of Specht modules and of simple modules are known only in a few cases. For a survey of known results, see [5]. For a partition λ of n , we denote the respective Specht $F\mathfrak{S}_n$ -module by S^λ , and when λ is p -regular, we denote the corresponding simple $F\mathfrak{S}_n$ -module by D^λ .

Although Specht modules and simple modules for two part partitions are studied a lot, so far their vertices and sources have only been found in a few cases; and the case $p = 2$ appears to be special. In [3], for $p > 2$, the first author determined the vertices of simple $F\mathfrak{S}_n$ -modules labelled by two part partitions $(n - m, m)$ where $m < (p + 1)p/2$. In these cases, the simple module $D^{(n-m,m)}$ has the defect groups of its block as vertices, unless $D^{(n-m,m)} \cong S^{(n-m,m)}$ in which case $D^{(n-m,m)}$ is a Young module.

In this paper, we determine the vertices of Specht modules and simple modules of \mathfrak{S}_n labelled by partitions $(n-2, 2)$ over fields of characteristic 2. In general terms, we show that the vertices are ‘usually’ Sylow 2-subgroups of \mathfrak{S}_n , and the exceptions occur either for very small degrees, or for the cases when the Specht module is simple and isomorphic to a Young module. Suppose that $n = \sum_{j=1}^s 2^{i_j}$, for some $s \geq 1$ and some $i_1 > \dots > i_s \geq 0$, is the 2-adic expansion of n . Recall from [14, 4.1.22, 4.1.24], that the Sylow 2-subgroups of the standard Young subgroup $\mathfrak{S}_{2^{i_1}} \times \dots \times \mathfrak{S}_{2^{i_s}}$ of \mathfrak{S}_n are also Sylow 2-subgroups of \mathfrak{S}_n . Throughout this paper, P_n will denote the Sylow 2-subgroup of \mathfrak{S}_n , generated by the following elements: for $i \in \mathbb{N}_0$, let $w_{2^i} := \prod_{k=1}^{2^{i-1}} (k, k + 2^{i-1})$. Moreover, for $j = 1, \dots, s$, let $g_j := \prod_{k=1}^{2^{i_j}} (k, k + 2^{i_1} + \dots + 2^{i_{j-1}})$.

In particular, $g_1 = 1$. Then $\{w_{2^i}^{g_j} \mid j = 1, \dots, s, i = 1, \dots, i_j\}$ is a generating set for $P_n = \prod_{j=1}^s (P_{2^{i_j}})^{g_j}$. For convenience, we just write $P_n = \prod_{j=1}^s P_{2^{i_j}}$.

We state now the precise results.

Theorem 1.1. *Let F be an algebraically closed field of characteristic 2, let $n \geq 4$, and let further $P \leq P_n$ be a vertex of the Specht $F\mathfrak{S}_n$ -module $S := S^{(n-2,2)}$.*

- (i) *If $n = 4$ then $S = S^{(2^2)} \cong D^{(3,1)}$, and P is the unique Sylow 2-subgroup Q_4 of \mathfrak{A}_4 . Moreover, S has trivial source.*
- (ii) *If $n \equiv 3 \pmod{4}$ then $S \cong D^{(n-2,2)}$, and $P \sim_{\mathfrak{S}_n} P_{n-5} \times P_2 \times P_2$. Furthermore, S has then trivial sources.*
- (iii) *If $n \equiv 0 \pmod{2}$ and $n > 4$ then $P = P_n$, and $\text{Res}_{P_n}^{\mathfrak{S}_n}(S)$ is a source of S .*
- (iv) *If $n \equiv 1 \pmod{4}$ with 2-adic expansion $n = 1 + \sum_{j=1}^s 2^{i_j}$, for $i_1 > \dots > i_s \geq 2$ then let $H := \mathfrak{S}_{2^{i_1}} \times \dots \times \mathfrak{S}_{2^{i_s}}$. For $1 \leq j \leq s$ let Ω_j denote the set of all 2-element subsets of the support of $\mathfrak{S}_{2^{i_j}}$. By U we denote the FH -module which is the kernel of the augmentation map $F(\Omega_1 \cup \dots \cup \Omega_s) \rightarrow F$. Then $P = P_n$, and $\text{Res}_{P_n}^H(U)$ is a source of S .*

Theorem 1.2. *Let F be an algebraically closed field of characteristic 2, let $n \geq 5$, and let $P \leq P_n$ be a vertex of the simple $F\mathfrak{S}_n$ -module $D := D^{(n-2,2)}$.*

- (i) *If $n = 5$ then P is the unique Sylow 2-subgroup Q_4 of \mathfrak{A}_4 . Moreover, $\text{Res}_{Q_4}^{\mathfrak{A}_5}(E_+^{(3,2)})$ and $\text{Res}_{Q_4}^{\mathfrak{A}_5}(E_-^{(3,2)})$ are then sources of D . Here $E_+^{(3,2)}$ and $E_-^{(3,2)}$ are non-isomorphic simple $F\mathfrak{A}_5$ -modules such that $\text{Res}_{\mathfrak{A}_5}^{\mathfrak{S}_5}(D) \cong E_+^{(3,2)} \oplus E_-^{(3,2)}$.*
- (ii) *If $n \equiv 3 \pmod{4}$ then $D \cong S^{(n-2,2)}$ with vertex $P \sim_{\mathfrak{S}_n} P_{n-5} \times P_2 \times P_2$ and trivial sources.*
- (iii) *Otherwise $P = P_n$, and if $n \equiv 0 \pmod{2}$ then $\text{Res}_{P_n}^{\mathfrak{S}_n}(D)$ is a source of D .*

We emphasize here that we are currently not able to determine the sources of D in the case where $n \equiv 1 \pmod{4}$. We will make some further comments on this question at the end of the article. To prove the theorems, we study restrictions of the modules in question to appropriate subgroups of \mathfrak{S}_n . It turns out that the case where n is odd and the case where n is even behave quite differently. We show that if n is even then both $S^{(n-2,2)}$ and $D^{(n-2,2)}$ restrict indecomposably to the elementary abelian subgroup $E_n := \langle (1, 2), (3, 4), \dots, (n-1, n) \rangle$ of \mathfrak{S}_n . This will be the key step in proving the theorems in the case where n is even. If n is odd then we investigate the restrictions of $D^{(n-2,2)}$ and $S^{(n-2,2)}$ to certain Young subgroups $\mathfrak{S}_{a_1} \times \dots \times \mathfrak{S}_{a_r}$ of \mathfrak{S}_n where a_1, \dots, a_r are divisible by 4. In doing this we will also gain some knowledge on the structure of these modules which may be of independent interest.

After summarizing our notation, in Section 2.2 we will collect some general facts about permutation modules. In Sections 3 and 4, respectively, we will prove Theorems 1.1 and 1.2, respectively.

For background on vertices and sources of indecomposable FG -modules we refer to [21, Chapter 4.3]. Details on the representation theory of the symmetric groups may be found in [13] and [14].

Any group G occurring here is supposed to be finite, and an FG -module is always understood to be a finitely generated left FG -module.

2 Prerequisites

2.1 Notation and known results

Let F be an algebraically closed field of prime characteristic p . Suppose that we are given some $n \in \mathbb{N}$ and some partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n . In addition we set $\lambda_{k+1} := 0$.

(1) Let G be a group, and let $M \neq 0$ be an FG -module which has a unique composition series, with composition factors D_1, D_2, \dots, D_r from head to socle. Then we write $M = \mathcal{U}(D_1, D_2, \dots, D_r)$.

(2) Let G and H be groups, let M be an FG -module, and let N be an FH -module. Then the outer tensor product of M and N becomes an $F[G \times H]$ -module in the obvious way and will be denoted by $M \boxtimes N$. If both M and N are indecomposable as FG -module and FH -module, respectively, then $M \boxtimes N$ is indecomposable as $F[G \times H]$ -module. If, moreover, M has vertex P and source V , and if N has vertex Q and source W then $M \boxtimes N$ has vertex $P \times Q$ and source $V \boxtimes W$. A proof for this can be found in [17, Prop. 1.2].

(3) Whenever we have a group G , an FG -module M and a finite subset $\mathcal{M} \subset M$ then we set $\mathcal{M}^+ := \sum_{m \in \mathcal{M}} m$.

(4) Let \mathfrak{S}_λ be the standard Young subgroup $\mathfrak{S}_{\lambda_1} \times \dots \times \mathfrak{S}_{\lambda_k}$ of \mathfrak{S}_n corresponding to λ . In this notation, for $j = 1, \dots, k$, the direct factor \mathfrak{S}_{λ_j} is always supposed to be acting on the set $\{(\sum_{i=1}^{j-1} \lambda_i) + 1, \dots, \sum_{i=1}^j \lambda_i\}$. The permutation $F\mathfrak{S}_n$ -module $M^\lambda := \text{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(F)$ has an F -basis consisting of all λ -*tabloids*. When $k = 2$, each λ -tabloid $\{t\}$ is uniquely determined by the entries in the second row of the λ -tableau t . If these are $i_1, \dots, i_{\lambda_k}$ then we simply identify $\{t\}$ with the set $\{i_1, \dots, i_{\lambda_k}\}$.

(5) The Specht $F\mathfrak{S}_n$ -module S^λ has an F -basis consisting of all *standard λ -polytabloids* e_t . Suppose that $\lambda_k = 2 = k$ and that $t = t(ij)$ is a standard λ -tableau with second row ij , for some $2 \leq i < j \leq n$. Then t and e_t are uniquely determined by these second row entries, and we thus also write $e(ij)$ rather than e_t .

(6) Fixing an indecomposable direct sum decomposition $M^\lambda = M_1 \oplus \dots \oplus M_r$ of the permutation $F\mathfrak{S}_n$ -module M^λ , there is precisely one $i \in \{1, \dots, r\}$ such that $S^\lambda \subseteq M_i$. The module M_i is unique up to isomorphism, and is called the *Young module* Y^λ corresponding to λ . For details we refer to [18, Sec. 4.6]. The vertices of the indecomposable Young modules have been determined by Grabmeier in [8, S. 7.8] (see also [7]). Namely, with the above notation, the module Y^λ has the Sylow p -subgroups of the Young subgroup

$$\prod_{i=1}^k (\mathfrak{S}_{\lambda_i - \lambda_{i+1}})^i$$

of \mathfrak{S}_n as vertices. Moreover, Y^λ has trivial sources.

(7) Suppose that $p = 2$, and let $n = \sum_{j=1}^s 2^{i_j}$, for some $s \geq 1$ and $i_1 > \dots > i_s \geq 0$, be the 2-adic expansion of n . Let $P_n := \prod_{j=1}^s P_{2^{i_j}}$ be the Sylow 2-subgroup of \mathfrak{S}_n fixed in the introduction. We set further $Q_n := P_n \cap \mathfrak{A}_n$ which is then a Sylow 2-subgroup of the alternating group \mathfrak{A}_n . Recall that if $s = 1$ and $i_1 = r$ then P_n can be identified with a wreath product. More precisely, $P_1 := \{1\}$, $P_2 = C_2 = \langle (1, 2) \rangle$, and $P_{2^r} \cong P_{2^{r-1}} \wr P_2 := \{(x_1, x_2; \sigma) \mid x_1, x_2 \in P_{2^{r-1}}, \sigma \in P_2\}$ if $r \geq 2$. In the case where $r \geq 2$, the base group of the wreath product

$P_{2^{r-1}} \wr P_2$ corresponds to the subgroup $P_{2^{r-1}} \times P_{2^{r-1}}$ of P_{2^r} , and this subgroup is from now on denoted by B_n .

We finish this subsection with the following result which we will use several times throughout this article.

Proposition 2.1 ([6]). *Let $p = 2$, let $r \geq 2$, and let $n = 2^r$. Then*

$$P_{2^{r-1}}\Phi(P_n) = B_n,$$

where B_n denotes the base group of the wreath product $P_n = P_{2^{r-1}} \wr P_2$.

Proof. First of all note that $\Phi(P_{2^k}) = [P_{2^k}, P_{2^k}]$, for $k \geq 1$. For $k = 1$ this is trivially true. If $k \geq 2$ then $[P_{2^k}, P_{2^k}] \leq \Phi(P_{2^k})$, by [11, Satz III.3.14], and, by construction, P_{2^k} can be generated by elements of order 2. Hence the same applies to $P_{2^k}/[P_{2^k}, P_{2^k}]$ so that $\Phi(P_{2^k}) = [P_{2^k}, P_{2^k}]$. Therefore, via the identification of P_n and $P_{2^{r-1}} \wr P_2$, by [22, L. 1.4], we have

$$\Phi(P_n) = \{(x_1, x_2; 1) \mid x_1 x_2 \in \Phi(P_{2^{r-1}})\}.$$

Now let $x \in B_n$. We may write $x = (x_1, x_2; 1)$, for appropriate $x_1, x_2 \in P_{2^{r-1}}$. That is, $x = (x_1 x_2, 1; 1)(x_2^{-1}, x_2; 1) \in P_{2^{r-1}}\Phi(P_n)$. Thus $P_{2^{r-1}}\Phi(P_n)$ is a proper subgroup of P_n containing the maximal subgroup B_n of P_n , and we get $P_{2^{r-1}}\Phi(P_n) = B_n$. \square

2.2 General FG -modules and permutation modules

Let p be any prime number. Let G be a group, and let M be a self-dual FG -module. Suppose further that there is a chain of submodules $0 \subset V \subset U \subset M$ such that

- (1) $V \cong F \cong M/U$, and
- (2) U/V does not have a trivial composition factor.

Then the following holds:

Lemma 2.2. *With the above notation, suppose that M does not have a direct summand of composition length 2 whose composition factors are both isomorphic to F . Then*

- (i) U does not have a trivial factor module, and
- (ii) M/V does not have a trivial submodule.

Proof. (i) Assume (for a contradiction) that there is some $0 \subsetneq X \subset U$ such that $U/X \cong F$. Consider $Z := M/X$, this has length two and both composition factors are isomorphic to F . Since M is self-dual, we get two exact sequences and a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^* & \xrightarrow{\iota} & M & \xrightarrow{\pi} & X^* & \longrightarrow & 0 \\ & & & & & & \downarrow 1 & & \\ 0 & \longrightarrow & X & \xrightarrow{\iota'} & M & \xrightarrow{\pi'} & Z & \longrightarrow & 0 \end{array}$$

We claim that $\pi' \circ \iota$ is injective.

Consider $\text{im}(\iota) \cap \ker(\pi')$. This is a submodule of X , and hence it does not have a trivial composition factor. On the other hand, it is contained in $\text{im}(\iota) \cong Z^*$ which has only trivial composition factors. So $\text{im}(\iota) \cap \ker(\pi') = 0$, that is the restriction of π' to $\text{im}(\iota)$ is injective. In other words, $\pi' \circ \iota$ is injective. Then it must be an isomorphism. Therefore $Z^* \cong Z$, and moreover $M \cong Z \oplus X$, a contradiction.

Dually one proves part (ii). \square

Remark 2.3. Let G be a group, and consider the category of finite G -sets. For such a G -set Ω , we denote the corresponding permutation module by $F\Omega$. In this way we obtain a category \mathcal{P} whose objects are the permutation modules $F\Omega$ with fixed permutation basis Ω . If Ω and Ω' are finite G -sets then the morphisms $\text{Hom}_{\mathcal{P}}(F\Omega, F\Omega')$ are the FG -homomorphisms $f : F\Omega \rightarrow F\Omega'$ such that $f(\omega) \in \Omega'$, for all $\omega \in \Omega$. The trivial FG -module is a permutation module for the trivial G -set $\{1_F\}$. Hence the augmentation map

$$F\Omega \rightarrow F, \omega \mapsto 1$$

on any permutation module $F\Omega$ in \mathcal{P} is a morphism in the category \mathcal{P} . We denote the kernel of the augmentation map $F\Omega \rightarrow F$ by $U_{F\Omega}$. Moreover, the trivial submodule of $F\Omega$ spanned by $\Omega^+ := \sum_{\omega \in \Omega} \omega$ is denoted by $V_{F\Omega}$. With these definitions we get:

Lemma 2.4. *Suppose that M and M' are objects in \mathcal{P} . If $f : M \rightarrow M'$ is an isomorphism in \mathcal{P} then $f(U_M) = U_{M'}$ and $f(V_M) = V_{M'}$.*

Remark 2.5. With the above notation, suppose that $M = F\Omega$ where $|\Omega| \equiv 0 \pmod{p}$. In this case, $V_M \subseteq U_M$, and we then set $\tilde{U}_M := U_M/V_M$. Furthermore, if $\Omega = \{\omega_1, \dots, \omega_n\}$ then, for $i = 1, \dots, n-1$, we set $\xi_i := \omega_i - \omega_n$. So $\{\xi_1, \dots, \xi_{n-1}\}$ is an F -basis of the FG -module U_M , and we also have

$$\Omega^+ = \sum_{i=1}^n \omega_i = \sum_{i=1}^{n-1} \xi_i.$$

The cosets of ξ_1, \dots, ξ_{n-2} span \tilde{U}_M . They are also linearly independent, and hence form an F -basis for \tilde{U}_M . Analogously, for each $i \in \{1, \dots, n-1\}$, the cosets of $\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_{n-1}$ form a basis of \tilde{U}_M . As a special case of Lemma 2.2 we now have the following lemma:

Lemma 2.6. *In the notation of Remark 2.5, suppose that \tilde{U}_M does not have a trivial composition factor. Then M/V_M does not have a trivial submodule, except when M has a direct summand of composition length 2 both of whose composition factors are trivial.*

Definition/Remark 2.7. Suppose $H = \prod_{i=1}^s H_i$, for some $s \geq 2$ and finite groups H_1, \dots, H_s . For $1 \leq i \leq s$, let Ω_i be an H_i -set, and consider the corresponding permutation module $F\Omega_i$. We assume that the sets Ω_i are pairwise disjoint.

For the remainder of this subsection, suppose that the following holds:

Hypothesis: For $1 \leq i \leq s$, let $\text{char}(F)$ divide $|\Omega_i|$. We assume that $F\Omega_i$ is indecomposable, and that $U_{F\Omega_i}/V_{F\Omega_i}$ is indecomposable and does not have a trivial composition factor. Note that this means, in particular, that $U_{F\Omega_i}/V_{F\Omega_i} \neq 0$.

For $1 \leq i \leq s$, we view $F\Omega_i$ as a module for H on which the factors H_j for $j \neq i$ act trivially. Let $\Omega := \bigcup_{i=1}^s \Omega_i$, then $F\Omega$ is a permutation module for H . Let further $W := U_{F\Omega}/V_{F\Omega} = \tilde{U}_{F\Omega}$. The aim is to show that W is indecomposable. Note that, in the case where $s = 1$, this is true, by our hypothesis.

We have a chain of FH -modules

$$V_{F\Omega} \subseteq \bigoplus_{i=1}^s V_{F\Omega_i} \subseteq \bigoplus_{i=1}^s U_{F\Omega_i} \subseteq U_{F\Omega} \subseteq F\Omega = \bigoplus_{i=1}^s F\Omega_i.$$

As the main step in proving the indecomposability of W we will first show that R is indecomposable where

$$R := U_{F\Omega} / \left(\bigoplus_{i=1}^s V_{F\Omega_i} \right). \quad (1)$$

Note that R has a submodule

$$Z = \left(\bigoplus_{i=1}^s U_{F\Omega_i} \right) / \left(\bigoplus_{i=1}^s V_{F\Omega_i} \right) \cong \bigoplus_{i=1}^s \tilde{U}_{F\Omega_i}, \quad (2)$$

and the quotient R/Z is the direct sum of $s-1$ trivial modules. We will identify Z and $\bigoplus_{i=1}^s \tilde{U}_{F\Omega_i}$ via the above isomorphism. Moreover, we will also identify R with a submodule of $\bigoplus_{i=1}^s (F\Omega_i/V_{F\Omega_i})$.

For $1 \leq i \leq s$, we have a basis for $\tilde{U}_{F\Omega_i}$ as in Remark 2.5. In addition, for each $1 \leq i \leq s$, we fix an element $\omega_i \in \Omega_i$ and we set

$$\tau_i := \omega_i + V_{F\Omega_i} \in F\Omega_i/V_{F\Omega_i}.$$

Then $R/Z \cong U_{F\Omega} / \left(\bigoplus_{i=1}^s U_{F\Omega_i} \right)$, and has a basis consisting of cosets of elements of the form

$$\Upsilon := \sum_{i=1}^s c_i \tau_i \quad (3)$$

with $c_1, \dots, c_s \in F$ such that $\sum_{i=1}^s c_i = 0$.

Lemma 2.8. *The module R does not have any trivial submodule.*

Proof. Suppose $\eta \in R$ spans a trivial submodule. Then we write $\eta = d\Upsilon + \sum_{i=1}^s \alpha_i$ with appropriate $d \in F$ and $\alpha_i \in \tilde{U}_{F\Omega_i}$. Let π be the projection onto $F\Omega_1/V_{F\Omega_1}$ which is a homomorphism of FH_1 -modules. Then $\pi(\eta) = dc_1\tau_1 + \alpha_1$. If this were non-zero, then it would span a trivial FH_1 -submodule of $F\Omega_1/V_{F\Omega_1}$. But, by Lemma 2.6 and Definition/Remark 2.7, such a submodule does not exist, so $\pi(\eta) = 0$. Similarly $dc_i\tau_i + \alpha_i = 0$ for $i = 2, \dots, s$, and so $\eta = 0$. \square

Lemma 2.9. *Any indecomposable direct summand of Z is equal to one of the FH -modules $\tilde{U}_{F\Omega_1}, \dots, \tilde{U}_{F\Omega_s}$.*

Proof. Suppose $Z = Z_1 \oplus Z_2$ where Z_1 is indecomposable. By the Krull-Schmidt Theorem, Z_1 is isomorphic to one of the modules $\tilde{U}_{F\Omega_1}, \dots, \tilde{U}_{F\Omega_s}$, say $Z_1 \cong \tilde{U}_{F\Omega_1}$. So, for $j \geq 2$, the group H_j acts trivially on Z_1 . Now consider $\zeta \in Z_1$. We can write $\zeta = \sum_{i=1}^s \alpha_i$, according to the decomposition of Z , in (2). If $g \in H_2$ then $0 = (g-1)\zeta$, but this is equal to $(g-1)\alpha_2$. If $\alpha_2 \neq 0$ then α_2 would span a trivial FH_2 -submodule of $\tilde{U}_{F\Omega_2}$, but there is no such submodule, by Lemma 2.6. So $\alpha_2 = 0$. Similarly $\alpha_j = 0$ for $3 \leq j \leq s$. This proves that $Z_1 \subseteq \tilde{U}_{F\Omega_1}$, and then, comparing dimensions, we must have equality. \square

Lemma 2.10 (Separation Lemma). *Let $\omega = \Upsilon + \sum_{i=1}^s \alpha_i \in R$ where Υ is as in (3), and where $\alpha_i \in \tilde{U}_{F\Omega_i}$ for $1 \leq i \leq s$. Let further $1 < t \leq s$. Suppose that, for all $g \in H$, we have $(g-1)\omega \in \bigoplus_{i=t}^s \tilde{U}_{F\Omega_i}$. Then $\alpha_1 = \alpha_2 = \dots = \alpha_{t-1} = 0$ and $c_1 = c_2 = \dots = c_{t-1} = 0$.*

Proof. Let $i < t$, and let $g \in H_i$. Then

$$(*) \quad (g-1)\omega = c_i(g-1)\tau_i + (g-1)\alpha_i.$$

Clearly $(g-1)\tau_i \in \tilde{U}_{F\Omega_i}$, and so $(g-1)\omega \in \tilde{U}_{F\Omega_i}$. By our hypothesis, $(g-1)\omega$ also lies in $\bigoplus_{j=t}^s \tilde{U}_{F\Omega_j}$, so that the element $(*)$ is equal to zero for $g \in H_i$.

Suppose $c_i\tau_i + \alpha_i \neq 0$. This element is then fixed by all $g \in H$, and therefore it spans a trivial submodule of R . This contradicts Lemma 2.8. Hence $c_i\tau_i + \alpha_i = 0$. Now, $\tau_i \notin \tilde{U}_{F\Omega_i}$, and therefore $c_i = 0$, and then also $\alpha_i = 0$. \square

Proposition 2.11. *In the notation of Definition/Remark 2.7, the FH -module R is indecomposable.*

Proof. Let Z be the submodule of R as defined in (2). That is, R/Z is isomorphic to the direct sum of $s-1$ copies of F . Moreover, Z does not have any trivial composition factor, and in fact, Z contains every submodule of R which does not have any trivial composition factor.

Let $e^2 = e \in \text{End}_{FH}(R)$. We have to show that $e = 0$ or $e = 1$. Note first that $e(Z) \cong Z/(\ker(e) \cap Z)$ is a submodule of R which does not have trivial composition factors, so $e(Z) \subseteq Z$, and the restriction of e to Z is also a projection. So we have

$$Z = e(Z) \oplus (1-e)(Z), \quad R = e(R) \oplus (1-e)(R),$$

$e(Z) \subseteq e(R)$, and $(1-e)(Z) \subseteq (1-e)(R)$. We may suppose $e(Z) \neq e(R)$. For otherwise we replace e by $(1-e)$. In particular, $e \neq 0$. We take $\omega \in e(R) \setminus e(Z)$, so that for all $g \in H$ we have

$$(g-1)\omega \in e(Z),$$

since $e(R)/e(Z)$ is the direct sum of trivial modules. The element ω has the form $\omega = \Upsilon + \sum_{i=1}^s \alpha_i$, as in the Separation Lemma. We now distinguish two cases.

(1) Assume that $e(Z) = 0$. Since $e \neq 0$, $e(R)$ then has a trivial submodule, a contradiction. If $e(Z) = Z$ then $(1-e)(Z) = 0$ so that $e = 1$. For otherwise we would get the contradiction that $(1-e)(R)$ has a trivial submodule.

(2) Hence, we may now assume that $e(Z) \neq 0 \neq (1-e)(Z)$. Using Lemma 2.9, we may assume further that $e(Z) = \bigoplus_{i=t}^s \tilde{U}_{F\Omega_i}$ for some $t > 1$, and $(1-e)(Z) = \bigoplus_{i=1}^{t-1} \tilde{U}_{F\Omega_i}$. Applying the Separation Lemma, we deduce

$$\omega = \sum_{i=t}^s c_i\tau_i + \sum_{i=t}^s \alpha_i,$$

and we recall that $\sum_{i=1}^s c_i = 0 = \sum_{i=t}^s c_i$.

This holds for any $\omega \in e(R) \setminus e(Z)$ so that $e(R)/e(Z)$ has dimension at most $s-t$ (which is the dimension of the space $\{\sum_{i=t}^s c_i\tau_i \mid \sum_{i=t}^s c_i = 0\}$).

Suppose that also $(1-e)(Z) \neq (1-e)(R)$. Then we use the analogous argument for $(1-e)(Z)$ which is the direct sum of $t-1$ indecomposable modules. We get that the dimension of

$(1 - e)(R)/(1 - e)(Z)$ is at most $t - 2$. On the other hand, $\dim((1 - e)(R)/(1 - e)(Z)) + \dim(e(R)/e(Z)) = s - 1$ but $s - t + t - 2 < s - 1$, a contradiction.

Thus we must have $(1 - e)(Z) = (1 - e)(R)$ in which case $s - 1 = \dim((1 - e)(R)/(1 - e)(Z)) + \dim(e(R)/e(Z)) \leq s - t \leq s - 2$, and we have again reached a contradiction.

To summarize, the only possibility left is $e = 1$, and hence R is indecomposable. \square

Lemma 2.12. *With the notation of Definition/Remark 2.7, let $R' := (\bigoplus_{i=1}^s U_{F\Omega_i})/V_{\bigoplus_{i=1}^s F\Omega_i}$. Then $R' \cong R^*$.*

Proof. We take the non-degenerate symmetric bilinear form of the permutation FH -module $F\Omega$ defined by $\langle \omega_i, \omega_j \rangle = \delta_{ij}$ for $\omega_i, \omega_j \in \Omega$. We identify $F\Omega^*$ with the set $\{\langle -, v \rangle \mid v \in F\Omega\}$. The vector space isomorphism $F\Omega \rightarrow F\Omega^* : v \mapsto \langle -, v \rangle$ is, in fact, an FH -module isomorphism.

For any submodule X of $F\Omega$, we have the exact sequence

$$0 \rightarrow X^0 \rightarrow F\Omega^* \xrightarrow{\pi} X^* \rightarrow 0$$

where π is restriction to X , and $X^0 := \{\langle -, v \rangle \in F\Omega^* \mid \langle X, v \rangle = 0\}$. One sees directly that $U_{F\Omega}^0 \cong V_{F\Omega}$ and also $V_{F\Omega}^0 \cong U_{F\Omega}$, under the isomorphism $F\Omega \rightarrow F\Omega^*$. So $U_{F\Omega}^* \cong F\Omega/V_{F\Omega}$ and $V_{F\Omega}^* \cong F\Omega/U_{F\Omega}$.

By definition, $R' = (\bigoplus_{i=1}^s U_{F\Omega_i})/V_{F\Omega}$. From the inclusion of submodules of $F\Omega$ given in Definition/Remark 2.7 we get an exact sequence

$$(*) \quad 0 \rightarrow R' \rightarrow F\Omega/V_{F\Omega} \rightarrow F\Omega/(\bigoplus_{i=1}^s U_{F\Omega_i}) \rightarrow 0.$$

By the previous considerations, $(F\Omega/V_{F\Omega})^* \cong U_{F\Omega}$. Moreover, since $F\Omega/(\bigoplus_{i=1}^s U_{F\Omega_i}) \cong \bigoplus_{i=1}^s (F\Omega_i/U_{F\Omega_i})$, we deduce that $(F\Omega/(\bigoplus_{i=1}^s U_{F\Omega_i}))^*$ is isomorphic to $\bigoplus_{i=1}^s V_{F\Omega_i}$. Hence dualizing $(*)$ shows that

$$(R')^* \cong U_{F\Omega}/(\bigoplus_{i=1}^s V_{F\Omega_i}) = R,$$

as claimed. \square

Retaining the notation introduced in Definition/Remark 2.7, we are now able to prove the following:

Proposition 2.13. *The FH -module W is indecomposable.*

Proof. Let $\varphi^2 = \varphi \in \text{End}_{FH}(W)$. We must show that $\varphi \in \{0, 1\}$. Let W_0 be the sum of all trivial submodules of W so that W_0 has dimension $s - 1$. Moreover W_0 is equal to $(\bigoplus_{i=1}^s V_{F\Omega_i})/V_{\bigoplus_{i=1}^s F\Omega_i}$. Since φ is a homomorphism, the submodule W_0 must be invariant under φ . Hence φ induces a projection $\bar{\varphi}$ on the quotient module $W/W_0 \cong R$. By Proposition 2.11, we know that R is indecomposable so that $\bar{\varphi} \in \{0, 1\}$. We may suppose that $\bar{\varphi} = 0$. This means that $\varphi(W) \subseteq W_0$.

Now we consider the submodule $R' := (\bigoplus_{i=1}^s U_{F\Omega_i})/V_{\bigoplus_{i=1}^s F\Omega_i}$ of W again. We have $W_0 \subset R' \subset W$, and hence $\varphi(R') \subseteq \varphi(W) \subseteq W_0 \subset R'$. Therefore φ restricts to a projection of R' which is not the identity. Lemma 2.12 above shows that $R' \cong R^*$. By Proposition 2.11, R' is therefore indecomposable. So the restriction of φ to R' is the zero map. This implies that, for every $w \in W$, we have $\varphi(w) = \varphi^2(w) \in \varphi(R') = 0$, and the assertion follows. \square

2.3 Permutation modules for symmetric groups

We will apply the previous results to direct summands of certain restrictions of the permutation $F\mathfrak{S}_{n-1}$ -module $M^{(n-3,2)}$ over an algebraically closed field of characteristic 2, and where $n \equiv 1 \pmod{4}$. Before doing this, we relate this module to S and D . The usual permutation basis for $M^{(n-3,2)}$ is the set Ω of $(n-3, 2)$ -tabloids, so that $M^{(n-3,2)} = F\Omega$. Later we will identify Ω with the set of 2-element subsets of $\{1, 2, \dots, n-1\}$. Throughout this subsection, let $n \geq 5$ with $n \equiv 1 \pmod{4}$.

Lemma 2.14. (i) *The restriction of the Young module $Y^{(n-2,2)}$ to \mathfrak{S}_{n-1} is isomorphic to $M^{(n-3,2)}$.*

(ii) *Let U be the kernel $U_{F\Omega}$ of the augmentation map $F\Omega \rightarrow F$, and let V be the trivial submodule of $F\Omega$ spanned by Ω^+ . Then the restriction of S to \mathfrak{S}_{n-1} is isomorphic to U , and the restriction of D to \mathfrak{S}_{n-1} is isomorphic to U/V . In particular, U/V is indecomposable and does not have trivial composition factors.*

Proof. We have $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(M^{(n-2,2)}) \cong M^{(n-2,1)} \oplus M^{(n-3,2)}$, by Mackey's Decomposition Theorem, and

$$M^{(n-2,2)} \cong Y^{(n-1,1)} \oplus Y^{(n-2,2)} \cong S^{(n-1,1)} \oplus Y^{(n-2,2)},$$

by [10]. By the Krull-Schmidt Theorem, it suffices to show that $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(S^{(n-1,1)}) \cong M^{(n-2,1)}$.

But this is clear from the fact that $S^{(n-1,1)}$ has F -basis $\{\omega_1 - \omega_n \mid 1 \leq i \leq n-1\}$ where $\{\omega_1, \dots, \omega_n\}$ is the permutation basis of the natural permutation $F\mathfrak{S}_n$ -module $M^{(n-1,1)}$.

By [19], the Young module $Y^{(n-2,2)}$ is uniserial and

$$Y^{(n-2,2)} = \mathcal{U}(F, D^{(n-2,2)}, F).$$

On the other hand, by [10], $F\Omega = M^{(n-3,2)} \cong Y^{(n-3,2)}$, and has thus a unique trivial submodule and a unique trivial quotient module, by [19]. This trivial submodule is then V , and the trivial quotient module is $F\Omega/U$. By [15, Thm., 11.2.7], we also know that $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^{(n-2,2)})$ is indecomposable. Moreover, the modular branching rules [15, Thm. 11.2.7], also yield that $\text{Res}_{\mathfrak{S}_4}^{\mathfrak{S}_5}(D^{(3,2)}) \cong \mathcal{U}(D^{(3,1)}, D^{(3,1)})$, and that $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^{(n-2,2)}) \cong \mathcal{U}(D^{(n-2,1)}, D^{(n-3,2)}, D^{(n-2,1)})$ for $n > 5$. From this assertion (ii) follows, and the lemma is proved. \square

Remark 2.15. (a) We now set $N := n-1$, and write $N = \sum_{i=1}^r a_i$, for some $r \geq 1$, and some $a_1 \equiv \dots \equiv a_r \equiv 0 \pmod{4}$. Moreover, we consider the (standard) Young subgroup $H := \prod_{i=1}^r \mathfrak{S}_{a_i}$ of \mathfrak{S}_N . Recall again that $\Omega = \{\{u, v\} \mid 1 \leq u < v \leq N\}$ is a permutation basis for the $F\mathfrak{S}_N$ -module $M^{(N-2,2)} = F\Omega$. We set $U := U_{F\Omega}$ and $V := V_{F\Omega}$.

(b) We have a direct sum decomposition

$$\text{Res}_H^{\mathfrak{S}_N}(M^{(N-2,2)}) = \bigoplus_{i=1}^r F\Omega_i \oplus \bigoplus_{1 \leq i < j \leq r} F\Delta_{ij},$$

where Ω_i denotes the set of all 2-element subsets of the support of \mathfrak{S}_{a_i} , and where Δ_{ij} denotes the set of all 2-element subsets $\{u, v\}$ of $\{1, \dots, N\}$ such that u belongs to the support of \mathfrak{S}_{a_i} and v belongs to the support of \mathfrak{S}_{a_j} . Note that this is, in fact, an indecomposable direct sum

decomposition. Namely, for $i, j \in \{1, \dots, r\}$ such that $i < j$, we have $\text{Res}_{\mathfrak{S}_{a_i}}^H(F\Omega_i) \cong M^{(a_i-2,2)}$ and $\text{Res}_{\mathfrak{S}_{a_i} \times \mathfrak{S}_{a_j}}^H(F\Delta_{ij}) \cong M^{(a_i-1,1)} \boxtimes M^{(a_j-1,1)}$ which are indecomposable modules. If $i \neq j$ then \mathfrak{S}_{a_j} acts trivially on $F\Omega_i$. We set $\Delta := \bigsqcup_{1 \leq i < j \leq r} \Delta_{ij}$ so that $\Omega = \bigsqcup_{i=1}^r \Omega_i \uplus \Delta$.

With this notation, the following holds:

Lemma 2.16. *There is a decomposition of FH -modules $\text{Res}_H^{\mathfrak{S}^N}(M^{(N-2,2)}) = \bigoplus_{i=1}^r F\Omega'_i \oplus F\Delta'$ where*

- (i) $F\Delta' \cong F\Delta$, $F\Delta' \subseteq U$, and $F\Delta' \cap V = 0$,
- (ii) $V \subseteq \bigoplus_{i=1}^r F\Omega'_i$, and $F\Omega'_i \cong F\Omega_i$ for $1 \leq i \leq r$.

In particular, $\text{Res}_H^{\mathfrak{S}^N}(U) \cong U_{\bigoplus_{i=1}^r F\Omega_i} \oplus F\Delta$, and $\text{Res}_H^{\mathfrak{S}^N}(U/V) \cong (U_{\bigoplus_{i=1}^r F\Omega_i} / V_{\bigoplus_{i=1}^r F\Omega_i}) \oplus F\Delta$.

Proof. We will show that there is an FH -isomorphism $\psi : M^{(N-2,2)} \longrightarrow M^{(N-2,2)}$ such that, for $1 \leq i < j \leq r$,

- (a) with $\Omega'_i := \psi(\Omega_i)$, we have $(\Omega'_1 \cup \dots \cup \Omega'_r)^+ = \Omega^+$.
- (b) $\Delta'_{ij} := \psi(\Delta_{ij}) \subseteq U$.

From (a) we then get $V \subseteq \bigoplus_{i=1}^r F\Omega'_i$, and from (b) we get $F\Delta' := \bigoplus_{1 \leq i < j \leq r} F\Delta'_{ij} \subseteq U$. Then this also forces $V \cap F\Delta' \subseteq (\bigoplus_{i=1}^r F\Omega'_i) \cap (\bigoplus_{1 \leq i < j \leq r} F\Delta'_{ij}) = 0$. Therefore, we obtain $\text{Res}_{P_N}^H(U_{F\Omega}) = (U_{F\Omega} \cap \bigoplus_{i=1}^r F\Omega'_i) \oplus \bigoplus_{1 \leq i < j \leq r} F\Delta'_{ij}$, and $\text{Res}_{P_N}^H(U_{F\Omega}/V_{F\Omega}) \cong (U_{F\Omega} \cap \bigoplus_{i=1}^r F\Omega'_i) / V_{F\Omega} \oplus \bigoplus_{1 \leq i < j \leq r} F\Delta'_{ij}$. Using the permutation basis $\Omega'_1 \cup \dots \cup \Omega'_r$ of $\bigoplus_{i=1}^r F\Omega'_i$ defined below, we deduce $U_{F\Omega} \cap \bigoplus_{i=1}^r F\Omega'_i = U_{\bigoplus_{i=1}^r F\Omega'_i}$. Furthermore, by (a), $(\Omega'_1 \cup \dots \cup \Omega'_r)^+ = \Omega^+$ so that Lemma 2.4 then also gives $(U_{F\Omega} \cap \bigoplus_{i=1}^r F\Omega'_i) / V_{F\Omega} \cong U_{\bigoplus_{i=1}^r F\Omega'_i} / V_{\bigoplus_{i=1}^r F\Omega'_i}$, as claimed.

We define ψ on the permutation basis Ω of $M^{(N-2,2)}$ as follows: for $j = 2, \dots, r$ and $\{u, v\} \in \Omega_j$, we set

$$\psi(\{u, v\}) := \sum_{\substack{i < j \\ \{t, u\}, \{t, v\} \in \Delta_{ij}}} \{t, u\} + \{t, v\}$$

and, for $\{u, v\} \in \Omega_1$, we set $\psi(\{u, v\}) := \{u, v\}$. If $\{u, v\} \in \Delta_{ij}$, for some $i < j$ and some $u < v$, then we set

$$(*) \quad \psi(\{u, v\}) := \{u, v\} + \sum_{\{t, u\} \in \Omega_i} \{t, u\}.$$

It is easily checked that ψ is an FH -homomorphism and that, for any $1 \leq i < j \leq r$, the restrictions of ψ to $F\Omega_i$ and $F\Delta_{ij}$, respectively, are injective. So, for $1 \leq i < j \leq r$ and with the notation as above, $F\Omega'_i \cong F\Omega_i$ and $F\Delta'_{ij} \cong F\Delta_{ij}$ as permutation FH -modules. For $1 \leq i < j \leq r$, there are $|\Omega_i|$ terms in the sum (*) which is even, hence $\Delta'_{ij} \subseteq U$. Furthermore, in the sum $\sum_{i=1}^r (\Omega'_i)^+$, each basis element $\{u, v\} \in \Omega$ occurs with odd multiplicity, giving (a).

Therefore, it remains to show that ψ is bijective. For this we write down the corresponding matrix Ψ which has block shape, and the blocks correspond to $\Omega_1, \dots, \Omega_r$, and then Δ_{ij} ordered via the lexicographic ordering on the indices. In the first block column there is only one non-zero block, namely the identity matrix of size $|\Omega_1|$ in position $(1, 1)$. Apart from this identity

matrix, the only non-zero blocks in the first block row of Ψ correspond to $\Delta_{12}, \dots, \Delta_{1r}$. These are in positions $(1, r+1), \dots, (1, 2r-1)$. So we can perform elementary column operations to make all blocks in the first row to 0. After that, we consider the block columns $r+1, \dots, 2r-1$ of the resulting matrix. Each of these has precisely one non-zero block, namely an identity matrix of size $|\Delta_{12}|, \dots, |\Delta_{1r}|$ in position $(r+1, r+1), \dots, (2r-1, 2r-1)$, respectively. Hence we can now perform elementary column operations such that all blocks in positions $(r+1, 2), \dots, (r+1, r), (r+2, 2), \dots, (r+2, r), \dots, (2r-1, 2), \dots, (2r-1, r)$ become 0. Now the second block column has one non-zero entry, namely the identity matrix of size $|\Omega_2|$ in position $(2, 2)$. The only non-zero entries in the second row of the new matrix are in positions $(2, 2), (2, 2r), \dots, (2, 3r-3)$. So performing elementary column operations, next the blocks in positions $(2, 2r), \dots, (2, 3r-3)$ become 0. Moreover, each of the block columns $2r, \dots, 3r-3$ of the resulting matrix has just one non-zero block. These blocks are the identity matrices of sizes $|\Delta_{23}|, \dots, |\Delta_{2r}|$ in positions $(2r, 2r), \dots, (3r-3, 3r-3)$, respectively. Inductively, this procedure yields a block diagonal matrix Ψ' whose diagonal blocks are identity matrices of sizes $|\Omega_1|, \dots, |\Omega_r|, |\Delta_{12}|, \dots, |\Delta_{1r}|, \dots, |\Delta_{r-1,r}|$. Consequently, both Ψ' and Ψ are invertible, and ψ is thus an FH -isomorphism. \square

Proposition 2.17. *With the notation as before, let $W := U_{\bigoplus_{i=1}^r F\Omega_i} / V_{\bigoplus_{i=1}^r F\Omega_i}$. Then W is indecomposable as FH -module.*

Proof. We apply Proposition 2.13 with $H = \prod_{i=1}^r \mathfrak{S}_{a_i}$ and $\Omega_1, \dots, \Omega_r$ as above. We only have to show that the hypotheses in Definition/Remark 2.7 are satisfied. Let $i \in \{1, \dots, r\}$. As we have already mentioned in Remark 2.15, $\text{Res}_{\mathfrak{S}_{a_i}}^H(F\Omega_i) \cong M^{(a_i-2, 2)}$ which is indecomposable, since $a_i \equiv 0 \pmod{4}$. Hence, by Lemma 2.14, both $\text{Res}_{\mathfrak{S}_{a_i}}^H(F\Omega_i)$ and $\text{Res}_{\mathfrak{S}_{a_i}}^H(U_{F\Omega_i}/V_{F\Omega_i})$ are indecomposable. Furthermore, by Lemma 2.14, $\text{Res}_{\mathfrak{S}_{a_i}}^H(U_{F\Omega_i}/V_{F\Omega_i})$ does not have any trivial composition factor. Since, for $j \neq i$ the direct factors H_j act trivially on $F\Omega_i$, we conclude that $F\Omega_i$ satisfies the conditions in Definition/Remark 2.7. \square

Remark 2.18. (a) Let $n = 1 + \sum_{j=1}^s 2^{i_j}$ be the 2-adic expansion of n , for some $s \geq 1$ and $i_1 > \dots > i_s \geq 2$, then, for $j = 1, \dots, s$, we set $n_j := 2^{i_j}$. Note that Proposition 2.17 holds, in particular, in the case where $r = s$ and $a_1 = n_1, \dots, a_s = n_s$. So, in this case, H is the Young subgroup $\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_s}$ of \mathfrak{S}_N containing $P_N = P_n$.

(b) To finish this subsection, we will prove an analogous statement for the FH -module $U_{\bigoplus_{j=1}^s F\Omega_j}$ where $H = \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_s}$, and $a_1 = n_1, \dots, a_s = n_s$. Namely, we will show below that, in fact, $U_{\bigoplus_{j=1}^s F\Omega_j}$ restricts indecomposably to P_N . For this, recall our convention concerning the fixed Sylow 2-subgroup P_N of \mathfrak{S}_N from the introduction. Let $j \in \{1, \dots, s\}$ and $l \in \{1, \dots, i_j\}$. The orbit of the element

$$\omega_{j,l} := \{2^{i_1} + \dots + 2^{i_{j-1}} + 1, 2^{i_1} + \dots + 2^{i_{j-1}} + 2^{l-1} + 1\} \in \Omega_j$$

under the P_N -action will be denoted by $\Omega_{j,l}$, so we have $\Omega_j = \bigsqcup_{l=1}^{i_j} \Omega_{j,l}$. Furthermore, $F\Omega_{j,l}$ is a permutation FP_N -module on which P_{n_j} acts transitively, and the other direct factors $P_{n_1}, \dots, P_{n_{j-1}}, P_{n_{j+1}}, \dots, P_{n_s}$ act trivially. The augmentation map $F\Omega_{j,l} \rightarrow F$ is denoted by $\alpha_{j,l}$ and, as usual, $U_{F\Omega_{j,l}}$ denotes the kernel of $\alpha_{j,l}$. Let $\alpha : \bigoplus_{j=1}^s F\Omega_j \rightarrow F$ be the augmentation map with respect to the permutation basis $\bigsqcup_{j=1}^s \Omega_j$.

Let $R_{j,l}$ be the stabilizer of $\omega_{j,l}$ under the P_N -action on $\bigsqcup_{j=1}^s \Omega_j$ so that

$$R_{j,l} = P_{n_1} \times \dots \times P_{n_{j-1}} \times \text{Stb}_{P_{n_j}}(\omega_{j,l}) \times P_{n_{j+1}} \times \dots \times P_{n_s},$$

and $F\Omega_{j,l} \cong \text{Ind}_{R_{j,l}}^{P_N}(F)$.

Proposition 2.19. *With the above notation, $\text{Res}_{P_N}^H(\bigoplus_{j=1}^s F\Omega_j) \cong \bigoplus_{j=1}^s \bigoplus_{l=1}^{i_j} F\Omega_{j,l}$. Moreover, this is an indecomposable direct sum decomposition, and $R_{j,l} \leq_{P_N} R_{j',l'}$ if and only if $j = j'$ and $l = l'$.*

Proof. (1) Suppose first that $s = 1$, and set $r := i_1$ so that $N = 2^r$. Recall that, in this case, $P_N = \langle \prod_{k=1}^{2^{i-1}} (k, k + 2^{i-1}) \mid i = 1, \dots, r \rangle$. From this it is clear that $\Omega_{1,1}, \dots, \Omega_{1,r}$ are precisely the P_N -orbits on Ω_1 . This gives the assertion concerning the decomposition of $\text{Res}_{P_N}^H(F\Omega_1)$. It remains to show that, for $l \neq l'$, we have $R_{1,l} \not\leq_{P_N} R_{1,l'}$. Assume, for a contradiction, that $gR_{1,l}g^{-1} \leq R_{1,l'} \leq P_N$, for some $l \neq l'$ and some $g \in P_N$. Comparing the group orders, we must have $1 \leq l' < l$. So $gR_{1,l}g^{-1}$ is the stabilizer of the element $g\{1, 1 + 2^{l-1}\} = \{g(1), g(1 + 2^{l-1})\}$.

Assume first that $l' > 1$. Since $\{g(1), g(1 + 2^{l-1})\}$ belongs to the P_N -orbit of $\{1, 1 + 2^{l-1}\}$, we have $\{1, 2\} \cap \{g(1), g(1 + 2^{l-1})\} = \emptyset$ or $\{1 + 2^{l'-1}, 2 + 2^{l'-1}\} \cap \{g(1), g(1 + 2^{l-1})\} = \emptyset$. In the first case, $(1, 2) \in \text{Stb}_{P_N}(\{g(1), g(1 + 2^{l-1})\}) \setminus R_{1,l'}$, and in the second case, $(1 + 2^{l'-1}, 2 + 2^{l'-1}) \in \text{Stb}_{P_N}(\{g(1), g(1 + 2^{l-1})\}) \setminus R_{1,l'}$. So this forces $l' = 1$. Note that $R_{1,1}$ is of shape $R_{1,1} = P_2 \times P_2 \times P_4 \times P_8 \times \dots \times P_{N/2}$. As usual, the direct factors are supposed to be acting on disjoint successive subsets of $\{1, \dots, N\}$. There are 2^{r-2} subgroups of P_N which are conjugate to $R_{1,1}$. The group $R_{1,l}$ contains the elements

$$\begin{aligned} & (1, 1 + 2^{l-1})(2, 2 + 2^{l-1}) \dots (2^{l-1}, 2^l), \\ & (1 + 2^l, 1 + 2^l + 2^{l-1})(2 + 2^l, 2 + 2^l + 2^{l-1}) \dots (2^l + 2^{l-1}, 2^{l+1}), \\ & \vdots \\ & (1 + 2^r - 2^l, 1 + 2^r - 2^{l-1})(2 + 2^r - 2^l, 2 + 2^r - 2^{l-1}) \dots (2^r - 2^{l-1}, 2^r). \end{aligned}$$

However, the product of these is not contained in any of the P_N -conjugates of $R_{1,1}$. Hence we have reached a contradiction.

(2) So we now suppose that $s \geq 1$ is arbitrary. Again, by the definition of $P_N = P_{n_1} \times \dots \times P_{n_s}$, we deduce that the P_N -sets $\Omega_{j,l}$, for $j = 1, \dots, s$ and $l = 1, \dots, i_j$, are the P_N -orbits on $\biguplus_{j=1}^s \Omega_j$. Thus

$$\text{Res}_{P_N}^H\left(\bigoplus_{j=1}^s F\Omega_j\right) = \bigoplus_{j=1}^s \bigoplus_{l=1}^{i_j} F\Omega_{j,l},$$

and this is of course an indecomposable direct sum decomposition.

In order to prove the second assertion, let $j, j' \in \{1, \dots, s\}$, let $l \in \{1, \dots, i_j\}$, and let $l' \in \{1, \dots, i_{j'}\}$. Let further $g = g_1 \dots g_s \in P_N$ where $g_1 \in P_{n_1}, \dots, g_s \in P_{n_s}$. If $j \neq j'$ then $P_{n_{j'}} \leq R_{j,l} \cap gR_{j,l}g^{-1}$ and $P_{n_j} \leq R_{j',l'} \cap gR_{j',l'}g^{-1}$, but $P_{n_{j'}} \not\leq R_{j',l'}$ and $P_{n_j} \not\leq R_{j,l}$. If $j = j'$ and $l \neq l'$ then $gR_{j,l}g^{-1} = g_j R_{j,l} g_j^{-1} \not\leq R_{j,l'}$, by (1) above. This completes the proof of the proposition. \square

Corollary 2.20. *In the notation of Remark 2.18, $\text{Res}_{P_N}^H(U \bigoplus_{j=1}^s F\Omega_j)$ is indecomposable.*

Proof. We know that the sets $\Omega_{j,l}$, for $j = 1, \dots, s$ and $l = 1, \dots, i_j$, are precisely the P_N -orbits on the permutation basis $\Omega_1 \cup \dots \cup \Omega_s$ of $\bigoplus_{j=1}^s F\Omega_j$. Moreover, by Proposition

2.19, we have $R_{j,l} \leq_{P_N} R_{j',l'}$ if and only if $j = j'$ and $l = l'$. Equivalently, there is an epimorphism of P_N -sets from $\Omega_{j,l}$ to $\Omega_{j',l'}$ if and only if $j = j'$ and $l = l'$. Therefore, $\text{Res}_{P_N}^H(U \bigoplus_{j=1}^s F\Omega_j) = U \bigoplus_{j=1}^s \bigoplus_{l=1}^{i_j} F\Omega_{j,l}$ is indecomposable, by [1, Thm. 2]. \square

3 The Specht module $S^{(n-2,2)}$

Throughout this section let F be an algebraically closed field of characteristic 2, let $n \geq 4$, and let S be the Specht $F\mathfrak{S}_n$ -module $S^{(n-2,2)}$.

3.1 Restrictions to elementary abelian subgroups

As we have already mentioned, in the proof of Theorem 1.1 we will distinguish between the cases $n \equiv 1 \pmod{2}$ and $n \equiv 0 \pmod{2}$. The key step in the proof of the latter case will be Lemma 3.5 below which asserts that, provided that $n \geq 6$ is even, S restricts indecomposably to the following elementary abelian subgroup of P_n :

$$E_n := \langle (2r-1, 2r) \mid r = 1, \dots, n/2 \rangle.$$

In order to prove this result, for the remainder of this subsection we suppose that $n \geq 6$ is even, and we aim to show that the endomorphism ring $\text{End}_{FE_n}(S)$ contains no idempotents other than 1 and 0. For this, we describe the action of E_n on the F -basis of S consisting of all standard $(n-2, 2)$ -polytabloids. After that we will determine the socle of $\text{Res}_{E_n}^{\mathfrak{S}_n}(S)$. For if $1 \neq \varphi \in \text{End}_{FE_n}(S)$ is an idempotent which is the zero map on $\text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(S))$ then, in fact, $\varphi = 0$.

The following result is easily verified:

Proposition 3.1. *Let $r \in \{1, \dots, n/2\}$.*

(i) *If $r = 1$ then*

$$(2r-1, 2r)e(ij) = (1, 2)e(ij) = \begin{cases} e(ij), & \text{if } i = 2, \\ e(ij) + e(2j), & \text{if } i = 3, \\ e(ij) + e(2i) + e(2j), & \text{if } i \geq 4. \end{cases}$$

(ii) *If $r = 2$ then*

$$(2r-1, 2r)e(ij) = (3, 4)e(ij) = \begin{cases} e(ij), & \text{if } i = 2 \text{ and } j = 4, \text{ or } i \geq 5, \\ e(ij) + e(24), & \text{if } i = 2, j \geq 5, \\ e(4, j), & \text{if } i = 3, j \geq 5, \\ e(3, j), & \text{if } i = 4, j \geq 5, \\ e(ij) + e(24), & \text{if } i = 3, j = 4. \end{cases}$$

(iii) If $r > 2$ then

$$(2r-1, 2r)e(ij) = \begin{cases} e(ij), & \text{if } i, j \notin \{2r-1, 2r\}, \\ e(i, 2r-1), & \text{if } j = 2r, i < 2r-1, \\ e(2r-1, j), & \text{if } i = 2r < j, \\ e(i, 2r), & \text{if } j = 2r-1, i < 2r-1, \\ e(2r, j), & \text{if } i = 2r-1 < j-1, \\ e(ij) + e(2i) + e(2j), & \text{if } i = 2r-1, j = 2r. \end{cases}$$

Proposition 3.2. For $2 \leq k < l \leq n/2$ let $\sigma_{k,l} := e(2k-1, 2l-1) + e(2k-1, 2l) + e(2k, 2l-1) + e(2k, 2l)$, and for $3 \leq l \leq n/2$ let $\rho_l := e(2, 2l-1) + e(2, 2l)$. Let

$$\mathfrak{B} := \{e(24)\} \cup \{\rho_l \mid 3 \leq l \leq n/2\} \cup \{\sigma_{k,l} \mid 2 \leq k < l \leq n/2\}.$$

Then \mathfrak{B} is an F -basis for $\text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(S))$.

Remark 3.3. Before proving the proposition, we introduce the following F -subspaces of S :

$$\begin{aligned} V(k, l) &:= F\{e(2k-1, 2l-1), e(2k-1, 2l), e(2k, 2l-1), e(2k, 2l)\}, \\ T &:= F\{e(2r-1, 2r) \mid 2 \leq r \leq n/2\}, \\ U &:= F\{e(2, j) \mid 4 \leq j \leq n\}. \end{aligned}$$

With this notation, we get:

Lemma 3.4. Let $E' := \langle (2r-1, 2r) \mid 2 \leq r \leq n/2 \rangle \leq E_n$.

(i) We have a direct sum decomposition

$$\text{Res}_{E'}^{\mathfrak{S}_n}(S) = (T + U) \oplus \bigoplus_{2 \leq k < l \leq n/2} V(k, l).$$

(ii) The socle of $\text{Res}_{E'}^{\mathfrak{S}_n}(S)$ has basis $\mathfrak{B} \cup \mathfrak{C}$. Here, \mathfrak{B} is as in Proposition 3.2, and $\mathfrak{C} := \{e(2k-1, 2k) + e(2, 2k) + e(34) \mid 3 \leq k \leq n/2\}$.

Proof. (1) From Proposition 3.1 we deduce that both U and $T + U$ are, in fact, FE' -submodules of S . The socle of the FE' -module $T + U$ clearly consists of all elements in $T + U$ fixed under E' . One checks that this socle has basis

$$\{e(24)\} \cup \{\rho_j \mid 3 \leq j \leq n/2\} \cup \mathfrak{C}.$$

(2) Now consider $V(k, l)$, for some $2 \leq k < l \leq n/2$. By Proposition 3.1, the subgroup $\langle (2k-1, 2k), (2l-1, 2l) \rangle$ of E' acts freely on $V(k, l)$, and any other generator of E' acts trivially on $V(k, l)$. This shows that $V(k, l)$ is also an FE' -submodule of S . With this assertion (i) follows. By Proposition 3.2, we deduce further that the socle of the FE' -module $V(k, l)$ is one-dimensional, and is spanned by $\sigma_{k,l}$. Together with our considerations in (1), we get (ii), proving the lemma. \square

Proof of Proposition 3.2. Let $x \in \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(S))$. Then x is, in particular, contained in the socle of $\text{Res}_{E'}^{\mathfrak{S}_n}(S)$. Hence, by Lemma 3.4, x lies in the F -span of $\mathfrak{B} \cup \mathfrak{C}$. Moreover, an element in $\text{Res}_{E'}^{\mathfrak{S}_n}(S)$ is contained in $\text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(S))$ if and only if it is fixed by $(1, 2)$. Therefore, by Proposition 3.1, x is in fact contained in the F -span of \mathfrak{B} . This proves the proposition. \square

Lemma 3.5. *Let $n \geq 6$ be even, and let $S := S^{(n-2, 2)}$. Then $\text{Res}_{E_n}^{\mathfrak{S}_n}(S)$ is indecomposable.*

Proof. Let $1 \neq \varphi \in \text{End}_{FE_n}(S)$ such that $\varphi^2 = \varphi$. We show that then $\varphi = 0$ so that $\text{End}_{FE_n}(S)$ is a local F -algebra and $\text{Res}_{E_n}^{\mathfrak{S}_n}(S)$ therefore indecomposable. It suffices to show that $\varphi(x) = 0$, for $x \in \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(S))$. For this then clearly forces $\varphi = 0$. By Proposition 3.2 we thus have to show

- (i) $\varphi(e(24)) = 0$,
- (ii) $\varphi(e(2, 2k-1) + e(2, 2k)) = 0$, for $k = 3, \dots, n/2$,
- (iii) $\varphi(e(2k-1, 2l-1) + e(2k-1, 2l) + e(2k, 2l-1) + e(2k, 2l)) = 0$, for $k = 2, \dots, (n-2)/2$ and $l = k+1, \dots, n/2$.

We repeatedly use the following equalities which hold by Proposition 3.1:

- (a) $(1 + (1, 2))e(3, j) = e(2, j)$ for $4 \leq j \leq n$, and $(1 + (1, 2))e(r, s) = e(2, r) + e(2, s)$ for $4 \leq r < s \leq n$.
- (b) Suppose that $3 \leq s \leq n/2$. Then $(2s-1, 2s)e(2, j) = e(2, j)$ for $2s-1 \neq j \neq 2s$.
- (c) For $3 \leq s \leq n/2$, we have $(1 + (3, 4))e(2, 2s-1) = e(2, 4) = (1 + (3, 4))e(2, 2s)$.
- (d) For $3 \leq l \leq n/2$, we have $(2l-1, 2l)e(2, 2l-1) = e(2, 2l)$.

Let $l \in \{3, \dots, n/2\}$, and consider $\varphi(e(2, 2l-1))$. Since $\varphi(e(2, 2l-1)) \in (1 + (1, 2))S \subseteq U$, by (a), we can write

$$(*) \quad \varphi(e(2, 2l-1)) = \sum_{j=4}^n \delta_j e(2, j),$$

for appropriate $\delta_j \in F$. Property (b) above shows that, for $r \notin \{1, 2, l\}$, the permutation $(2r-1, 2r)$ fixes $e(2, 2l-1)$. Therefore, $(2r-1, 2r)$ fixes $\varphi(e(2, 2l-1))$ which means that the coefficients of $e(2, 2r-1)$ and $e(2, 2r)$ in (*) must be equal so that

$$\delta_{2r-1} = \delta_{2r} \quad \text{for } r \notin \{1, 2, l\}. \quad (4)$$

Using this and applying $(1 + (3, 4))$ to (*), we deduce from (c) that

$$\varphi(e(2, 4)) = \left(\sum_{j=5}^n \delta_j \right) e(2, 4) = (\delta_{2l-1} + \delta_{2l}) e(2, 4). \quad (5)$$

This shows that $e(2, 4)$ is an eigenvector of φ . Since φ is a projection, the eigenvalue is either 1 or 0. We continue with the case where $\varphi(e(2, 4)) = 0$, and we will show that then $\varphi = 0$. This then also implies that $\varphi = 1$ in the case where $\varphi(e(2, 4)) = 1$ which, by our assumption,

is not possible.

So part (i) above holds. Furthermore, we have $\delta_{2l-1} = \delta_{2l}$, and hence

$$\delta_{2r-1} = \delta_{2r}, \quad (6)$$

for all $r = 3, \dots, n/2$. Using (a) and (b), into (*), this gives

$$\varphi(e(2, 2l-1)) = \varphi(e(2, 2l)).$$

This shows that (ii) holds true. Thus, in the notation of Remark 3.3, for $r = 3, \dots, n/2$ the socle element ρ_r lies in the kernel of φ . Consequently, since $\varphi = \varphi^2$ and $\varphi(e(2, 4)) = 0$, (*) and (6) yield

$$\varphi(e(2, 2l-1)) = 0 = \varphi(e(2, 2l)) \quad \text{and} \quad \delta_j = 0 \text{ for } j = 4, \dots, n. \quad (7)$$

In order to prove (iii), let $2 \leq k < l \leq n/2$, and note that $\sigma_{k,l} = \eta e(2k-1, 2l-1)$ where $\eta = (1 + (2k-1, 2k))(1 + (2l-1, 2l))$ and where $\sigma_{k,l}$ is as in Proposition 3.2. We have to show that $\eta\varphi(e(2k-1, 2l-1)) = 0$. For this we use the vector space decomposition of S given in Lemma 3.4. That is, we can write

$$\varphi(e(2k-1, 2l-1)) = \tau + u + \sum_{2 \leq s < t \leq n/2} v(s, t)$$

with $\tau \in T$, $u \in U$ and $v(s, t) \in V(s, t)$. We claim that $\eta\tau = 0$, $\eta u = 0$, and $\eta v(s, t) = 0$ for $(s, t) \neq (k, l)$. To see this, we use property (b). Namely, τ is a linear combination of the elements $e(2r-1, 2r)$ for $r = 2, \dots, n/2$. If $r \neq k$ then $(1 + (2k-1, 2k))e(2r-1, 2r) = 0$, and $(1 + (2l-1, 2l))e(2k-1, 2k) = 0$. Hence $\eta\tau = 0$. Similar arguments show that also $\eta u = 0$ and $\eta v(s, t) = 0$ for $(s, t) \neq (k, l)$. Thus $\eta\varphi(e(2k-1, 2l-1)) = \eta v(k, l)$. We write

$$\varphi(e(2k-1, 2l-1)) = \sum_{i,j} r(ij)e(ij),$$

for some $r(ij) \in F$. Then $\eta v(k, l) = \eta\varphi(e(2k-1, 2l-1)) = \lambda\sigma_{k,l}$ where

$$\lambda = r(2k-1, 2l-1) + r(2k-1, 2l) + r(2k, 2l-1) + r(2k, 2l).$$

By (a), we have $(1 + (1, 2))e(2k-1, 2l-1) = e(2, 2k-1) + e(2, 2l-1)$ if $k > 2$, and $(1 + (1, 2))e(2k-1, 2l-1) = e(2, 2l-1)$ if $k = 2$. So using (7) we get

$$0 = (1 + (1, 2))\varphi(e(2k-1, 2l-1)) = \sum_{j=4}^n \gamma_j e(2j).$$

Here, for $j = 4, \dots, n$, we have

$$0 = \gamma_j = \sum_{2 < i < j} r(ij) + \sum_{j < i} r(ji). \quad (8)$$

Let $t \notin \{1, k, l\}$, and apply $(2t-1, 2t)$ to $\varphi(e(2k-1, 2l-1))$. Then (b) implies:

$$\begin{aligned} r(2s-1, 2k-1) &= r(2s, 2k-1), & \text{for } 2 < 2s < 2k-1, \\ r(2k-1, 2s-1) &= r(2k-1, 2s), & \text{for } 2k-1 < 2s-1 \neq 2l-1. \end{aligned}$$

Substituting this into (8) with $j = 2k-1$, we get $0 = r(2k-1, 2k) + r(2k-1, 2l-1) + r(2k-1, 2l)$. Similarly, if we take $j = 2k$ then we get $0 = r(2k-1, 2k) + r(2k, 2l-1) + r(2k, 2l)$. Therefore, $\lambda = 0$. This completes the proof of (iii). \square

Remark 3.6. Notice that the above lemma also holds without any restrictions on the field F as long as it has characteristic 2.

3.2 Proof of Theorem 1.1

We will now prove Theorem 1.1. For this, let $n \geq 4$ be arbitrary again. The statement for $n = 4$ is well-known (cf. [20]). Thus, from now on, let $n \geq 5$. We recall from [13, Thm. 20.1] that

$$\dim(S) = \binom{n}{2} - n.$$

Hence if $n \equiv 1 \pmod{4}$ or if $n \equiv 2 \pmod{4}$ then $\dim(S) \not\equiv 0 \pmod{2}$ so that we have $P = P_n$ in these cases. In order to prove the assertion concerning the sources of S in the case where $n \equiv 1 \pmod{4}$, suppose that $n = 1 + \sum_{j=1}^s 2^{i_j}$ is the 2-adic expansion of n , and set $n_j := 2^{i_j}$, for $j = 1, \dots, s$. Denote the Young subgroup $\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_s}$ of \mathfrak{S}_n by H . Moreover, for $j = 1, \dots, s$, we denote the set of all 2-element subsets of the support of \mathfrak{S}_{n_j} by Ω_j , as in Remark 2.15. Then, by Lemma 2.14 and Lemma 2.16, we know that $U_{\bigoplus_{j=1}^s F\Omega_j} \mid \text{Res}_H^{\mathfrak{S}_n}(S)$. By Corollary 2.20, we know further that $\text{Res}_{P_n}^H(U_{\bigoplus_{j=1}^s F\Omega_j})$ is indecomposable. Since the dimension of $U_{\bigoplus_{j=1}^s F\Omega_j}$ is odd, $\text{Res}_{P_n}^H(U_{\bigoplus_{j=1}^s F\Omega_j})$ must have vertex P_n , and is thus a source of S .

If $n \equiv 3 \pmod{4}$ then $S = S^{(n-2,2)} \cong D^{(n-2,2)}$, by Carter's Criterion (cf. [14, Thm. 7.3.23]). Thus we then also have $S = S^{(n-2,2)} \cong Y^{(n-2,2)}$ with vertex $P_{n-4} \times P_2 \times P_2 = P_{n-5} \times P_2 \times P_2$ and trivial sources, by Grabmeier's Theorem [8, S. 7.8]. So it remains to settle the case where $4 < n \equiv 0 \pmod{4}$. Note that then always $\dim(S) \equiv 0 \pmod{2}$.

Take $n \equiv 0 \pmod{4}$ with 2-adic expansion $n = \sum_{j=1}^s 2^{i_j}$, for some $s \geq 1$ and some $i_1 > \dots > i_s \geq 2$. Furthermore, for $j = 1, \dots, s$ we set $n_j := 2^{i_j}$. By the modular branching rules for Specht modules, we obtain

$$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(S) \cong S^{(n-2,1)} \oplus S^{(n-3,2)}.$$

Since $n-1 \equiv 3 \pmod{4}$, we already know that $S^{(n-3,2)}$ has vertex $P_{n-6} \times P_2 \times P_2$ and trivial source. Moreover, $S^{(n-2,1)} \cong D^{(n-2,1)}$ has vertex P_{n-4} and trivial source, by [20]. By Lemma 3.5, we know further that $\text{Res}_{E_n}^{\mathfrak{S}_n}(S)$ is indecomposable, where E_n is the elementary abelian subgroup

$$E_n := \langle (2r-1, 2r) \mid r = 1, \dots, n/2 \rangle$$

of P_n . In particular, also $L := \text{Res}_{P_n}^{\mathfrak{S}_n}(S)$ is indecomposable. Now suppose that S has vertex $P < P_n$, and let R be a maximal subgroup of P_n containing P . Then P is a vertex of L as well, and the trivial sources of $S^{(n-3,2)}$ and $S^{(n-2,1)}$, respectively, are isomorphic to indecomposable direct summands of $\text{Res}_{P_{n-6} \times P_2 \times P_2}^{P_n}(L)$ and $\text{Res}_{P_{n-4}}^{P_n}(L)$, respectively. Hence

$$P_{n-6} \times P_2 \times P_2 \leq_{P_n} P \leq R < P_n \quad \text{and} \quad P_{n-4} \leq_{P_n} P \leq R < P_n.$$

Since R is normal in P_n , this in fact yields $P_{n-6} \times P_2 \times P_2 \leq R$ and $P_{n-4} \leq R$. Of course we also have $\Phi(P_{n_1}) \times \dots \times \Phi(P_{n_s}) = \Phi(P_n) \leq R$. Since

$$P_{n-4} = \begin{cases} P_{n_1} \times \dots \times P_{n_{s-1}} \times P_{n_s/2} \times \dots \times P_4, & \text{if } n_s > 4, \\ P_{n_1} \times \dots \times P_{n_{s-1}}, & \text{if } n_s = 4, \end{cases}$$

from Proposition 2.1 we deduce that

$$\begin{aligned} P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{n_s} &= P_{n-4} \Phi(P_n) = R, & \text{if } n_s > 4, \\ P_{n_1} \times \cdots \times P_{n_{s-1}} \times \bar{B}_{n_s} &\leq P_{n-4} \Phi(P_n) \leq R, & \text{if } n_s = 4. \end{aligned}$$

Again $B_{n_j} \cong P_{n_j/2} \times P_{n_j/2}$ is understood to be the base group of the wreath product $P_{n_j} = P_{n_j/2} \wr P_2$, and $\bar{B}_{n_j} := B_{n_j} \cap \mathfrak{A}_n$, for $j = 1, \dots, s$. But, since also $P_{n-6} \times P_2 \times P_2 \leq R$, we get

$$P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{n_s} = R$$

in any case. In particular, $E_n \leq R$ so that $\text{Res}_R^{\mathfrak{S}_n}(S)$ is indecomposable, by Lemma 3.5. But this contradicts Green's Indecomposability Theorem [9]. Therefore, we indeed have $P = P_n$, for $n \equiv 0 \pmod{4}$.

Finally, by Lemma 3.5, we also know that $\text{Res}_{P_n}^{\mathfrak{S}_n}(S)$ is a source of S if n is even. This proves the theorem.

4 The simple module $D^{(n-2,2)}$

The aim of this section is to give a proof of Theorem 1.2. Therefore, from now on, let $n \geq 5$. As before, we denote the Specht $F\mathfrak{S}_n$ -module $S^{(n-2,2)}$ by S and the corresponding simple $F\mathfrak{S}_n$ -module $D^{(n-2,2)} = S/\text{Rad}(S)$ by D . In analogy to the proof of Theorem 1.1, we will also distinguish between the case when n is odd and the case when n is even.

4.1 Outline of the proof of Theorem 1.2

(1) Suppose that $n \geq 5$ is odd. If $n \equiv 3 \pmod{4}$ then $S \cong D$ with vertex $P_{n-5} \times P_2 \times P_2$ and trivial source, as we have already proved in Theorem 1.1. If $n = 5$ then D is the spin $F\mathfrak{S}_5$ -module $D^{(3,2)}$ whose vertices and sources have been determined in [4]. Namely, the Sylow 2-subgroup Q_4 of \mathfrak{A}_4 is then a vertex of D , and the restrictions of the simple $F\mathfrak{A}_5$ -modules $E_+^{(3,2)}$ and $E_-^{(3,2)}$ to Q_4 are sources of D .

(2) Therefore, we may now suppose that $n \equiv 1 \pmod{4}$ and that $n > 5$. If $n = 1 + \sum_{j=1}^s 2^{i_j}$, with appropriate $s \geq 1$ and $i_1 > \dots > i_s \geq 2$, is the 2-adic expansion of n then we set $n_j := 2^{i_j}$, for $j = 1, \dots, s$. That is, $P_n = P_{n_1} \times \cdots \times P_{n_s}$. From [15, Thm. 11.2.10] we deduce that $D^{(n-3,1)} \mid \text{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D)$. Moreover $D^{(n-3,1)} \cong S^{(n-3,1)} \cong Y^{(n-3,1)}$ with vertex P_{n-5} and trivial source, by [20]. Consequently, there is an indecomposable direct summand X of $\text{Res}_{P_n}^{\mathfrak{S}_n}(D)$ such that $F \mid \text{Res}_{P_{n-5}}^{P_n}(X)$. Hence $P_{n-5} \leq_{P_n} Q$, for any vertex $Q \leq P_n$ of X . Now assume that D has vertex $P < P_n$. Since $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D)$ is, by [15, Thm. 11.2.7], indecomposable, we may suppose that P is also a vertex of $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D)$. In particular, $P_{n-5} \leq_{P_n} Q \leq_{\mathfrak{S}_{n-1}} P < P_n$. If $n_s > 4$ then

$$P_{n-5} = P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_{2^{i_{s-1}}} \times \cdots \times P_4,$$

and if $n_s = 4$ then

$$P_{n-5} = P_{n_1} \times \cdots \times P_{n_{s-1}}.$$

This shows that, in fact, $P_{n-5} \leq_{P_n} P < P_n$. Furthermore, there is a maximal subgroup R of P_n such that $P_{n-5} \leq_{P_n} P \leq R < P_n$. Since R is normal in P_n , we have $P_{n-5} \leq R$. Of course

we also have $\Phi(P_{n_1}) \times \cdots \times \Phi(P_{n_s}) = \Phi(P_n) \leq R$. Therefore, from Proposition 2.1 we obtain

$$\begin{aligned} R &= P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{n_s}, \text{ if } n_s > 4, \\ R &> P_{n_1} \times \cdots \times P_{n_{s-1}} \times \overline{B}_{n_s}, \text{ if } n_s = 4. \end{aligned}$$

Recall that $B_{n_s} \cong P_{2^{i_s-1}} \times P_{2^{i_s-1}}$ denotes the base group of the wreath product P_{n_s} and that $\overline{B}_{n_s} = B_{n_s} \cap \mathfrak{A}_n$. If $n_s > 4$ then

$$R = P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{n_s} \leq \mathfrak{S}_{n-1-n_s} \times \mathfrak{S}_{\frac{n_s}{2}} \times \mathfrak{S}_{\frac{n_s}{2}} =: H,$$

so that D has to be relatively H -projective. If $n_s = 4$ then $T := P_{n_1} \times \cdots \times P_{n_{s-1}} \times \overline{B}_{n_s} \triangleleft P_n$ such that P_n/T is elementary abelian of order $4 = 2^2$. Hence there are precisely $2^2 - 1 = 3$ maximal subgroups of P_n containing T , namely $R_1 := P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{n_s}$, $R_2 := P_{n_1} \times \cdots \times P_{n_{s-1}} \times Q_{n_s}$ and $R_3 := P_{n_1} \times \cdots \times P_{n_{s-1}} \times \langle (n-4, n-2, n-3, n-1) \rangle$. Consequently, in order to prove the assertion of Theorem 1.2 in the case when $n \equiv 1 \pmod{4}$, it remains to show that D is not relatively projective with respect to any of the maximal subgroups R_1, R_2, R_3 of P_n if $n_s = 4$, and that D is not relatively H -projective if $n_s > 4$.

(3) We now suppose that the assertion of Theorem 1.2 has already been proved for all odd $n \geq 5$. Then we may suppose further that $n \geq 6$ is even with 2-adic expansion $n = \sum_{j=1}^s 2^{i_j}$, for appropriate $s \geq 1$ and $i_1 > \dots > i_s \geq 1$. If $n = 6$ then D is the spin $F\mathfrak{S}_6$ -module $D^{(4,2)}$ which has vertex P_6 and source $\text{Res}_{P_6}^{\mathfrak{S}_6}(D^{(4,2)})$, by [4]. Let now $n > 6$. We claim that it suffices to prove that D restricts indecomposably to the elementary abelian subgroup

$$E_n := \langle (2r-1, 2r) \mid r = 1, \dots, n/2 \rangle$$

of \mathfrak{S}_n . For if we know this then $\text{Res}_{P_n}^{\mathfrak{S}_n}(D)$ is also indecomposable. Therefore, the vertex $P \leq P_n$ of D is also a vertex of the module $\text{Res}_{P_n}^{\mathfrak{S}_n}(D)$. By [15, Thm. 11.2.7], we have $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D) \cong D^{(n-3,2)}$. Since $n-1$ is odd, we already know that $D^{(n-3,2)}$ has vertex $P_{n-6} \times P_2 \times P_2$ in case that $n \equiv 0 \pmod{4}$, and that it has vertex $P_{n-2} = P_{n_1} \times \cdots \times P_{n_{s-1}}$ in case that $n \equiv 2 \pmod{4}$. Hence

$$P_{n_1} \times \cdots \times P_{n_{s-1}} \leq_{P_n} P \leq P_n = P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_2$$

if $n \equiv 2 \pmod{4}$,

$$P_{n_1} \times \cdots \times P_{n_{s-2}} \times P_{2^{i_{s-1}-1}} \times \cdots \times P_4 \times P_2 \times P_2 \times P_2 = P_{n-6} \times P_2 \times P_2 \leq_{P_n} P \leq P_n$$

if $n \equiv 0 \pmod{4}$ and $n_s = 4$, and

$$P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_{2^{i_{s-1}}} \times \cdots \times P_8 \times P_2 \times P_2 \times P_2 = P_{n-6} \times P_2 \times P_2 \leq_{P_n} P \leq P_n$$

if $n \equiv 0 \pmod{4}$ and $n_s > 4$. As a direct consequence of Knörr's Theorem [16] we therefore get

$$P_{n_1} \times \cdots \times P_{n_{s-2}} \times P_{2^{i_{s-1}-1}} \times \cdots \times P_4 \times P_2 \times P_2 \times P_2 \times P_2 \leq_{P_n} P \leq P_n$$

if $n \equiv 0 \pmod{4}$ and $n_s = 4$,

$$P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_{2^{i_{s-1}}} \times \cdots \times P_8 \times P_2 \times P_2 \times P_2 \times P_2 \leq_{P_n} P \leq P_n$$

if $n \equiv 0 \pmod{4}$ and $n_s > 4$, and $P = P_n$ if $n \equiv 2 \pmod{4}$. In any case, P contains a P_n -conjugate of the elementary abelian group $E_n = \langle (2r-1, 2r) \mid r = 1, \dots, n/2 \rangle$. If P were a proper subgroup of P_n then we would have a maximal subgroup R of P_n such that $E_n \leq_{P_n} P \leq R < P_n$, and $\text{Res}_R^{\mathfrak{S}_n}(D)$ would have to be indecomposable. But this is a contradiction to Green's Indecomposability Theorem [9]. Hence $P = P_n$ when n is even. Moreover, $\text{Res}_{P_n}^{\mathfrak{S}_n}(D)$ then also has to be a source of D .

(4) This shows that it now remains to prove the following:

- (a) If $5 < n \equiv 1 \pmod{4}$ with 2-adic expansion $n = 1 + \sum_{j=1}^s 2^{i_j}$, for some $s \geq 1$ and some $i_1 > \dots > i_s = 2$ then D is not relatively R_i -projective, for $i = 1, 2, 3$. Again R_1, R_2 and R_3 denote the maximal subgroups of P_n appearing in paragraph (2) above.
- (b) If $5 < n \equiv 1 \pmod{4}$ with 2-adic expansion $n = 1 + \sum_{j=1}^s 2^{i_j}$, for some $s \geq 1$ and some $i_1 > \dots > i_s > 2$ then D is not relatively H -projective. Here $H := \mathfrak{S}_{n-1-2^{i_s}} \times \mathfrak{S}_{2^{i_s-1}} \times \mathfrak{S}_{2^{i_s-1}}$.
- (c) If n is even then D restricts indecomposably to the elementary abelian subgroup $E_n = \langle (2r-1, 2r) \mid r = 1, \dots, n/2 \rangle$ of \mathfrak{S}_n .

In the next subsections we will prove (a), (b) and (c).

4.2 The case when n is odd

In this subsection let $n > 5$ such that $n \equiv 1 \pmod{4}$. Moreover, let $n = 1 + \sum_{j=1}^s 2^{i_j}$ be the 2-adic expansion of n , for appropriate $s \geq 1$ and $i_1 > \dots > i_s \geq 2$. Again we set $n_j := 2^{i_j}$, for $j = 1, \dots, s$. We will prove parts (4)(a) and (4)(b) from the previous subsection which will then complete the proof of Theorem 1.2 in the case where n is odd. In order to show (4)(b) where $n_s > 4$, we will make use of Proposition 2.13 and Lemma 2.16 which determine the restrictions of D to Young subgroups of the form

$$\mathfrak{S}_{a_1} \times \dots \times \mathfrak{S}_{a_r},$$

where $n-1 = \sum_{i=1}^r a_i$ and $a_1 \equiv \dots \equiv a_r \equiv 0 \pmod{4}$. Furthermore, if $n_s = 4$ then we will determine the restriction of D to

$$\mathfrak{S}_a \times \mathfrak{S}_2 \times \mathfrak{S}_2$$

where $a+4 = n-1$. From this we will then also derive (4)(a).

Firstly we have:

Proposition 4.1. *Let $n = 1 + \sum_{j=1}^s 2^{i_j}$, for some $s \geq 1$ and some $i_1 > \dots > i_s > 2$. Then the simple $F\mathfrak{S}_n$ -module D is not relatively projective with respect to the Young subgroup $\mathfrak{S}_{n-1-2^{i_s}} \times \mathfrak{S}_{2^{i_s-1}} \times \mathfrak{S}_{2^{i_s-1}}$.*

Proof. Let $H := \mathfrak{S}_{n-1-2^{i_s}} \times \mathfrak{S}_{2^{i_s-1}} \times \mathfrak{S}_{2^{i_s-1}}$, let $a_1 := n-1-2^{i_s}$, and let $a_2 := a_3 := 2^{i_s-1}$. If $s = 1$ then, by Lemma 2.16 and Proposition 2.17, $\text{Res}_H^{\mathfrak{S}_n}(D)$ splits into the direct sum of two indecomposable modules of dimension $2a_1^2$ and $2a_1^2 - 2a_1 - 2 \neq 2a_1^2$, respectively. If $s \geq 2$ then, by Lemma 2.16 and Proposition 2.17, $\text{Res}_H^{\mathfrak{S}_n}(D)$ splits into the direct sum of two indecomposable modules of dimension $a_1 a_2$ each, an indecomposable module of dimension a_2^2 ,

and an indecomposable module of dimension $(a_1^2 - a_1 + 2a_2^2 - 2a_2 - 4)/2$.

Set $K := \mathfrak{S}_{n-1-2^is} \times (\mathfrak{S}_{2^{i_s-1}} \wr \mathfrak{S}_2)$. Then $|\mathfrak{S}_n : K|$ is odd, so that D is clearly relatively K -projective. Assume that D is also relatively H -projective. Then there is an indecomposable direct summand X of $\text{Res}_K^{\mathfrak{S}_n}(D)$ which is relatively H -projective and has common vertices with D . Since $|K : H| = 2$, the restriction of this summand X to H in turn has to be the direct sum of two indecomposable and in K conjugate FH -modules, by [12, Thm. VII.9.3], each of which has common vertices with X and thus with D . This immediately leads to a contradiction in the case where $s = 1$. If $s \geq 2$ then we can only have

$$\text{Res}_H^K(X) \cong (M^{(a_1-1,1)} \boxtimes M^{(a_2-1,1)} \boxtimes F) \oplus (M^{(a_1-1,1)} \boxtimes F \boxtimes M^{(a_2-1,1)}).$$

But, by Grabmeier's Theorem [8], [7], the FH -module $M^{(a_1-1,1)} \boxtimes M^{(a_2-1,1)} \boxtimes F \cong Y^{(a_1-1,1)} \boxtimes Y^{(a_2-1,1)} \boxtimes F$ has vertex $P_{a_1-2} \times P_{a_2-2} \times P_{a_2} \leq_{\mathfrak{S}_n} \mathfrak{S}_{n-2}$. By Knörr's Theorem [16], the module D cannot be relatively \mathfrak{S}_{n-2} -projective, and we have a contradiction also in this case. This finishes the proof of the proposition. \square

Remark 4.2. Finally, we prove part (4)(a) of Section 4.1. That is, we suppose that $n > 5$ with 2-adic expansion $n = 1 + \sum_{j=1}^s 2^{i_j}$, for some $s \geq 1$ and some $i_1 > \dots > i_s = 2$. We further set $n_j := 2^{i_j}$, for $j = 1, \dots, s$, and show that D is not relatively projective with respect to any of the following maximal subgroups of P_n :

$$\begin{aligned} R_1 &:= P_{n_1} \times \dots \times P_{n_{s-1}} \times B_{n_s}, & R_2 &:= P_{n_1} \times \dots \times P_{n_{s-1}} \times Q_{n_s}, \\ R_3 &:= P_{n_1} \times \dots \times P_{n_{s-1}} \times \langle (n-4, n-2, n-3, n-1) \rangle. \end{aligned}$$

As in Section 2.3, we denote the set of all 2-element subsets of $\{1, \dots, n-1\}$ by Ω , and we define

$$\begin{aligned} \Omega_1 &:= \{\{i, j\} \mid 1 \leq i < j \leq a\}, & \Omega_2 &:= \{\{a+1, a+2\}\}, & \Omega_3 &:= \{\{a+3, a+4\}\}, \\ \Delta_{12} &:= \{\{i, a+1\}, \{i, a+2\} \mid 1 \leq i \leq a\}, & \Delta_{13} &:= \{\{i, a+3\}, \{i, a+4\} \mid 1 \leq i \leq a\} \\ \Delta_{23} &:= \{\{i, j\} \mid a+1 \leq i \leq a+2 < a+3 \leq j \leq a+4\}. \end{aligned}$$

So $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Delta_{12} \cup \Delta_{13} \cup \Delta_{23}$, and $F\Omega \cong M^{(N-2,2)}$. In analogy to Lemma 2.16, we have the following:

Lemma 4.3. *Suppose that $5 < n \equiv 1 \pmod{4}$, and let $N := n - 1$. Let further $a := N - 4$ and $H := \mathfrak{S}_a \times \mathfrak{S}_2 \times \mathfrak{S}_2$. Then*

- (i) $\text{Res}_H^{\mathfrak{S}_N}(M^{(N-2,2)}) \cong \text{Res}_H^{\mathfrak{S}_N}(F\Omega) = F\Omega_1 \oplus F\Omega_2 \oplus F\Omega_3 \oplus F\Delta_{12} \oplus F\Delta_{13} \oplus F\Delta_{23}$ is an indecomposable direct sum decomposition, and
- (ii) $\text{Res}_H^{\mathfrak{S}_n}(D) \cong F\Omega_1 \oplus F\Delta_{12} \oplus F\Delta_{13} \oplus F\Delta_{23}$ is an indecomposable direct sum decomposition.

Proof. Note that, in fact, $\Omega_1, \Omega_2, \Omega_3, \Delta_{12}, \Delta_{13}$ and Δ_{23} are disjoint H -sets. So we have a decomposition of FH -modules

$$F\Omega = F\Omega_1 \oplus F\Omega_2 \oplus F\Omega_3 \oplus F\Delta_{12} \oplus F\Delta_{13} \oplus F\Delta_{23}. \quad (9)$$

Furthermore,

$$\begin{aligned} F\Omega_1 &\cong M^{(a-2,2)} \boxtimes F \boxtimes F, & F\Omega_2 &\cong F \cong F\Omega_3, & F\Delta_{23} &\cong F \boxtimes M^{(1^2)} \boxtimes M^{(1^2)}, \\ F\Delta_{12} &\cong M^{(a-1,1)} \boxtimes M^{(1^2)} \boxtimes F, & F\Delta_{13} &\cong M^{(a-1,1)} \boxtimes F \boxtimes M^{(1^2)}. \end{aligned}$$

Thus, since $a \equiv 0 \pmod{4}$, (9) is, by [10], a decomposition of $\text{Res}_H^{\mathfrak{S}_N}(M^{(N-2,2)})$ into indecomposable direct summands, proving (i).

By Lemma 2.14, $\text{Res}_{\mathfrak{S}_N}^{\mathfrak{S}_n}(D) \cong U_{F\Omega}/V_{F\Omega}$. For each $\{i, j\} \in \Omega \setminus \{\{N-1, N\}\}$, we define

$$\xi(i, j) := \{i, j\} + \{N-1, N\},$$

and we set $\Xi := \{\bar{\xi}(i, j) \mid (i, j) \neq (a+1, a+2)\}$. Here $\bar{\cdot} : U_{F\Omega} \rightarrow U_{F\Omega}/V_{F\Omega}$ denotes the canonical epimorphism. Then, as mentioned in Remark 2.5, Ξ is an F -basis of $U_{F\Omega}/V_{F\Omega}$. With this notation, we obtain the following H -sets:

$$\Omega'_1 := \{\bar{\xi}(i, j) \mid 1 \leq i < j \leq a\},$$

$$\Delta'_{12} := \{\bar{\xi}(i, a+1), \bar{\xi}(i, a+2) \mid 1 \leq i \leq a\},$$

$$\Delta'_{13} := \{\bar{\xi}(i, a+3), \bar{\xi}(i, a+4) \mid 1 \leq i \leq a\},$$

$$\Delta'_{23} := \{\bar{\xi}(a+1, a+3), \bar{\xi}(a+1, a+4), \bar{\xi}(a+2, a+3), \bar{\xi}(a+2, a+4)\},$$

and a decomposition $U_{F\Omega}/V_{F\Omega} = F\Omega'_1 \oplus F\Delta'_{12} \oplus F\Delta'_{13} \oplus F\Delta'_{23}$ as FH -modules. The F -linear map

$$U_{F\Omega}/V_{F\Omega} \rightarrow F\Omega, \quad \bar{\xi}(i, j) \mapsto \{i, j\}$$

induces FH -isomorphisms $F\Omega'_1 \cong F\Omega_1$, $F\Delta'_{12} \cong F\Delta_{12}$, $F\Delta'_{13} \cong F\Delta_{13}$, and $F\Delta'_{23} \cong F\Delta_{23}$. This proves (ii), and the assertion of the lemma follows. \square

Corollary 4.4. *With the notation of Remark 4.2, the simple $F\mathfrak{S}_n$ -module D is not relatively R_i -projective, for $i = 1, 2, 3$.*

Proof. We set $N := n-1$, $a := n-1-4 = n_1 + \dots + n_{s-1}$ and $H := \mathfrak{S}_a \times \mathfrak{S}_2 \times \mathfrak{S}_2 \leq \mathfrak{S}_N$. Then $R_1 \leq H \leq \mathfrak{S}_a \times (\mathfrak{S}_2 \wr \mathfrak{S}_2) =: K$. Assume first that D is relatively R_1 -projective. Since $|\mathfrak{S}_n : K|$ is odd, D is relatively K -projective. Hence there is some indecomposable direct summand X of $\text{Res}_K^{\mathfrak{S}_n}(D)$ which has common vertices with D and is also relatively H -projective. Since $|K : H| = 2$, from [12, Thm. VII.9.3] we deduce that $\text{Res}_H^K(X)$ has to be the direct sum of two conjugate indecomposable modules both of which have common vertices with X and thus with D . By Lemma 4.3, we know that

$$(*) \quad \text{Res}_H^{\mathfrak{S}_n}(D) \cong F\Omega_1 \oplus F\Delta_{12} \oplus F\Delta_{13} \oplus F\Delta_{23}$$

is an indecomposable direct sum decomposition, and $\dim(F\Omega_1) = \binom{a}{2}$, $\dim(F\Delta_{23}) = 4$ and $\dim(F\Delta_{12}) = 2a = \dim(F\Delta_{13})$. This forces $\text{Res}_H^K(X) \cong F\Delta_{12} \oplus F\Delta_{13}$. But $F\Delta_{12} \cong M^{(a-1,1)} \boxtimes M^{(1^2)} \boxtimes F$ and has thus vertex $P_{a-2} \times 1 \times P_2 \leq_{\mathfrak{S}_n} \mathfrak{S}_{n-4}$, by Grabmeier's Theorem [8], [7]. On the other hand, in consequence of Knörr's Theorem [16], D cannot be relatively \mathfrak{S}_{n-2} -projective, a contradiction. Thus D is not relatively H -projective and then not relatively R_1 -projective either.

Furthermore, the decomposition $(*)$ above also shows that $F\Omega_1 \mid \text{Res}_H^{\mathfrak{S}_n}(D)$. Since $a \equiv 0 \pmod{4}$, we have $F\Omega_1 \cong M^{(a-2,2)} \boxtimes F \boxtimes F \cong Y^{(a-2,2)} \boxtimes F \boxtimes F$, by [10], and so $F\Omega_1$ has vertex $P_{a-4} \times P_2 \times P_2 \times P_2 \times P_2$, by Grabmeier's Theorem [8], [7]. Hence if $P \leq P_n$ is a vertex of D then also

$$E_{n-1} := \langle (2r-1, 2r) \mid r = 1, \dots, (n-1)/2 \rangle \leq P_{a-4} \times P_2 \times P_2 \times P_2 \times P_2 \leq_{\mathfrak{S}_n} P \leq P_n.$$

But neither R_2 nor R_3 contains a subgroup conjugate to E_{n-1} . Consequently, D is not relatively R_i -projective, for $i = 2, 3$, and the corollary is proved. \square

Corollary 4.4 completes the proof of Theorem 1.2 in the case where n is odd.

4.3 The case when n is even

Throughout this subsection, let $n \geq 6$ be even. In order to complete our proof of Theorem 1.2, we need to show that $\text{Res}_{E_n}^{\mathfrak{S}_n}(D)$ is indecomposable where $E_n := \langle (2r-1, 2r) \mid r = 1, \dots, n/2 \rangle$. We begin our investigation of $D = S/\text{Rad}(S)$ and its restriction to E_n by giving a convenient F -basis of D . This will be done in the following two propositions.

Proposition 4.5. *Let $n \geq 6$ be even. For $a \in \{1, \dots, n\}$, we define $\gamma_a := \sum_{i \neq a \neq j} \{i, j\} \in M^{(n-2,2)}$.*

(i) *If $n \equiv 2 \pmod{4}$ then $\text{Rad}(S) = \{\sum_{a=1}^n c_a \gamma_a \mid c_a \in F\}$.*

(ii) *If $n \equiv 0 \pmod{4}$ then $\text{Rad}(S) = \{\sum_{a=1}^n c_a \gamma_a \mid c_a \in F, \sum_{a=1}^n c_a = 0\}$.*

Proof. Suppose first that $n \equiv 2 \pmod{4}$. From [10], we deduce that $M^{(n-2,2)} \cong F \oplus Y^{(n-2,2)}$. Moreover, by [19], the Young module $Y^{(n-2,2)}$ is uniserial with composition series

$$Y^{(n-2,2)} = \mathcal{U}(D^{(n-1,1)}, F, D^{(n-2,2)}, F, D^{(n-1,1)}). \quad (10)$$

In particular, $Y^{(n-2,2)}$ is isomorphic to the kernel of the augmentation map $\nu : M^{(n-2,2)} \rightarrow F$, mapping each tabloid to 1. We now define

$$\rho : M^{(n-1,1)} \rightarrow M^{(n-2,2)}, \{a\} \mapsto \sum_{i \neq a \neq j} \{i, j\} = \gamma_a,$$

for $a \in \{1, \dots, n\}$. This is a non-zero homomorphism of $F\mathfrak{S}_n$ -modules. For each $a \in \{1, \dots, n\}$, the sum $\sum_{i \neq a \neq j} \{i, j\}$ has $\binom{n-1}{2} \equiv 0 \pmod{2}$ terms so that $\text{im}(\rho) \subseteq \ker(\nu) = Y^{(n-2,2)}$. It remains to show that, in fact, $\text{im}(\rho) = \text{Rad}(S)$. Since n is even, the permutation module $M^{(n-1,1)}$ is uniserial, with composition series $M^{(n-1,1)} \supset S^{(n-1,1)} \supset F \supset 0$, by [13, Ex. 5.1]. Thus either $\ker(\rho) \cong F$ or $\ker(\rho) \cong S^{(n-1,1)}$. In the latter case, we would have $\rho(\{a\} + \{b\}) = 0$, for all $a, b \in \{1, \dots, n\}$ which is obviously not the case. Therefore, $\ker(\rho) \cong F$, so that $\text{im}(\rho)$ has composition factors $D^{(n-1,1)}$ and F . More precisely, $\text{im}(\rho)$ is indecomposable of composition length 2, with head isomorphic to F and socle isomorphic to $D^{(n-1,1)}$. Since $Y^{(n-2,2)}$ is uniserial, this forces $\text{im}(\rho) = \text{Rad}(S)$, and (i) follows.

Next, let $n \equiv 0 \pmod{4}$. Then from [10] we get $M^{(n-2,2)} = Y^{(n-2,2)}$. As above, we again consider the augmentation map $\nu : M^{(n-2,2)} \rightarrow F$ and the homomorphism $\rho : M^{(n-1,1)} \rightarrow M^{(n-2,2)}$, $\{a\} \mapsto \gamma_a := \sum_{i \neq a \neq j} \{i, j\}$. Since each $(n-2, 2)$ -polytabloid is the sum of four $(n-2, 2)$ -tabloids, we have $S \subseteq \ker(\nu)$. As in the previous case, also here we deduce that $\text{im}(\rho)$ is indecomposable of composition length 2, with socle isomorphic to $D^{(n-1,1)}$ and head isomorphic to F . Furthermore, the definitions of ρ and ν immediately yield $0 \neq \text{im}(\rho) \cap \ker(\nu) \neq \text{im}(\rho)$. Hence $\text{im}(\rho) \cap \ker(\nu) \cong D^{(n-1,1)}$. By [19], $M^{(n-2,2)}$ has a unique submodule isomorphic to $D^{(n-1,1)}$, namely $\text{Rad}(S)$. Thus it now remains to prove

$$\ker(\nu) \cap \text{im}(\rho) = \left\{ \sum_{a=1}^n c_a \gamma_a \mid c_a \in F, \sum_{a=1}^n c_a = 0 \right\}. \quad (11)$$

For this let $x \in \ker(\nu) \cap \text{im}(\rho)$, that is $x = \sum_{a=1}^n c_a \gamma_a$ where $c_a \in F$ for $a = 1, \dots, n$, and $\sum_{a=1}^n c_a \nu(\gamma_a) = 0$. Since, for each $a = 1, \dots, n$, the sum γ_a has $\binom{n-1}{2} \equiv 1 \pmod{2}$ terms, we get $0 = \sum_{a=1}^n c_a \nu(\gamma_a) = \sum_{a=1}^n c_a$. Conversely, the set on the right hand side of (11) is of course contained in $\ker(\nu) \cap \text{im}(\rho)$. This completes the proof of (ii). \square

Proposition 4.6. *Let $n \geq 6$ be even, and let $\bar{\cdot} : S \rightarrow S/\text{Rad}(S) = D$ be the canonical epimorphism. For $j \in \{4, \dots, n\}$ and $i \in \{2, \dots, j-1\}$, we set $e(ij) =: f(ij)$.*

(i) *If $n \equiv 0 \pmod{4}$ then*

$$\mathfrak{B}_n := \{f(ij) \mid 4 \leq j \leq n-1, 2 \leq i \leq j-1\}$$

is an F -basis of D .

(ii) *If $n \equiv 2 \pmod{4}$ then*

$$\mathfrak{B}_n := \{f(ij) \mid 4 \leq j \leq n-1, 2 \leq i \leq \min\{j-1, n-3\}\}$$

is an F -basis of D .

Proof. Let $n \equiv 0 \pmod{4}$. Then $\dim(D) = \dim(S) - \dim(D^{(n-1,1)}) = \binom{n}{2} - n - (n-2) = |\mathfrak{B}_n|$. It thus suffices to prove that \mathfrak{B}_n is F -linearly independent. For this, suppose that

$$0 = \sum_{i,j} r(ij)f(ij)$$

where $r(ij) \in F$ for $j = 4, \dots, n-1, i = 2, \dots, j-1$. By Proposition 4.5, there are $r_1, \dots, r_n \in F$ such that

$$\sum_{i,j} r(ij)e(ij) = \sum_{a=1}^n r_a \gamma_a \in \text{Rad}(S), \quad (12)$$

that is $\sum_{a=1}^n r_a = 0$. We show that this implies $r(ij) = 0$, for all admissible i and j . For this, let $l \in \{2, \dots, n\}$ and $k \in \{1, \dots, l-1\}$. We compare the coefficients of the tabloid $\{k, l\}$ in both sums of (12). Consider the cases where $j \in \{n-1, n\}$ first. Then we have

tabloid	LHS	RHS
$\{k, n\}$ for $1 \leq k \leq n-1$	0	$\sum_{k \neq a \neq n} r_a$
$\{1, n-1\}$	$\sum_{i=2}^{n-2} r(i, n-1)$	$\sum_{1 \neq a \neq n-1} r_a$
$\{k, n-1\}$ for $2 \leq k \leq n-2$	$r(k, n-1)$	$\sum_{k \neq a \neq n-1} r_a$

Since $\sum_{a=1}^n r_a = 0$, this implies $r := r_1 = \dots = r_n$, and $r(k, n-1) = (n-2)r = 0$ for $k = 2, \dots, n-2$. We may now suppose that $4 \leq l < n-1$ and argue with reverse induction on l , in order to show that $r(k, l) = 0$, for all $k = 2, \dots, l-1$. Again we compare the coefficients of the tabloids on both sides of (12). This yields:

tabloid	LHS	RHS
$\{k, l\}$ for $3 \leq k \leq l-1$	$r(k, l)$	$(n-2)r = 0$
$\{2, l\}$	$r(2, l) + \sum_{j=l+1}^{n-1} r(l, j)$	$(n-2)r = 0$
$\{1, l\}$	$\sum_{i=2}^{l-1} r(i, l)$	$(n-2)r = 0$

Since, by induction, $r(l, l+1) = \dots = r(l, n-1) = 0$, we then also get $r(k, l) = 0$, for $k = 2, \dots, l-1$. This proves (i).

Now let $n \equiv 2 \pmod{4}$. Then $\dim(D) = \dim(S) - \dim(D^{(n-1,1)}) - 1 = \binom{n}{2} - n - (n-2) - 1 = |\mathfrak{B}_n|$ so that also in this case we only need to show that \mathfrak{B}_n is F -linearly independent. For this, let $0 = \sum_{i,j} r(ij)f(ij)$ where $r(ij) \in F$ for $j = 4, \dots, n-1$ and $k = 2, \dots, \min\{n-3, j-1\}$. Hence, by Proposition 4.5, there are $r_1, \dots, r_n \in F$ such that

$$\sum_{i,j} r(ij)f(ij) = \sum_{a=1}^n r_a \gamma_a \in \text{Rad}(S). \quad (13)$$

As above, for $l = 2, \dots, n$ and $k = 1, \dots, l-1$, we compare the coefficients of the tabloid $\{k, l\}$ on both sides of (13), in order to show $r(ij) = 0$, for $j = 4, \dots, n-1$ and $i = 2, \dots, \min\{n-3, j-1\}$. Again we argue with reverse induction on l . If $l \geq n-2$ then we have

tabloid	LHS	RHS
$\{k, n\}$ for $1 \leq k \leq n-1$	0	$\sum_{k \neq a \neq n} r_a$
$\{1, n-1\}$	$\sum_{i=2}^{n-3} r(i, n-1)$	$\sum_{1 \neq a \neq n-1} r_a$
$\{k, n-1\}$ for $2 \leq k \leq n-3$	$r(k, n-1)$	$\sum_{k \neq a \neq n-1} r_a$
$\{n-2, n-1\}$	0	$\sum_{n-2 \neq a \neq n-1} r_a$
$\{1, n-2\}$	$\sum_{i=2}^{n-3} r(i, n-2)$	$\sum_{1 \neq a \neq n-2} r_a$
$\{k, n-2\}$ for $2 \leq k \leq n-3$	$r(k, n-2)$	$\sum_{k \neq a \neq n-2} r_a$

This implies $r := r_1 = r_2 = \dots = r_n$ and $r(k, n-2) = r(k, n-1) = 0$, for $k = 2, \dots, n-3$. We may now suppose that $4 \leq l < n-2$, and that we have already proved $r(k, j) = 0$, for $j = l+1, \dots, n-1$ and $k = 2, \dots, \min\{j-1, n-3\}$. Comparing coefficients on both sides of (13) we obtain:

tabloid	LHS	RHS
$\{k, l\}$ for $3 \leq k \leq l-1$	$r(k, l)$	0
$\{2, l\}$	$r(2, l) + \sum_{j=l+1}^{n-1} r(l, j)$	0
$\{1, l\}$	$\sum_{i=2}^{l-1} r(i, l)$	0

Therefore, we also get $0 = r(2, l) = r(3, l) = \dots = r(l-1, l)$, and assertion (ii) follows. \square

Using the F -basis \mathfrak{B}_n of D given in Proposition 4.6, we can describe the action of the elementary abelian group E_n on D .

Proposition 4.7. *Let $n \geq 6$ be even, and let $E_n := \langle (2r-1, 2r) \mid r = 1, \dots, n/2 \rangle$. If $n \equiv 0 \pmod{4}$ then let $i \in \{2, \dots, n-2\}$, and if $n \equiv 2 \pmod{4}$ then let $i \in \{2, \dots, n-3\}$. Furthermore, set $\Sigma'_i := \sum_{l=i+1}^{n-1} f(il) + \sum_{l=3}^{i-1} f(li)$.*

(i) If $n \equiv 2 \pmod{4}$ and $r = (n-2)/2$ then

$$(2r-1, 2r)f(n-3, n-1) = \sum_{\substack{j=4 \\ j \text{ odd}}}^{n-1} f(2j) + \sum_{i=3}^{j-1} \sum_{j=4}^{n-2} f(ij) + \sum_{i=3}^{n-3} f(i, n-1).$$

(ii) If $r = n/2$ then

$$(2r-1, 2r)f(i, n-1) = \begin{cases} \sum_{l=4}^{i-1} f(2l) + \Sigma'_i, & \text{if } i \text{ is odd,} \\ \sum_{l=4}^i f(2l) + \Sigma'_i, & \text{if } i \text{ is even.} \end{cases}$$

(iii) Otherwise, suppose $(2r-1, 2r)e(ij) = \sum_{a,b} r(ab)e(ab)$. Then $r(ab) = 0$ whenever $f(ab) \notin \mathfrak{B}_n$. In particular, $(2r-1, 2r)f(ij) = \sum_{r(ab) \neq 0} r(ab)f(ab)$.

Proof. Assertion (iii) follows immediately from Proposition 3.1.

Now let $n \equiv 2 \pmod{4}$. Then $(n-3, n-2)e(n-3, n-1) = e(n-2, n-1)$, by Proposition 3.1. Moreover, one checks that

$$e(n-2, n-1) + \gamma_n = \sum_{\substack{j=4 \\ j \text{ odd}}}^{n-1} e(2j) + \sum_{i=3}^{j-1} \sum_{j=4}^{n-2} e(ij) + \sum_{i=3}^{n-3} e(i, n-1).$$

From this and Proposition 4.6 we deduce (i).

In order to prove (ii), let $n \geq 6$ be even. Furthermore, if $n \equiv 0 \pmod{4}$ then let $i \in \{2, \dots, n-2\}$, and if $n \equiv 2 \pmod{4}$ then let $i \in \{2, \dots, n-3\}$, and set

$$\Sigma_i := \sum_{l=i+1}^{n-1} e(il) + \sum_{l=3}^{i-1} e(li).$$

Then,

$$\Sigma_i = \begin{cases} \gamma_1 + \gamma_i, & \text{if } i \in \{2, 3, 4\}, \\ \gamma_1 + \gamma_i + \sum_{j=4}^r e(2j), & \text{if } i \geq 5 \text{ odd,} \\ \gamma_1 + \gamma_i + \sum_{j=4}^{r-1} e(2j), & \text{if } i \geq 6 \text{ even.} \end{cases}$$

Together with Proposition 3.1 and Proposition 4.6 this gives (ii). \square

Proposition 4.8. Let $n \geq 6$ be even. In the notation of Proposition 4.6,

$$\begin{aligned} \mathfrak{B} := & \{f(24)\} \cup \{f(2, 2l-1) + f(2, 2l) \mid 3 \leq l \leq (n-2)/2\} \\ & \cup \left\{ \sum_{i=2k-1}^{2k} \sum_{j=2l-1}^{2l} f(ij) \mid 2 \leq k \leq (n-4)/2, k+1 \leq l \leq (n-2)/2 \right\} \end{aligned}$$

is an F -basis of $\text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D))$.

Proof. By Proposition 4.7, all the elements in \mathfrak{B} are fixed under E_n , and hence belong to $\text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D))$. Moreover, \mathfrak{B} is linearly independent. Conversely, let $x = \sum_{i,j} r(ij)f(ij) \in \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D))$ with $r(ij) \in F$, for all admissible i and j . We show that we can write x as an F -linear combination of elements in \mathfrak{B} . For this, we will distinguish between the cases $n \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{4}$.

Consider the subspace $V := F\{f(ij) \mid j = 4, \dots, n-2, i = 2, \dots, j-1\}$ of D . By Propositions 3.1 and 4.7, this is an FE_{n-2} -submodule of D , and is isomorphic to $\text{Res}_{E_{n-2}}^{\mathfrak{S}_{n-2}}(S^{(n-4,2)})$. An explicit isomorphism is given by mapping $f(ij)$ to the standard $(n-4, 2)$ -polytabloid $e(ij)$, for $j = 4, \dots, n-2, i = 2, \dots, j-1$. Hence, by Proposition 3.2, \mathfrak{B} is an F -basis of the socle of the FE_{n-2} -module V . Notice that the transposition $(n-1, n)$ fixes every element in V so that V is, in fact, an FE_n -submodule of D . Now suppose that $n \equiv 2 \pmod{4}$, and let $x = \sum_{i,j} r(ij)f(ij) \in \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D))$, with $r(ij)$ for $j = 4, \dots, n-1$ and $i = 2, \dots, \min(j-1, n-3)$. Since $(n-1, n)x = x$, for $i = 2, \dots, n-3$, the coefficients of $f(i, n-2)$ in x and $(n-1, n)x$ must coincide. This yields $r(i, n-2) = r(i, n-2) + r(i, n-1)$, for $i = 2, \dots, n-3$, and thus

$$0 = r(2, n-1) = r(3, n-1) = \dots = r(n-3, n-1).$$

This shows that $x \in V$ so that also $x \in \text{Soc}(V) = F\mathfrak{B}$. This proves the assertion of the proposition in the case $n \equiv 2 \pmod{4}$.

Now suppose that $n \equiv 0 \pmod{4}$. Let V be as above, and let further $W := F\{f(i, n-1) \mid i = 2, \dots, n-2\}$. Then, by Proposition 4.7, W is also an FE_{n-2} -submodule of D , and we have

$$\text{Res}_{E_{n-2}}^{\mathfrak{S}_n}(D) = V \oplus W.$$

As we have seen above, V is already an FE_n -submodule of D . The FE_n -socle of V equals the FE_{n-2} -socle, and has thus basis \mathfrak{B} . Therefore it suffices to show that the socle of $\text{Res}_{E_n}^{\mathfrak{S}_n}(D)$ is contained in the socle of the FE_n -module V . For this, let $x \in \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D))$. We may write $x = x_0 + x_1$, for some x_0 in the socle of the FE_{n-2} -module V and some x_1 in the socle of the FE_{n-2} -module W . By our previous considerations, both x and x_0 are fixed by E_n , and so also x_1 is fixed by E_n . We show that this implies $x_1 = 0$. We may write $x_1 = \sum_{i=2}^{n-2} r_i f(i, n-1)$, for some $r_i \in F$ and $i = 2, \dots, n-2$. Then

$$0 = (1 + (1, 2))x_1 = \left(\sum_{i=3}^{n-2} r_i \right) f(2, n-1) + \sum_{j=4}^{n-2} r_j f(2j).$$

It follows that $\sum_{i=3}^{n-2} r_i = 0 = r_4 = r_5 = \dots = r_{n-2}$. Thus also $r_3 = 0$, and we are left with $x_1 = r_2 f(2, n-1)$. Consequently,

$$r_2 f(2, n-1) = x_1 = (n-1, n)x_1 = r_2 \sum_{j=4}^{n-1} f(2, j).$$

This now shows that $r_2 = 0$ so that $x_1 = 0$, proving the assertion also in the case that $n \equiv 0 \pmod{4}$. \square

Lemma 4.9. *Let $n \geq 6$ be even, and let $E_n := \langle (2r-1, 2r) \mid r = 1, \dots, n/2 \rangle$. Then $\text{Res}_{E_n}^{\mathfrak{S}_n}(D)$ is indecomposable.*

Proof. In the case that $n = 6$ the assertion trivially holds, since then $\text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D)) \cong F$, by Proposition 4.8. Therefore, for the remainder of the proof we may suppose that $n \geq 8$. We show that the endomorphism algebra $\text{End}_{FE_n}(D)$ is local. For this, let $\varphi^2 = \varphi \in \text{End}_{FE_n}(D)$. We need to show that $\varphi \in \{0, 1\}$.

Let $l \in \{4, \dots, n-2\}$ and $k \in \{2, \dots, l-1\}$. We write

$$\varphi(f(kl)) = \sum_{i,j} r(ij)f(ij),$$

with $r(ij) \in F$ for all admissible i and j . As in the proof of Proposition 4.8, consider the FE_n -submodule $V := F\{f(ij) \mid j = 4, \dots, n-2, i = 2, \dots, j-1\}$ of D which, as an FE_{n-2} -module is isomorphic to $\text{Res}_{E_{n-2}}^{\mathfrak{S}_{n-2}}(S^{(n-4,2)})$. Since $(n-1, n)$ acts trivially on V , it acts trivially on $\varphi(V)$ as well. Comparing the coefficients of $f(i, n-2)$, for $i = 2, \dots, n-3$, in $\varphi(f(kl))$ and in $(n-1, n)\varphi(f(kl))$, we obtain:

$$r(i, n-2) = r(i, n-2) + r(i, n-1) + r(n-2, n-1), \text{ if } n \equiv 0 \pmod{4}, \quad (14)$$

$$r(i, n-2) = r(i, n-2) + r(i, n-1), \text{ if } n \equiv 2 \pmod{4}. \quad (15)$$

Hence, $r(2, n-1) = r(3, n-1) = \dots = r(n-3, n-1) = 0$ if $n \equiv 2 \pmod{4}$, and $r_{kl} := r(2, n-1) = r(3, n-1) = \dots = r(n-2, n-1)$ if $n \equiv 0 \pmod{4}$. In particular, for $n \equiv 2 \pmod{4}$ this means $\varphi(V) \subseteq V$. That is, in this case $\psi := \varphi|_V$ is an idempotent in $\text{End}_{FE_{n-2}}(V) \cong \text{End}_{FE_{n-2}}(S^{(n-4,2)})$. Thus either $\psi = 0$ or $\psi = 1$, by Lemma 3.5. If $\psi = 0$ then, since $\text{Soc}(\text{Res}_{E_{n-2}}^{E_n}(V)) = \text{Soc}(V) = \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D))$, we have $0 = \varphi(\text{Soc}(V)) = \varphi(\text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D)))$, and hence $\varphi = 0$. If ψ is the identity map on V then we analogously get $1 - \varphi = 0$. This proves the lemma in the case when $n \equiv 2 \pmod{4}$.

Therefore, from now on we may suppose that $n \equiv 0 \pmod{4}$. We suppose further that $\varphi \neq 1$, and we show that $\varphi(\text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D))) = 0$. Our calculations will be similar to those done in the proof of Lemma 3.5, so we will omit some details here. By Proposition 4.8, it suffices to show that

- (i) $\varphi(f(24)) = 0$,
- (ii) $\varphi(f(2, 2l-1)) + \varphi(f(2, 2l)) = 0$, for $l = 3, \dots, (n-2)/2$,
- (iii) $\varphi(f(2k-1, 2l-1) + f(2k-1, 2l) + f(2k, 2l-1) + f(2k, 2l)) = 0$, for $k = 2, \dots, (n-4)/2$ and $l = k+1, \dots, (n-2)/2$.

By the first part of the proof, we already know that, for all $2 \leq k < l \leq n-2$, we have

$$\varphi(f(kl)) = v_{kl} + r_{kl} \sum_{j=2}^{n-2} f(j, n-1),$$

where $v_{kl} \in V$ and $r_{kl} \in F$ is as above. We claim that

$$(*) \quad r_{2l} = 0,$$

for all $4 \leq l \leq n-2$. Namely, $(1 + (1, 2))f(3l) = f(2l)$, and therefore $\varphi(f(2, l)) = (1 + (1, 2))\varphi(f(3l))$ lies in $F\{f(2j) \mid 4 \leq j \leq n-1\}$. So the coefficient of $f(3, n-1)$ is 0, and hence $r_{2l} = 0$.

We now list the properties we use in proving (i)-(iii).

(A) $(1 + (1, 2))f(3j) = f(2j)$ for $4 \leq j \leq n - 1$, and $(1 + (1, 2))f(i, j) = f(2, i) + f(2, j)$ for $3 < i < j \leq n - 1$.

(B) Suppose that $3 \leq l \leq n/2$. Then $(2l - 1, 2l)$ fixes $f(2, j)$ for $2l - 1 \neq j \neq 2l$.

(C) For $3 \leq l$ and $2l - 1 \leq n - 1$, we have $(1 + (3, 4))f(2, 2l - 1) = f(24)$.

(D) For $6 \leq 2l \leq n - 2$, the permutation $(2l - 1, 2l)$ swaps $f(2, 2l - 1)$ and $f(2, 2l)$.

We fix some $l \in \{3, \dots, (n - 2)/2\}$, and consider $\varphi(f(2, 2l - 1))$. Since $f(2, 2l - 1)$, by (A), is contained in $(1 + (1, 2))D \subseteq F\{f(2, j) \mid 4 \leq j \leq n - 1\}$, we can write

$$(**) \quad \varphi(f(2, 2l - 1)) = \sum_{j=4}^{n-1} \delta_j f(2, j),$$

for some $\delta_j \in F$. First note that $\delta_{n-1} = 0$, by (*). Furthermore, property (B) shows that, for $r \notin \{1, 2, l\}$, the permutation $(2r - 1, 2r)$ fixes $f(2, 2l - 1)$. Therefore it fixes $\varphi(f(2, 2l - 1))$ which means that the coefficients of $f(2, 2r - 1)$ and $f(2, 2r)$ in (**) are equal. So

$$\delta_{2r-1} = \delta_{2r}, \quad \text{for } r \notin \{1, 2, l\}.$$

Using this and applying $(1 + (3, 4))$ to (**) we get, by (C), that

$$\varphi(f(2, 4)) = f(2, 4) \sum_{j=5}^{n-2} \delta_j = f(2, 4)(\delta_{2l-1} + \delta_{2l}).$$

This shows that $f(2, 4)$ is an eigenvector of φ . Since φ is a projection, the corresponding eigenvalue is either 0 or 1. We continue with the case $\varphi(f(2, 4)) = 0$, and will show that then $\varphi = 0$. This then gives free that in the case $\varphi(f(2, 4)) = f(2, 4)$ we must have $\varphi = 1$ which was excluded.

So part (i) of our statement above holds. Furthermore, we also have $\delta_{2l-1} = \delta_{2l}$, and hence $\delta_{2r-1} = \delta_{2r}$, for all $3 \leq r \leq (n - 2)/2$. Now using the fact that $(2l - 1, 2l)f(2, 2l - 1) = f(2, 2l)$ we get

$$\varphi(f(2, 2l - 1)) = \varphi(f(2, 2l)).$$

This proves part (ii). Actually, we have something stronger, namely $\varphi(f(2, 2l - 1)) = \varphi(f(2, 2l)) = 0$. To see this, recall that $\delta_{2r-1} = \delta_{2r}$, for $3 \leq r \leq (n - 2)/2$ so that (**) implies

$$\varphi(f(2, 2l - 1)) = \varphi^2(f(2, 2l - 1)) = \delta_4 \varphi(f(2, 4)) = 0.$$

Lastly, we prove (iii). For this, let $2 \leq k < l \leq (n - 2)/2$, and write

$$\varphi(f(2k - 1, 2l - 1)) = \sum_{i,j} r(ij) f(ij),$$

for some $r(ij) \in F$ and all admissible i and j . We know, by (A), that $(1 + (1, 2))f(2k - 1, 2l - 1) = f(2, 2k - 1) + f(2, 2l - 1)$ if $k > 2$, and $(1 + (1, 2))f(2k - 1, 2l - 1) = f(2, 2l - 1)$ if $k = 2$. Hence, by (ii),

$$0 = (1 + (1, 2))\varphi(f(2k - 1, 2l - 1)) = \sum_{j=4}^{n-2} \gamma_j \varphi(f(2, j)).$$

Here, for each $4 \leq j \leq n-2$, we have $0 = \gamma_j = \sum_{2 < i < j} r(ij) + \sum_{j < i \leq n-1} r(ji)$. Furthermore, from Proposition 4.7, we get

$$\begin{aligned} \varphi\left(\sum_{i=2k-1}^{2k} \sum_{j=2l-1}^{2l} f(ij)\right) &= (1 + (2k-1, 2k))(1 + (2l-1, 2l))\varphi(f(2k-1, 2l-1)) \\ &= \lambda\left(\sum_{i=2k-1}^{2k} \sum_{j=2l-1}^{2l} f(ij)\right), \end{aligned}$$

where $\lambda := r(2k-1, 2l-1) + r(2k, 2l-1) + r(2k-1, 2l) + r(2k, 2l)$. It remains to show that $\lambda = 0$. For this let $r \in \{2, \dots, (n-2)/2\} \setminus \{k, l\}$ so that $(2r-1, 2r)(1 + (2l-1, 2l))\varphi(f(2k-1, 2l-1)) = (1 + (2l-1, 2l))\varphi(f(2k-1, 2l-1))$, by (B). This implies

$$\begin{aligned} 0 &= r(2r-1, 2l-1) + r(2r-1, 2l) + r(2r, 2l-1) + r(2r, 2l), \text{ for } 4 < 2r < 2l-1, \\ 0 &= r(2l-1, 2r-1) + r(2l, 2r-1) + r(2l-1, 2r) + r(2l, 2r), \text{ for } 2l < 2r-1 < n-2. \end{aligned}$$

Hence $0 = \gamma_{2l-1} + \gamma_{2l} = \lambda$, and (iii) follows. This completes the proof of the Proposition. \square

5 Closing remarks

Suppose further that F is an algebraically closed field of characteristic 2, and let $n \in \mathbb{N}$.

Remark 5.1. As mentioned in the introduction, it remains open to determine the sources of the simple $F\mathfrak{S}_n$ -module $D := D^{(n-2,2)}$ in the case where $5 < n \equiv 1 \pmod{4}$. In this situation consider, as before, the 2-adic expansion $n = 1 + \sum_{j=1}^s 2^{i_j}$, for appropriate $s \geq 1$ and $i_1 > \dots > i_s \geq 2$. By Theorem 1.2, we know that $P_n = P_{2^{i_1}} \times \dots \times P_{2^{i_s}}$ is a vertex of D . Denote the Young subgroup $\mathfrak{S}_{2^{i_1}} \times \dots \times \mathfrak{S}_{2^{i_s}}$ of \mathfrak{S}_n by H . As in Remark 2.15, for $j = 1, \dots, s$, let Ω_j be the set of all 2-element subsets of the support of the direct factor $\mathfrak{S}_{2^{i_j}}$. Then, by Lemma 2.14 and Lemma 2.16,

$$\text{Res}_H^{\mathfrak{S}_n}(D) \cong (U_{\bigoplus_{j=1}^s F\Omega_j} / V_{\bigoplus_{j=1}^s F\Omega_j}) \oplus F\Delta.$$

Each of the indecomposable direct summands of the permutation FH -module $F\Delta$ has a vertex strictly contained in P_n . Moreover, $W := U_{\bigoplus_{j=1}^s F\Omega_j} / V_{\bigoplus_{j=1}^s F\Omega_j}$ is, by Proposition 2.17, indecomposable. In particular, W and D have common vertex P_n . Computer calculations with MAGMA [2] suggest the following:

Conjecture 5.2. In the notation of Remark 5.1 above, the indecomposable FH -module $W = U_{\bigoplus_{j=1}^s F\Omega_j} / V_{\bigoplus_{j=1}^s F\Omega_j}$ restricts indecomposably to P_n . In particular, $\text{Res}_{P_n}^H(W)$ is a source of D .

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