The vertices of a class of Specht modules and simple modules for symmetric groups in characteristic 2

Susanne Danz and Karin Erdmann

Abstract

We study Specht modules $S^{(n-2,2)}$ and simple modules $D^{(n-2,2)}$ for symmetric groups $S_n$ of degree $n$ over a field of characteristic 2. In particular, we determine the vertices of these modules, and also provide some information on their sources.

Subject classification: 20C20, 20C30

Keywords: Specht module, simple module, symmetric group, two part partition, vertex

1 Introduction and results

Let $G$ be a finite group and $F$ a field of characteristic $p > 0$. Any finite-dimensional indecomposable $FG$-module $M$ has a vertex, which is a group theoretic invariant. A vertex is a subgroup $P$ of $G$, minimal with respect to the property that $M$ is relatively $P$-projective, that is, the canonical map $FG \otimes FP M \to M$ splits. Such a vertex $P$ of $M$ is a $p$-subgroup of $G$, and it is unique up to conjugation in $G$. Moreover, there is an indecomposable $FP$-module $L$, unique up to isomorphism and conjugation with elements in $N_G(P)$ such that $M$ is isomorphic to a direct summand of $\text{Ind}_G^P(L)$. The $FP$-module $L$ has also vertex $P$, and is called a source of $M$.

One would like to know the vertices for distinguished classes of modules, in particular for simple modules. For symmetric groups, there are further classes of distinguished modules, such as Specht modules, and Young modules. Vertices of Young modules have been determined by Grabmeier a while ago (see [6]). On the other hand, vertices of Specht modules and of simple modules are known only in a few cases. For a survey of known results, see [3]. For a partition $\lambda$ of $n$, we denote the respective Specht $F\mathfrak{S}_n$-module by $S^{\lambda}$, and when $\lambda$ is $p$-regular, we denote the corresponding simple $F\mathfrak{S}_n$-module by $D^{\lambda}$.

Although Specht modules and simple modules for two part partitions are studied a lot, so far their vertices and sources have only been found in a few cases; and the case $p = 2$ appears to be special. In [1], for $p > 2$, the first author determined the vertices of simple $F\mathfrak{S}_n$-modules labelled by two part partitions $(n - m, m)$ where $m < (p + 1)p/2$. In these cases, the simple module $D^{(n-m,m)}$ has the defect groups of its block as vertices, unless $D^{(n-m,m)} \cong S^{(n-m,m)}$ in which case $D^{(n-m,m)}$ is a Young module.

In this paper, we determine the vertices of Specht modules and simple modules of $\mathfrak{S}_n$ labelled by partitions $(n - 2, 2)$ over fields of characteristic 2. In general terms, we show that the vertices are ‘usually’ Sylow 2-subgroups of $\mathfrak{S}_n$, and the exceptions occur either for very small degrees, or for the cases when the Specht module is simple and isomorphic to a Young module. We state now the precise results, here $P_n$ is a Sylow 2-subgroup of $\mathfrak{S}_n$.

Theorem 1.1. Let $F$ be an algebraically closed field of characteristic 2, let $n \geq 4$, and let further $P \leq P_n$ be a vertex of the Specht $F\mathfrak{S}_n$-module $S := S^{(n-2,2)}$.

(i) If $n = 4$ then $S = S^{(2)} \cong D^{(3,1)}$, and $P$ is the unique Sylow 2-subgroup $Q_4$ of $A_4$. Moreover, $S$ has trivial source.
(ii) If \( n \equiv 3 \pmod{4} \) then \( S \cong D^{(n-2,2)} \), and \( P \simeq \text{S}_{n-5} \times P_2 \times P_2 \). Furthermore, \( S \) has then trivial sources.

(iii) Otherwise, \( P = P_n \). If \( n \equiv 0 \pmod{2} \) then \( \text{Res}^\text{S}_n(S) \) is a source of \( S \).

**Theorem 1.2.** Let \( F \) be an algebraically closed field of characteristic 2, let \( n \geq 5 \), and let \( P \leq P_n \) be a vertex of the simple \( F\text{S}_n \)-module \( D := D^{(n-2,2)} \).

(i) If \( n = 5 \) then \( P \) is the unique Sylow 2-subgroup \( Q_4 \) of \( \text{S}_4 \). Moreover, \( \text{Res}^\text{S}_4(E_+^{(3,2)}) \) and \( \text{Res}^\text{S}_4(E_-^{(3,2)}) \) are then sources of \( D \). Here \( E_+^{(3,2)} \) and \( E_-^{(3,2)} \) are non-isomorphic simple \( F\text{S}_4 \)-modules such that \( \text{Res}^\text{S}_4(D) \cong E_+^{(3,2)} \oplus E_-^{(3,2)} \).

(ii) If \( n \equiv 3 \pmod{4} \) then \( D \cong S^{(n-2,2)} \) with vertex \( P \simeq \text{S}_{n-5} \times P_2 \times P_2 \) and trivial sources.

(iii) Otherwise \( P = P_n \), and if \( n \equiv 0 \pmod{2} \) then \( \text{Res}^\text{S}_n(D) \) is a source of \( D \).

To prove these theorems, we study restrictions of the modules in question to appropriate subgroups of \( \text{S}_n \). It turns out that the case where \( n \) is odd and the case where \( n \) is even behave quite differently. We show that if \( n \) is even then both \( S^{(n-2,2)} \) and \( D^{(n-2,2)} \) restrict indecomposably to the elementary abelian subgroup \( E_n := \langle (1,2), (3,4), \ldots, (n-1,n) \rangle \) of \( \text{S}_n \). This will be the key step in proving the theorems in the case where \( n \) is even. If \( n \) is odd then we investigate the restriction of \( D^{(n-2,2)} \) to certain Young subgroups \( \text{S}_a \times \text{S}_b \times \text{S}_c \) of \( \text{S}_n \) where \( a, b \) and \( c \) are divisible by 4. In doing this we will also gain some knowledge on the structure of these modules which may be of independent interest.

After summarizing our notation, in Section 2.2 we will collect some general facts about permutation modules. In Sections 3 and 4, respectively, we will prove Theorems 1.1 and 1.2, respectively.

For background on vertices and sources of indecomposable \( FG \)-modules we refer to [19], Chapter 4.3. Details on the representation theory of the symmetric groups may be found in [11] and [12].

## 2 Prerequisites

### 2.1 Notation and known results

Let \( F \) be an algebraically closed field of prime characteristic \( p \). Suppose that we are given some \( n \in \mathbb{N} \) and some partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of \( n \). In addition we set \( \lambda_{k+1} := 0 \).

1. Let \( \text{S}_\lambda \) be the Young subgroup \( \text{S}_{\lambda_1} \times \cdots \times \text{S}_{\lambda_k} \) of \( \text{S}_n \) corresponding to \( \lambda \). In this notation, for \( j = 1, \ldots, k \), the direct factor \( \text{S}_{\lambda_j} \) is always supposed to be acting on the set \( \{ \sum_{i=1}^{j-1} \lambda_i + 1, \ldots, \sum_{i=1}^{j} \lambda_i \} \). The permutation \( F\text{S}_n \)-module \( M^\lambda := \text{Ind}^\text{S}_n(\text{F}) \) has an \( F \)-basis consisting of all \( \lambda \)-tabloids. When \( k = 2 \), each \( \lambda \)-tabloid \( \{ t \} \) is uniquely determined by the entries in the second row of the \( \lambda \)-tableau \( t \). If these are \( i_1, \ldots, i_{\lambda_2} \) then we simply identify \( \{ t \} \) with the set \( \{ i_1, \ldots, i_{\lambda_2} \} \).
(2) The Specht $FG_{n}$-module $S^\lambda$ has an $F$-basis consisting of all standard $\lambda$-polytabloids $e_t$. Suppose again that $k = 2$ and that $t := t(ij)$ is a standard $\lambda$-tableau with second row $ij$, for some $2 \leq i < j \leq n$. Then $t$ and $e_t$ are uniquely determined by these second row entries, and we thus also write $e(ij)$ rather than $e_t$.

(3) Fixing an indecomposable direct sum decomposition $M^\lambda = M_1 \oplus \cdots \oplus M_r$ of the permutation $FG_{n}$-module $M^\lambda$, there is precisely one $i \in \{1, \ldots, r\}$ such that $S^\lambda \subseteq M_i$. The module $M_i$ is unique up to isomorphism, and is called the Young module $Y^\lambda$ corresponding to $\lambda$. For details we refer to [16], Sec. 4.6. The vertices of the indecomposable Young modules have been determined by Grabmeier in [6] S. 7.8 (see also [5]). Namely, with the above notation, the module $Y^\lambda$ has the Sylow $p$-subgroups of the Young subgroup

$$\prod_{i=1}^{k}(\mathfrak{S}_{\lambda_i-\lambda_{i+1}})^i$$

of $\mathfrak{S}_n$ as vertices. Moreover, $Y^\lambda$ has trivial sources.

(4) We set $P_1 := \{1\}, P_p := C_p := \langle(1, \ldots, p)\rangle$, and

$$P_{p^i} := P_{p^{i-1}} \wr P_p := \{(x_1, \ldots, x_p; \sigma) \mid x_1, \ldots, x_p \in P_{p^{i-1}}, \sigma \in P_p\},$$

for $i \geq 2$. Then, for $i \geq 0$, we can identify $P_{p^i}$ with a subgroup of the symmetric group $\mathfrak{S}_{p^i}$ in the obvious way. Via this identification, $P_{p^i}$ is then a Sylow $p$-subgroup of $\mathfrak{S}_{p^i}$, by [12] 4.1.22 and 4.1.24. For $i \geq 2$, the base group of the wreath product $P_{p^i} := P_{p^{i-1}} \wr P_p$ is isomorphic to $P_{p^{i-1}} \times \cdots \times P_{p^2}$, and will be denoted by $B_{p^i}$.

Let $n = \sum_{j=1}^{s} \alpha_j p^{i_j}$ be the $p$-adic expansion of $n$, with $i_1 > \cdots > i_s \geq 0$ and $\alpha_j > 0$ for $j = 1, \ldots, s$. Then

$$P_n := \prod_{j=1}^{s} (P_{p^{i_j}})^{\alpha_j}$$

is a Sylow $p$-subgroup of $\mathfrak{S}_n$, by [12], 4.1.22. The different direct factors of $P_n$ are understood to be acting on disjoint successive subsets of $\{1, \ldots, n\}$. Furthermore, we set $Q_n := P_n \cap \mathfrak{A}_n$ which is then a Sylow $p$-subgroup of the alternating group $\mathfrak{A}_n$.

(5) Now suppose that $G$ is an arbitrary group and that $M \neq 0$ is an $FG$-module which has a unique composition series, with composition factors $D_1, D_2, \ldots, D_r$ from top to socle. Then we write

$$M = U(D_1, D_2, \ldots, D_r).$$

(6) Let $G$ and $H$ be groups, let $M$ be an $FG$-module, and let $N$ be an $FH$-module. Then the outer tensor product of $M$ and $N$ becomes an $F[G \times H]$-module in the obvious way and will be denoted by $M \boxtimes N$. If both $M$ and $N$ are indecomposable as $FG$-module and $FH$-module, respectively, then $M \boxtimes N$ is indecomposable as $F[G \times H]$-module. If, moreover, $M$ has vertex $P$ and source $V$, and if $N$ has vertex $Q$ and source $W$ then $M \boxtimes N$ has vertex $P \times Q$ and source $V \boxtimes W$. A proof for this can be found in [15], Prop. 1.2.
(7) Whenever we have a group $G$, an $FG$-module $M$ and a finite subset $\mathcal{M} \subset M$ then we set $\mathcal{M}^+ := \sum_{m \in \mathcal{M}} m$.

We conclude this subsection with the following result which we will use several times throughout this article.

**Proposition 2.1** ([4]). Let $p = 2$, let $r \geq 2$, and let $n = 2^r$. Then

$$P_{2^{r-1}} \Phi(P_n) = B_n,$$

where $B_n$ denotes the base group of the wreath product $P_n = P_{2^{r-1}} \wr P_2$.

**Proof.** First of all note that $\Phi(P_{2^k}) = [P_{2^k}, P_{2^k}]$, for $k \geq 1$. For $k = 1$ this is trivially true. If $k \geq 2$ then $[P_{2^k}, P_{2^k}] \leq \Phi(P_{2^k})$, by [9] Satz III.3.14, and, by construction, $P_{2^k}$ can be generated by elements of order 2. Hence the same applies to $P_{2^k}/[P_{2^k}, P_{2^k}]$ so that $\Phi(P_{2^k}) = [P_{2^k}, P_{2^k}]$. Therefore, by [20] L. 1.4, we have

$$\Phi(P_n) = \{(x_1, x_2; 1) \mid x_1x_2 \in \Phi(P_{2^{r-1}})\}.$$

Now let $x \in B_n$. We may write $x = (x_1, x_2; 1)$, for appropriate $x_1, x_2 \in P_{2^{r-1}}$. That is, $x = (x_1x_2, 1; 1)(x_2^{-1}, x_2; 1) \in P_{2^{r-1}}\Phi(P_n)$. Thus $P_{2^{r-1}}\Phi(P_n)$ is a proper subgroup of $P_n$ containing the maximal subgroup $B_n$ of $P_n$, and we get $P_{2^{r-1}}\Phi(P_n) = B_n$. \hfill $\square$

### 2.2 General $FG$-modules and permutation modules

Let $G$ be a group, and let $M$ be a self-dual $FG$-module. Suppose further that there is a chain of submodules $0 \subset V \subset U \subset M$ such that

(1) $V \cong F \cong M/U$, and

(2) $U/V$ does not have a trivial composition factor.

Then the following holds:

**Lemma 2.2.** With the above notation, suppose that $M$ does not have a direct summand of composition length 2 whose composition factors are both isomorphic to $F$. Then

(i) $U$ does not have a trivial factor module, and

(ii) $M/V$ does not have a trivial submodule.

**Proof.** (i) Assume (for a contradiction) that there is some $0 \subset X \subset U$ such that $U/X \cong F$. Consider $Z := M/X$, this has length 2 and both composition factors are isomorphic to $F$. Since $M$ is self-dual, we get two exact sequences and a diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & Z^* & \overset{\iota}{\longrightarrow} & M & \overset{\pi}{\longrightarrow} & X^* & \longrightarrow & 0 \\
& & 1 & & & & & \\
0 & \longrightarrow & X & \overset{\iota'}{\longrightarrow} & M & \overset{\pi'}{\longrightarrow} & Z & \longrightarrow & 0
\end{array}
$$

We claim that $\pi' \circ \iota$ is injective.
Consider $\text{im}(\iota) \cap \ker(\pi')$. This is a submodule of $X$, and hence it does not have a trivial composition factor. On the other hand, it is contained in $\text{im}(\iota) \cong Z^*$ which has only trivial composition factors. So $\text{im}(\iota) \cap \ker(\pi') = 0$, that is the restriction of $\pi'$ to $\text{im}(\iota)$ is injective. In other words, $\pi' \circ \iota$ is injective. Then it must be an isomorphism. Therefore $Z^* \cong Z$, and moreover $M \cong Z \oplus X$, a contradiction.

Dually one proves part (ii).

\begin{proof}

\end{proof}

**Remark 2.3.** Let $G$ be a group, and consider the category of finite $G$-sets. For such $G$-set $\Omega$, we denote the corresponding permutation module by $F\Omega$. In this way we obtain a category, $\mathcal{P}$ whose objects are the permutation modules $F\Omega$ with fixed permutation basis $\Omega$. If $\Omega$ and $\Omega'$ are finite $G$-sets then the morphisms $\text{Hom}_\mathcal{P}(F\Omega, F\Omega')$ are the $FG$-homomorphisms $f : F\Omega \to F\Omega'$ such that $f(\omega) \in \Omega'$, for all $\omega \in \Omega$. The trivial $FG$-module is a permutation module for the trivial $G$-set $\{1\}$. Hence the augmentation map

$$F\Omega \to F, \omega \mapsto 1$$

on any permutation module $F\Omega$ in $\mathcal{P}$ is a morphism in the category $\mathcal{P}$. We denote the kernel of the augmentation map $F\Omega \to F$ by $U_{F\Omega}$. Moreover, the trivial submodule of $F\Omega$ spanned by $\Omega^+ := \sum_{\omega \in \Omega} \omega$ is denoted by $V_{F\Omega}$. With these definitions we get:

**Lemma 2.4.** Suppose that $M$ and $M'$ are objects in $\mathcal{P}$. If $f : M \to M'$ is an isomorphism in $\mathcal{P}$ then $f(U_M) = U_{M'}$ and $f(V_M) = V_{M'}$.

**Remark 2.5.** With the above notation, suppose that $M = F\Omega$ where $|\Omega| \equiv 0 \pmod{p}$. In this case, $V_M \subseteq U_M$, and we then set

$$\tilde{U}_M := U_M/V_M.$$ 

Furthermore, if $\Omega = \{\omega_1, \ldots, \omega_n\}$ then, for $i = 1, \ldots, n - 1$, we set $\xi_i := \omega_i - \omega_n$. Then $\{\xi_1, \ldots, \xi_{n-1}\}$ is an $F$-basis of the $FG$-module $U_M$, and we also have

$$\Omega^+ = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n-1} \xi_i.$$ 

So the cosets of $\xi_1, \ldots, \xi_{n-2}$ span $\tilde{U}_M$. They are also linearly independent, and hence form an $F$-basis for $\tilde{U}_M$. Analogously, for each $i \in \{1, \ldots, n-1\}$, the cosets of $\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{n-1}$ form a basis of $\tilde{U}_M$. As a special case of Lemma 2.2 we now have the following Lemma:

**Lemma 2.6.** In the notation of Remark 2.5 suppose that $\tilde{U}_M$ does not have a trivial composition factor. Then $M/V_M$ does not have a trivial submodule, except when $M$ has a direct summand of composition length 2 both of whose composition factors are trivial.

**Remark 2.7.** Suppose $H = \mathfrak{S}_\lambda$ where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ (possibly $\lambda_3 = 0$). Assume $F\Omega_i$ is a permutation module of $\mathfrak{S}_\lambda$ for $i = 1, 2, 3$.

**Hypothesis** Let $\text{char}(F)$ divide $|\Omega_i|$, for $i = 1, 2, 3$. We assume that $F\Omega_i$ is indecomposable, and that $U_{F\Omega_i}/V_{F\Omega_i}$ is indecomposable and does not have a trivial composition factor, for $i = 1, 2, 3$.

We view $F\Omega_i$ as a module for $H$ on which the factors $\mathfrak{S}_{\lambda_j}$ for $j \neq i$ act trivially.
Let $W := U_{F\Omega_1 \oplus F\Omega_2 \oplus F\Omega_3} / V_{F\Omega_1 \oplus F\Omega_2 \oplus F\Omega_3}$. The aim is to show that the $FH$-module $W$ is indecomposable.
Note that we have a chain of $FH$-modules

$$V_{F\Omega_1 \oplus F\Omega_2 \oplus F\Omega_3} \subset V_{F\Omega_1} \oplus V_{F\Omega_2} \oplus V_{F\Omega_3} \subset U_{F\Omega_1} \oplus U_{F\Omega_2} \oplus U_{F\Omega_3} \subset U_{F\Omega_1 \oplus F\Omega_2 \oplus F\Omega_3} \subset F\Omega_1 \oplus F\Omega_2 \oplus F\Omega_3.$$ 

As the main step in proving the indecomposability of $W$ we will first show that $R$ is indecomposable where

$$R := U_{F\Omega_1 \oplus F\Omega_2 \oplus F\Omega_3} / (V_{F\Omega_1} \oplus V_{F\Omega_2} \oplus V_{F\Omega_3}).$$ \hfill (1)

Note that $R$ has a submodule

$$Z = (U_{F\Omega_1} \oplus U_{F\Omega_2} \oplus U_{F\Omega_3}) / (V_{F\Omega_1} \oplus V_{F\Omega_2} \oplus V_{F\Omega_3}) \cong \tilde{U}_{F\Omega_1} \oplus \tilde{U}_{F\Omega_2} \oplus \tilde{U}_{F\Omega_3},$$ \hfill (2)

and the quotient $R/Z$ is the direct sum of either one or two trivial modules, depending on whether or not $C = 0$. We will identify $Z$ and $\tilde{U}_{F\Omega_1} \oplus \tilde{U}_{F\Omega_2} \oplus \tilde{U}_{F\Omega_3}$ via the above isomorphism. Moreover, we will also identify $R$ with a submodule of $\bigoplus_{i=1}^3 (F\Omega_i / V_{F\Omega_i})$.

We have a basis for $\tilde{U}_{F\Omega_3}$ as in Remark 2.5. In addition, for each $i \in \{1, 2, 3\}$, we fix an element $\omega_i \in \Omega_i$ and we fix elements

$$\tau_i := \omega_i + V_{F\Omega_i} \in F\Omega_i / V_{F\Omega_i}.$$ 

Then $R/Z \cong U_{F\Omega_1 \oplus F\Omega_2 \oplus F\Omega_3} / (U_{F\Omega_1} \oplus U_{F\Omega_2} \oplus U_{F\Omega_3})$, and has a basis consisting of cosets of elements of the form

$$\Upsilon := c_1 \tau_1 + c_2 \tau_2 + c_3 \tau_3$$ \hfill (3)

with $c_i \in F$ such that $\sum_{i=1}^3 c_i = 0$.

**Lemma 2.8.** The module $R$ does not have any trivial submodule.

**Proof.** Suppose $\eta \in R$ spans a trivial submodule. Then we write $\eta = s \Upsilon + \sum_{i=1}^3 \alpha_i$ with appropriate $s \in F$ and $\alpha_i \in \tilde{U}_{F\Omega_i}$, for $i = 1, 2, 3$. Let $\pi$ be the projection onto $F\Omega_1 / V_{F\Omega_1}$ which is a homomorphism of $F\mathfrak{S}_{\lambda_1}$-modules. Then $\pi(\eta) = s c_1 \tau_1 + \alpha_1$. If this were non-zero, then it would span a trivial $F\mathfrak{S}_{\lambda_1}$-submodule of $F\Omega_1 / V_{F\Omega_1}$. But, by Lemma 2.6 and Remark 2.7, such a submodule does not exist, so $\pi(\eta) = 0$. Similarly $sc_i \tau_i + \alpha_i = 0$ for $i = 2, 3$, and so $\eta = 0$, a contradiction. \hfill $\square$

**Lemma 2.9.** Any indecomposable direct summand of $Z$ is equal to one of the modules $\tilde{U}_{F\Omega_1}$, $\tilde{U}_{F\Omega_2}$, $\tilde{U}_{F\Omega_3}$.

**Proof.** Suppose $Z = Z_1 \oplus Z_2$ where $Z_1$ is indecomposable. By the Krull-Schmidt Theorem, $Z_1$ is isomorphic to one of the three modules listed, say $Z_1 \cong \tilde{U}_{F\Omega_1}$. So $\mathfrak{S}_{\lambda_2}$ and $\mathfrak{S}_{\lambda_3}$ act trivially on $Z_1$. Now consider $\zeta \in Z_2$. We can write $\zeta = \sum_{i=1}^3 \alpha_i$ according to the decomposition of $Z$, in (2). If $g \in \mathfrak{S}_{\lambda_2}$ then $0 = (g - 1)\zeta$, but this is equal to $(g - 1)\alpha_2$. If $\alpha_2 \neq 0$ then $\alpha_2$ would span a trivial $F\mathfrak{S}_{\lambda_2}$-submodule of $\tilde{U}_{F\Omega_2}$, but there is no such submodule, by Lemma 2.6. So $\alpha_2 = 0$. Similarly $\alpha_3 = 0$. This proves that $Z_1 \subseteq \tilde{U}_{F\Omega_1}$, and then, comparing dimensions, we must have equality. \hfill $\square$
Lemma 2.10 (Separation Lemma). Let \( \omega = \Upsilon + \sum_{i=1}^{3} \alpha_i \in R \) where \( \Upsilon \) is as in (3), and where \( \alpha_i \in \tilde{U}_{\Omega_1} \) for \( i = 1, 2, 3 \).

(i) Suppose that, for all \( g \in H \), we have \( (g - 1)\omega \in \tilde{U}_{\Omega_2} \oplus \tilde{U}_{\Omega_3} \). Then \( \alpha_1 = 0 \) and \( c_1 = 0 \).

(ii) Suppose that, for all \( g \in H \), we have \( (g - 1)\omega \in \tilde{U}_{\Omega_2} \). Then \( \alpha_1 = \alpha_3 = 0 \) and \( c_1 = c_3 = 0 \).

In part (i) it follows that \( c_2 + c_3 = 0 \) if \( \lambda_3 \neq 0 \), and if \( \lambda_3 = 0 \) then it follows that \( c_2 = 0 \) and \( \Upsilon = 0 \). In part (ii) it follows that \( c_2 = 0 \) and hence \( \Upsilon = 0 \).

Proof. (i) Let \( g \in \mathfrak{S}_{\lambda_1} \). Then

\[
(g - 1)\omega = c_1 (g - 1)\tau_1 + (g - 1)\alpha_1.
\]

Clearly \( (g - 1)\tau_1 \in \tilde{U}_{\Omega_1} \) and so \( (g - 1)\omega \in \tilde{U}_{\Omega_1} \). By our hypothesis, \( (g - 1)\omega \) also lies in \( \tilde{U}_{\Omega_2} \oplus \tilde{U}_{\Omega_3} \), so that the element \((*)\) is equal to zero for \( g \in \mathfrak{S}_{\lambda_1} \).

Suppose \( c_1\tau_1 + \alpha_1 \neq 0 \). This is then fixed by all \( g \in H \) and therefore it spans a trivial submodule of \( R \). This contradicts 2.8. Hence \( c_1\tau_1 + \alpha_1 = 0 \). Now, \( \tau_1 \notin \tilde{U}_{\Omega_1} \) and therefore \( c_1 = 0 \) and then also \( \alpha_1 = 0 \). This proves (i).

(ii) With the assumption of part (ii), we have that, for all \( g \in \mathfrak{S}_{\lambda_1} \),

\[
(g - 1)\omega = c_1 (g - 1)\tau_1 + (g - 1)\alpha_1 \in \tilde{U}_{\Omega_1} \cap \tilde{U}_{\Omega_2} = 0.
\]

As in (i) we get \( \alpha_1 = 0 \) and \( c_1 = 0 \). The same argument works with \( g \in \mathfrak{S}_{\lambda_3} \) (in case \( \lambda_3 \neq 0 \)). We get therefore \( \alpha_3 = 0 \) and \( c_3 = 0 \).

The last statement holds since \( \sum_{i=1}^{3} c_i = 0 \). \( \square \)

Proposition 2.11. In the notation of Remark 2.7, the FH-module \( R \) is indecomposable.

Proof. Let \( Z \) be the submodule of \( R \) as defined in (2). That is, \( R/Z \cong F \oplus F \) if \( \lambda_3 \neq 0 \), and \( R/Z = F \) if \( \lambda_3 = 0 \). Moreover, \( Z \) does not have any trivial composition factor, and in fact, \( Z \) contains every submodule of \( R \) which does not have any trivial composition factor.

Let \( e^2 = e \in \text{End}_{FH}(R) \). We have to show that \( e = 0 \) or \( e = 1 \). Note first that \( e(Z) \cong Z/(\ker(e) \cap Z) \) is a submodule of \( R \) which does not have trivial composition factors, so \( e(Z) \subseteq Z \), and the restriction of \( e \) to \( Z \) is also a projection. So we have

\[
Z = e(Z) \oplus (1 - e)(Z), \quad R = e(R) \oplus (1 - e)(R),
\]

\( e(Z) \subseteq e(R) \), and \( (1 - e)(Z) \subseteq (1 - e)(R) \). We may suppose \( e(Z) \neq e(R) \). For otherwise we replace \( e \) by \( (1 - e) \). In particular, \( e \neq 0 \). We take \( \omega \in e(R) \setminus e(Z) \), so that for all \( g \in H \) we have

\[
(g - 1)\omega \in e(Z),
\]

since \( e(R)/e(Z) \) is the direct sum of trivial modules. The element \( \omega \) has the form \( \omega = \Upsilon + \sum_{i=1}^{3} \alpha_i \), as in the Separation Lemma. We now distinguish two cases.

(1) Assume that \( e(Z) = 0 \). Since \( e \neq 0 \), \( e(R) \) then has a trivial submodule, a contradiction. If \( e(Z) = Z \) then \( (1 - e)(Z) = 0 \) so that \( e = 1 \). For otherwise we would get the contradiction
that \((1-e)(R)\) has a trivial submodule.

(2) Hence, we may now assume that \(e(Z) \neq 0 \neq (1-e)(Z)\). That is \(e(Z)\) is either indecomposable or the direct sum of two indecomposable modules. Assume first that \(e(Z)\) is indecomposable. Then, by Lemma 2.9, \(e(Z)\) must be equal to one of the modules \(\tilde{U}_{F\Omega_i}\) for some \(i \in \{1, 2, 3\}\), say \(e(Z) = \tilde{U}_{F\Omega_2}\). Applying part (ii) of the Separation Lemma, we deduce that \(\Upsilon = 0\) and \(\omega \in e(Z)\) which contradicts our hypothesis. Note that, at this stage, we have now already completed the proof in the case where \(\lambda_3 = 0\).

Therefore, we may now assume that \(\lambda_3 \neq 0\) and that \(e(Z)\) is the sum of two indecomposable modules and \((1-e)(Z) \neq 0\). Using Lemma 2.9, we may suppose \(e(Z) = \tilde{U}_{F\Omega_2} \oplus \tilde{U}_{F\Omega_3}\), and hence \((1-e)(Z) = \tilde{U}_{F\Omega_1}\). We apply part (i) of the Separation Lemma and get

\[
\omega = c_2(\tau_2 - \tau_3) + \alpha_2 + \alpha_3.
\]

We claim that \(e(R)/e(Z)\) must have dimension 1. Namely, otherwise we would have some \(\omega' \in e(R) \setminus e(Z)\) such that \(\omega + Z \) and \(\omega' + Z \) form a basis for the two-dimensional quotient module \(R/Z\). The element \(\omega'\) could then also be written as \(\omega' = \Upsilon' + \alpha_1' + \alpha_2' + \alpha_3'\) with appropriate \(\alpha_1' \in \tilde{U}_{F\Omega_1}, \alpha_2' \in \tilde{U}_{F\Omega_2}, \alpha_3' \in \tilde{U}_{F\Omega_3}\) and \(\Upsilon' = c_1' \tau_1 + c_2' \tau_2 + c_3' \tau_3\). The same arguments as above would apply, and so

\[
\omega' = c_2(\tau_2 - \tau_3) + \alpha_2' + \alpha_3',
\]

But this shows that \(\omega\) and \(\omega'\) would be linearly dependent modulo \(Z\), a contradiction. So both \(e(R)/e(Z)\) and \((1-e)(R)/(1-e)(Z)\) have dimension 1. Then, for any \(\omega'' \in (1-e)(R) \setminus (1-e)(Z)\) and any \(g \in H\), we get \((g-1)\omega'' \in (1-e)(Z)\). But, again by part (ii) of the Separation Lemma, this is not possible. So case (2) cannot occur.

To summarize, the only possibility left is \(e = 1\), and hence \(R\) is indecomposable.

\[\square\]

**Lemma 2.12.** Let \(W\) be as in Remark 2.7, and let \(R' := U_{F\Omega_1} \oplus U_{F\Omega_2} \oplus U_{F\Omega_3}/V_{F\Omega_1} \oplus V_{F\Omega_2} \oplus V_{F\Omega_3}\). Then \(R' \cong R^*\).

**Proof.** Suppose \(F\Omega\) is a (finite-dimensional) permutation module of some group \(G\). We take the non-degenerate symmetric bilinear form on \(F\Omega\) defined by \(\langle \alpha_i, \alpha_j \rangle = \delta_{ij}\) for \(\alpha_i, \alpha_j \in \Omega\). We identify \(F\Omega^*\) with the set \(\{(\langle -v \rangle | v \in F\Omega)\}\). The vector space isomorphism \(F\Omega \longrightarrow F\Omega^*: v \longmapsto \langle -v \rangle\) is, in fact, an \(FG\)-module isomorphism.

For any submodule \(X\) of \(F\Omega\), we have the exact sequence

\[
0 \rightarrow X^0 \longrightarrow F\Omega^* \xrightarrow{\pi} X^* \rightarrow 0
\]

where \(\pi\) is restriction to \(X\), and \(X^0 := \{(\langle -v \rangle | v \in F\Omega^* \big| \langle X, v \rangle = 0\}\}. One sees directly that \(U^0_{F\Omega} \cong V_{F\Omega}\) and also \(V^0_{F\Omega} \cong U_{F\Omega}\), under the isomorphism \(F\Omega \longrightarrow F\Omega^*\). So \(U_{F\Omega} \cong F\Omega/V_{F\Omega}\) and \(V_{F\Omega} \cong F\Omega/U_{F\Omega}\).

Now let \(\Omega = \bigcup_{i=1}^3 \Omega_i\), where the \(\Omega_i\) are as in Remark 2.7, then by definition \(R' = U_{F\Omega_1} \oplus U_{F\Omega_2} \oplus U_{F\Omega_3}/V_{F\Omega}\). From the inclusion of submodules of \(F\Omega\) given in Remark 2.7 we get an exact sequence

\[
(*) \quad 0 \rightarrow R' \longrightarrow F\Omega/V_{F\Omega} \longrightarrow F\Omega/(\bigoplus_{i=1}^3 U_{F\Omega_i}) \longrightarrow 0.
\]

8
By the previous considerations, \((F \Omega/V_{F \Omega})^* \cong U_{F \Omega}\). Moreover, since for \(i = 1, 2, 3\) we have \(F \Omega/(\bigoplus_{i=1}^3 U_{F \Omega}) \cong \bigoplus_{i=1}^3 (F \Omega/U_{F \Omega})\), we deduce that \((F \Omega/(\bigoplus_{i=1}^3 U_{F \Omega}))^*\) is isomorphic to \(\bigoplus_{i=1}^3 V_{F \Omega}\). Hence dualizing (*) shows that

\[
(R')^* \cong U_{F \Omega}/(\bigoplus_{i=1}^3 V_{F \Omega}) = R,
\]
as claimed. \(\square\)

Retaining the notation introduced in Remark 2.7, we are now able to prove the following:

**Proposition 2.13.** The \(FH\)-module \(W\) is indecomposable.

**Proof.** Let \(\varphi^2 = \varphi \in \text{End}_{FH}(W)\). We must show that \(\varphi \in \{0, 1\}\). Let \(W_0\) be the sum of all trivial submodules of \(W\) so that \(W_0\) has dimension 2 if \(\lambda_3 \neq 0\), and it has dimension 1 if \(\lambda_3 = 0\). Moreover \(W_0\) is equal to \((\bigoplus_{i=1}^3 V_{F \Omega})/V_{\bigoplus_{i=1}^3 F \Omega}\). Since \(\varphi\) is a homomorphism, the submodule \(W_0\) must be invariant under \(\varphi\). Hence \(\varphi\) induces a projection \(\varphi\) on the quotient module \(W/W_0 \cong R\). By Proposition 2.11, we know that \(R\) is indecomposable so that \(\varphi \in \{0, 1\}\). We may suppose that \(\varphi = 0\). This means that \(\varphi(W) \subseteq W_0\).

Now we consider the submodule \(R' := (\bigoplus_{i=1}^3 U_{F \Omega})/V_{\bigoplus_{i=1}^3 F \Omega}\) of \(W\). We have \(W_0 \subseteq R' \subseteq W\), and hence \(\varphi(R') \subseteq \varphi(W) \subseteq W_0 \subseteq R'\). Therefore \(\varphi\) restricts to a projection of \(R'\) which is not the identity. Lemma 2.12 above shows that \(R' \cong R^*\). By Proposition 2.11, \(R'\) is therefore indecomposable. So the restriction of \(\varphi\) to \(R'\) is the zero map. This implies that, for every \(w \in W\), we have \(\varphi(w) = \varphi^2(w) \in \varphi(R') = 0\), and the assertion follows. \(\square\)

### 3 The Specht module \(S^{(n-2,2)}\)

Throughout this section let \(F\) be an algebraically closed field of characteristic 2, let \(n \geq 4\), and let \(S\) be the Specht \(FG_n\)-module \(S^{(n-2,2)}\).

#### 3.1 Restrictions to elementary abelian subgroups

As we have already mentioned, in the proof of Theorem 1.1 we will distinguish between the cases \(n \equiv 1\) (mod 2) and \(n \equiv 0\) (mod 2). The key step in the proof of the latter case will be Lemma 3.5 below which asserts that, provided that \(n \geq 6\) is even, \(S\) restricts indecomposably to the following elementary abelian subgroup of \(P_n\):

\[
E_n := \langle (2r - 1, 2r) \mid r = 1, \ldots, n/2 \rangle.
\]

In order to prove this result, we aim to show that the endomorphism ring \(\text{End}_{E_n}(S)\) contains no idempotents other than 1 and 0. For this, we describe the action of \(E_n\) on the \(F\)-basis of \(S\) consisting of all standard \((n - 2, 2)\)-polytabloids. After that we will determine the socle of \(\text{Res}^{E_n}_{E_n}(S)\). For if \(1 \neq \varphi \in \text{End}_{E_n}(S)\) is an idempotent which is the zero map on \(\text{Soc}(\text{Res}^{E_n}_{E_n}(S))\) then, in fact, \(\varphi = 0\).

The following result is easily verified:
Lemma 3.4. We get:

Proposition 3.2. Let \( \sigma_{k,l} := e(2k - 1, 2l - 1) + e(2k - 1, 2l) + e(2k, 2l - 1) + e(2k, 2l) \), for \( 2 \leq k < l \leq n/2 \), and let \( \rho_l := e(2, 2l - 1) + e(2, 2l) \) for \( 3 \leq l \leq n/2 \). Let

\[ \mathcal{B} := \{e(24)\} \cup \{\rho_l \mid 3 \leq l \leq n/2\} \cup \{\sigma_{k,l} \mid 2 \leq k < l \leq n/2\}. \]

Then \( \mathcal{B} \) is an \( F \)-basis for \( \text{Soc}(\text{Res}_E^S(S)) \).

Remark 3.3. The \( F \)-subspace \( F\{e(2k - 1, 2l - 1), e(2k - 1, 2l), e(2k, 2l - 1), e(2k, 2l)\} \) of \( S \) is denoted by \( V(k,l) \). Moreover, we denote the \( F \)-subspace \( F\{e(2r - 1, 2r) \mid 2 \leq r \leq n/2\} \) of \( S \) by \( T \), and we denote the \( F \)-subspace \( F\{e(2,j) \mid 4 \leq j \leq n\} \) of \( S \) by \( U \). With this notation, we get:

Lemma 3.4. Let \( E' := \langle (2r - 1, 2r) \mid 2 \leq r \leq n/2 \rangle \leq E_n \).

(i) We have a direct sum decomposition

\[ \text{Res}_E^S(S) = (T + U) \oplus \bigoplus_{2 \leq k < l \leq n/2} V(k,l). \]

(ii) The socle of \( \text{Res}_E^S(S) \) has basis \( \mathcal{B} \cup \mathcal{C} \). Here, \( \mathcal{B} \) is as in Proposition 3.2, and \( \mathcal{C} := \{e(2k - 1, 2k) + e(2, 2k) \mid 2 \leq k \leq n/2\} \).
**Proof.** (1) From Proposition 3.1 we deduce that both $U$ and $T + U$ are, in fact, $FE'$-submodules of $S$. The socle of the $FE'$-module $T + U$ clearly consists of all elements in $T + U$ fixed under $E'$. It is easily checked that this socle has basis

$$\{ e(24) \} \cup \{ \rho_j \mid 3 \leq j \leq n/2 \} \cup \mathcal{C}. $$

(2) Now consider $V(k, l)$, for some $2 \leq k < l \leq n/2$. By Proposition 3.1, the subgroup $\langle (2k-1, 2k), (2l-1, 2l) \rangle$ of $E'$ acts freely on $V(k, l)$, and any other generator of $E'$ acts trivially on $V(k, l)$. This shows that $V(k, l)$ is also an $FE'$-submodule of $S$. With this assertion (i) follows. By Proposition 3.2, we deduce further that the socle of the $FE'$-module $V(k, l)$ is one-dimensional, and is spanned by $\sigma_{k,l}$. Together with our considerations in (1), we get (ii), proving the lemma. 

**Proof of Proposition 3.2.** Let $x \in \text{Soc}(\text{Res}_{E_n}^{S_n}(S))$. Then $x$ is, in particular, contained in the socle of $\text{Res}_{E_n}^{S_n}(S)$. Hence, by Lemma 3.4, $x$ lies in the $F$-span of $\mathcal{B} \cup \mathcal{C}$. Moreover, an element in $\text{Res}_{E_n}^{S_n}(S)$ in contained in $\text{Soc}(\text{Res}_{E_n}^{S_n}(S))$ if and only if it is fixed by $(1, 2)$. Therefore, by Proposition 3.1, $x$ is in fact contained in the $F$-span of $\mathcal{B}$. This proves the proposition. 

**Lemma 3.5.** Let $n \geq 6$ be even, and let $S := S^{(n-2,2)}$. Then $\text{Res}_{E_n}^{S_n}(S)$ is indecomposable.

**Proof.** Let $1 \neq \varphi \in \text{End}_{E_n}(S)$ such that $\varphi^2 = \varphi$. We show that then $\varphi = 0$ so that $\text{End}_{E_n}(S)$ is a local $F$-algebra and $\text{Res}_{E_n}^{S_n}(S)$ therefore indecomposable. It suffices to show that $\varphi(x) = 0$, for $x \in \text{Soc}(\text{Res}_{E_n}^{S_n}(S))$. For this then clearly forces $\varphi = 0$. By Proposition 3.2 we thus have to show

(i) $\varphi(e(24)) = 0$,

(ii) $\varphi(e(2, 2k - 1) + e(2, 2k)) = 0$, for $k = 3, \ldots, n/2$,

(iii) $\varphi(e(2k - 1, 2l - 1) + e(2k - 1, 2l) + e(2k, 2l - 1) + e(2k, 2l)) = 0$, for $k = 2, \ldots, (n - 2)/2$ and $l = k + 1, \ldots, n/2$.

We repeatedly use the following equalities which hold by Proposition 3.1:

(a) $(1 + (1, 2))e(3, j) = e(2, j)$ for $4 \leq j \leq n$, and $(1 + (1, 2))e(r, s) = e(2, r) + e(2, s)$ for $4 \leq r < s \leq n.$

(b) Suppose that $3 \leq s \leq n/2$. Then $(2s - 1, 2s)e(2, j) = e(2, j)$ for $2s - 1 \neq j \neq 2s$.

(c) For $3 \leq s \leq n/2$, we have $(1 + (3, 4))e(2, 2s - 1) = e(2, 4) = (1 + (3, 4))e(2, 2s).$

(d) For $3 \leq l \leq n/2$, we have $(2l - 1, 2l)e(2, 2l - 1) = e(2, 2l).$

Let $l \in \{3, \ldots, n/2\}$, and consider $\varphi(e(2, 2l - 1))$. Since $\varphi(e(2, 2l - 1)) \in (1 + (1, 2))S \subseteq U$, by (a), we can write

$$\varphi(e(2, 2l - 1)) = \sum_{j=4}^{n} \delta_j e(2, j),$$

(*)
for appropriate $\delta_j \in F$. Property (b) above shows that, for $r \notin \{1, 2, l\}$, the permutation $(2r - 1, 2r)$ fixes $e(2, 2l - 1)$. Therefore, $(2r - 1, 2r)$ fixes $\varphi(e(2, 2l - 1))$ which means that the coefficients of $e(2, 2r - 1)$ and $e(2, 2r)$ in $(\ast)$ must be equal so that

$$\delta_{2r-1} = \delta_{2r} \quad \text{for} \quad r \notin \{1, 2, l\}. \quad (4)$$

Using this and applying $(1 + (3, 4))$ to $(\ast)$, we deduce from (c) that

$$\varphi(e(2, 4)) = \left(\sum_{j=5}^{n} \delta_j\right)e(2, 4) = (\delta_{2l-1} + \delta_{2l})e(2, 4). \quad (5)$$

This shows that $e(2, 4)$ is an eigenvector of $\varphi$. Since $\varphi$ is a projection, the eigenvalue is either 1 or 0. We continue with the case where $\varphi(e(2, 4)) = 0$, and we will show that then $\varphi = 0$. This then also implies that $\varphi = 1$ in the case where $\varphi(e(2, 4)) = 1$ which, by our assumption, is not possible.

So part (i) above holds. Furthermore, we have $\delta_{2l-1} = \delta_{2l}$, and hence

$$\delta_{2r-1} = \delta_{2r}, \quad (6)$$

for all $r = 3, \ldots, n/2$. Using (a) and (b), into $(\ast)$, this gives

$$\varphi(e(2, 2l - 1)) = \varphi(e(2, 2l)).$$

This shows that (ii) holds true. Thus, in the notation of Remark 3.3, for $r = 3, \ldots, n/2$ the socle element $\rho_r$ lies in the kernel of $\varphi$. Consequently, since $\varphi = \varphi^2$ and $\varphi(e(2, 4)) = 0$, $(\ast)$ and (6) yield

$$\varphi(e(2, 2l - 1)) = 0 = \varphi(e(2, 2l)) \quad \text{and} \quad \delta_j = 0 \quad \text{for} \quad j = 4, \ldots, n. \quad (7)$$

In order to prove (iii), let $2 \leq k < l \leq n/2$, and note that $\sigma_{k,l} = \eta e(2k - 1, 2l - 1)$ where $\eta = (1 + (2k - 1, 2k))(1 + (2l - 1, 2l))$ and where $\sigma_{k,l}$ is as in Proposition 3.2. We have to show that $\eta \varphi(e(2k - 1, 2l - 1)) = 0$. For this we use the vector space decomposition of $S$ given in Lemma 3.4. That is, we can write

$$\varphi(e(2k - 1, 2l - 1)) = \tau + u + \sum_{2 \leq s < t \leq n/2} v(s, t)$$

with $\tau \in T, u \in U$ and $v(s, t) \in V(s, t)$. We claim that $\eta \tau = 0$, $\eta u = 0$, and $\eta v(s, t) = 0$ for $(s, t) \neq (k, l)$. To see this, we use property (b). Namely, $\tau$ is a linear combination of the elements $e(2r - 1, 2r)$ for $r = 2, \ldots, n/2$. If $r \neq k$ then $(1 + (2k - 1, 2k))e(2r - 1, 2r) = 0$, and $(1 + (2l - 1, 2l))e(2k - 1, 2k) = 0$. Hence $\eta \tau = 0$. Similar arguments show that also $\eta u = 0$ and $\eta v(s, t) = 0$ for $(s, t) \neq (k, l)$. Thus $\eta \varphi(e(2k - 1, 2l - 1)) = \eta \varphi(k, l)$.

We write

$$\varphi(e(2k - 1, 2l - 1)) = \sum_{i,j} r(ij)e(ij),$$

for some $r(ij) \in F$. Then $\eta v(k, l) = \eta \varphi(e(2k - 1, 2l - 1)) = \lambda \sigma_{k,l}$ where

$$\lambda = r(2k - 1, 2l - 1) + r(2k - 1, 2l) + r(2k, 2l - 1) + r(2k, 2l).$$
By (a), we have \((1 + (1, 2))e(2k - 1, 2l - 1) = e(2, 2k - 1) + e(2, 2l - 1)\) if \(k > 2\), and 
\((1 + (1, 2))e(2k - 1, 2l - 1) = e(2, 2l - 1)\) if \(k = 2\). So using (7) we get 
\[0 = (1 + (1, 2))\varphi(e(2k - 1, 2l - 1)) = \sum_{j=1}^{n} \gamma_{j}e(2j).\]

Here, for \(j = 4, \ldots, n\), we have 
\[0 = \gamma_{j} = \sum_{2<i<j} r(ij) + \sum_{j<i} r(ji).\] (8)

Let \(t \notin \{1, k, l\}\), and apply \((2t - 1, 2t)\) to \(\varphi(e(2k - 1, 2l - 1))\). Then (b) implies:
\[
\begin{align*}
    r(2s - 1, 2k - 1) &= r(2s, 2k - 1), \quad \text{for } 2 < 2s < 2k - 1, \\
    r(2k - 1, 2s - 1) &= r(2k - 1, 2s), \quad \text{for } 2k - 1 < 2s - 1 \neq 2l - 1.
\end{align*}
\]
Substituting this into (8) with \(j = 2k - 1\), we get 
\[0 = r(2k - 1, 2k) + r(2k - 1, 2l - 1) + r(2k - 1, 2l).\]
Similarly, if we take \(j = 2k\) then we get 
\[0 = r(2k - 1, 2k) + r(2k, 2l - 1) + r(2k, 2l).\]
Therefore, \(\lambda = 0\). This completes the proof of (iii). \(\square\)

**Remark 3.6.** Notice that the above lemma also holds without any restrictions on the field 
\(F\) as long as it has characteristic 2.

### 3.2 Proof of Theorem 1.1

We will now prove Theorem 1.1. The statement for \(n = 4\) is well-known (cf. [18]). Thus, from 
now on, let \(n \geq 5\). We recall from [11], Thm. 20.1 that 
\[\dim(S) = \binom{n}{2} - n.\]

Hence if \(n \equiv 1 \pmod{4}\) or if \(n \equiv 2 \pmod{4}\) then \(\dim(S) \not\equiv 0 \pmod{2}\) so that we have 
\(P = P_{n}\) in these cases. Furthermore, if \(n \equiv 3 \pmod{4}\) then \(S = S^{(n-2,2)} \cong D^{(n-2,2)}\), by 
Carter’s Criterion (cf. [12], Thm. 7.3.23). Thus we then also have 
\(S = S^{(n-2,2)} \cong Y^{(n-2,2)}\) with vertex \(P_{n-4} \times P_{2} \times P_{2} = P_{n-5} \times P_{2} \times P_{2}\) and trivial sources, by Grabmeier’s Theorem 
[6] S. 7.8. So it remains to settle the case where \(4 < n \equiv 0 \pmod{4}\). Note that then always 
\(\dim(S) \equiv 0 \pmod{2}\).

Take \(n \equiv 0 \pmod{4}\) with 2-adic expansion \(n = \sum_{j=1}^{s} 2^{i_{j}}\), for some \(s \geq 1\) and some \(i_{1} > \ldots > i_{s} \geq 2\). Furthermore, for \(j = 1, \ldots, s\) we set 
\(n_{j} := 2^{i_{j}}\). By the modular branching rules for Specht modules, we obtain 
\[\text{Res}_{S_{n-1}}^{S_{n}}(S) \cong S^{(n-2,1)} \oplus S^{(n-3,2)}.\]
Since \(n - 1 \equiv 3 \pmod{4}\), we already know that \(S^{(n-3,2)}\) has vertex \(P_{n-6} \times P_{2} \times P_{2}\) and trivial source. Moreover, \(S^{(n-2,1)} \cong D^{(n-2,1)}\) has vertex \(P_{n-4}\) and trivial source, by [18]. By Lemma 
3.5, we know further that \(\text{Res}_{E_{n}}^{S_{n}}(S)\) is indecomposable, where \(E_{n}\) is the elementary abelian 
subgroup 
\[E_{n} := \langle (2r - 1, 2r) \mid r = 1, \ldots, n/2 \rangle\]
of \( P_n \). In particular, also \( L := \text{Res}^{\mathfrak{S}_n}_P(S) \) is indecomposable. Now suppose that \( S \) has vertex \( P < P_n \), and let \( R \) be a maximal subgroup of \( P_n \) containing \( P \). Then \( P \) is a vertex of \( L \) as well, and the trivial sources of \( S^{(n-2,1)} \), respectively, are isomorphic to indecomposable direct summands of \( \text{Res}^{P_n}_{P_{n-6} \times P_2} \) and \( \text{Res}^{P_n}_{P_{n-4}}(L) \), respectively. Hence

\[
P_{n-6} \times P_2 \times P_2 \leq P \leq R < P_n \quad \text{and} \quad P_{n-4} \leq P \leq R < P_n.
\]

Since \( R \) is normal in \( P_n \), this in fact yields \( P_{n-6} \times P_2 \times P_2 \leq R \) and \( P_{n-4} \leq R \). Of course we also have \( \Phi(P_{i_1}) \times \cdots \times \Phi(P_{i_s}) = \Phi(P_n) \leq R \). Since

\[
P_{n-4} = \begin{cases} P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_{n_s/2} \times \cdots \times P_4, & \text{if } n_s > 4, \\ P_{n_1} \times \cdots \times P_{n_{s-1}}, & \text{if } n_s = 4,
\end{cases}
\]

from Proposition 2.1 we deduce that

\[
P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{n_s} = P_{n-4} \Phi(P_n) = R, \quad \text{if } n_s > 4,
\]

\[
P_{n_1} \times \cdots \times P_{n_{s-1}} \times \mathcal{B}_{n_s} \leq P_{n-4} \Phi(P_n) \leq R, \quad \text{if } n_s = 4.
\]

Again \( B_{n_j} \cong P_{n_j/2} \times P_{n_j/2} \) is understood to be the base group of the wreath product \( P_{n_j} = P_{n_j/2} \wr P_2 \), and \( \mathcal{B}_{n_j} := B_{n_j} \cap \mathfrak{A}_n \), for \( j = 1, \ldots, s \). But, since also \( P_{n-6} \times P_2 \times P_2 \leq R \), we get

\[
P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{n_s} = R
\]

in any case. In particular, \( E_n \leq R \) so that \( \text{Res}^{\mathfrak{S}_n}_P(S) \) is indecomposable, by Lemma 3.5. But this contradicts Green’s Indecomposability Theorem [7]. Therefore, we indeed have \( P = P_n \), for \( n \equiv 0 \pmod{4} \).

Finally, by Lemma 3.5, we also know that \( \text{Res}^{\mathfrak{S}_n}_P(S) \) is a source of \( S \) if \( n \) is even. This proves the theorem.

4 The simple module \( D^{(n-2,2)} \)

The aim of this section is to give a proof of Theorem 1.2. Therefore, from now on, let \( n \geq 5 \). As before, we denote the Specht \( F \mathfrak{S}_n \)-module \( S^{(n-2,2)} \) by \( S \) and the corresponding simple \( F \mathfrak{S}_n \)-module \( D^{(n-2,2)} = S/\text{Rad}(S) \) by \( D \). In analogy to the proof of Theorem 1.1, we will also distinguish between the case when \( n \) is odd and the case when \( n \) is even.

4.1 Outline of the proof of Theorem 1.2

(1) Suppose that \( n \geq 5 \) is odd. If \( n \equiv 3 \pmod{4} \) then \( S \cong D \) with vertex \( P_{n-5} \times P_2 \times P_2 \) and trivial source, as we have already proved in Theorem 1.1. If \( n = 5 \) then \( D \) is the spin \( F \mathfrak{S}_5 \)-module \( D^{(3,2)} \) whose vertices and sources have been determined in [2]. Namely, the Sylow 2-subgroup \( Q_4 \) of \( \mathfrak{A}_4 \) is then a vertex of \( D \), and the restrictions of the simple \( F \mathfrak{S}_5 \)-modules \( E_+^{(3,2)} \) and \( E_-^{(3,2)} \) to \( Q_4 \) are sources of \( D \).

(2) Therefore, we may now suppose that \( n \equiv 1 \pmod{4} \) and that \( n > 5 \). If \( n = 1 + \sum_{j=1}^s 2^{i_j} \), with appropriate \( s \geq 1 \) and \( i_1 > \ldots > i_s \geq 2 \), is the 2-adic expansion of \( n \) then we set \( n_j := 2^{i_j} \), for \( j = 1, \ldots, s \). That is, \( P_n = P_{n_1} \times \cdots \times P_{n_s} \). From [13] Thm. 11.2.10 we deduce that \( D^{(n-3,1)} \mid \text{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-2}}(D) \). Moreover \( D^{(n-3,1)} \cong S^{(n-3,1)} \cong Y^{(n-3,1)} \) with vertex \( P_{n-5} \) and trivial
source, by [18]. Consequently, there is an indecomposable direct summand $X$ of $\text{Res}_{P_n}^\mathcal{S}_n(D)$ such that $F \mid \text{Res}_{P_n}^\mathcal{S}_n(X)$. Hence $P_{n-5} \leq P_n Q$, for any vertex $Q \leq P_n$ of $X$. Now assume that $D$ has vertex $P < P_n$. Since $\text{Res}_{\mathcal{S}_n-1}(D)$ is, by [13] Thm. 11.2.7, indecomposable, we may suppose that $P$ is also a vertex of $\text{Res}_{\mathcal{S}_n-1}(D)$. In particular, $P_{n-5} \leq P_n Q \leq \mathcal{S}_n-1 P < P_n$. If $n_s > 4$ then

$$P_{n-5} = P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_{2s-1} \times \cdots \times P_4,$$

and if $n_s = 4$ then

$$P_{n-5} = P_{n_1} \times \cdots \times P_{n_{s-1}}.$$

This shows that, in fact, $P_{n-5} \leq P_n P < P_n$. Furthermore, there is a maximal subgroup $R$ of $P_n$ such that $P_{n-5} \leq P_n P \leq R < P_n$. Since $R$ is normal in $P_n$, we have, in fact, $P_{n-5} \leq R$. Of course we also have $\Phi(P_{n_1}) \times \cdots \times \Phi(P_{n_s}) = \Phi(P_n) \leq R$. Therefore, from Proposition 2.1 we obtain

$$R = P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{ns}, \text{ if } n_s > 4, \quad R = P_{n_1} \times \cdots \times P_{n_{s-1}} \times \overline{B_{ns}} , \text{ if } n_s = 4.$$

Recall that $B_{ns} \cong P_{2s-1} \times P_{2n_{s-1}}$ denotes the base group of the wreath product $P_{ns}$ and that $\overline{B_{ns}} = B_{ns} \cap \mathfrak{A}_s$. If $n_s > 4$ then

$$R = P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{ns} \leq \mathcal{S}_{n-1-ns} \times \mathcal{S}_{2s} \times \mathcal{S}_{2s} =: H,$$

so that $D$ has to be relatively $H$-projective. If $n_s = 4$ then $T := P_{n_1} \times \cdots \times P_{n_{s-1}} \times \overline{B_{ns}} \triangleleft P_n$ such that $P_n/T$ is elementary abelian of order $4 = 2^2$. Hence there are precisely $2^2 - 1 = 3$ maximal subgroups of $P_n$ containing $T$, namely $R_1 := P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{ns}$, $R_2 := P_{n_1} \times \cdots \times P_{n_{s-1}} \times Q_{ns}$ and $R_3 := P_{n_1} \times \cdots \times P_{n_{s-1}} \times \langle (n-4, n-2, n-3, n-1) \rangle$. Consequently, in order to prove the assertion of Theorem 1.2 in the case when $n \equiv 1 \pmod{4}$, it remains to show that $D$ is not relatively projective with respect to any of the maximal subgroups $R_1, R_2, R_3$ of $P_n$ if $n_s = 4$, and that $D$ is not relatively $H$-projective if $n_s > 4$.

(3) We now suppose that the assertion of Theorem 1.2 has already been proved for all odd $n \geq 5$. Then we may suppose further that $n \geq 6$ is even with 2-adic expansion $n = \sum_{j=1}^s 2^{i_j}$, for appropriate $s \geq 1$ and $i_1 > \cdots > i_s \geq 1$. If $n = 6$ then $D$ is the spin $F\mathcal{G}_6$-module $D^{(4,2)}$ which has vertex $P_6$ and source $\text{Res}_{P_6}^\mathcal{S}_6(D^{(4,2)})$, by [2]. Let now $n > 6$. We claim that is suffices to prove that $D$ restricts indecomposably to the elementary abelian subgroup

$$E_n := \langle (2r - 1, 2r) \mid r = 1, \ldots, n/2 \rangle$$

of $\mathcal{S}_n$. For if we know this then $\text{Res}_{P_n}^\mathcal{S}_n(D)$ is also indecomposable. Therefore, the vertex $P \leq P_n$ of $D$ is also a vertex of the module $\text{Res}_{P_n}^\mathcal{S}_n(D)$. By [13] Thm. 11.2.7, we have $\text{Res}_{\mathcal{S}_n-1}(D) \cong D^{(n-3,2)}$. Since $n-1$ is odd, we already know that $D^{(n-3,2)}$ has vertex $P_{n-6} \times P_2 \times P_2$ in case that $n \equiv 0 \pmod{4}$, and that has vertex $P_{n-2} = P_{n_1} \times \cdots \times P_{n_{s-1}}$ in case that $n \equiv 2 \pmod{4}$. Hence

$$P_{n_1} \times \cdots \times P_{n_{s-1}} \leq P_n P \leq P_n = P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_2$$

if $n \equiv 2 \pmod{4}$,

$$P_{n_1} \times \cdots \times P_{n_{s-2}} \times P_2^{s-1} \times \cdots \times P_4 \times P_2 \times P_2 \times P_2 = P_{n-6} \times P_2 \times P_2 \leq P_n P \leq P_n$$
if $n \equiv 0 \pmod{4}$ and $n_s = 4$, and
\[
P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_{2s-1} \times \cdots \times P_8 \times P_2 \times P_2 \times P_2 = P_{n-6} \times P_2 \times P_2 \leq P_n \ P \leq P_n
\]
if $n \equiv 0 \pmod{4}$ and $n_s > 4$. As a direct consequence of Knörr’s Theorem [14] we therefore get
\[
P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_{2s-1} \times \cdots \times P_8 \times P_2 \times P_2 \times P_2 \leq P_n \ P \leq P_n
\]
if $n \equiv 0 \pmod{4}$ and $n_s = 4$, and
\[
P_{n_1} \times \cdots \times P_{n_{s-1}} \times P_{2s-1} \times \cdots \times P_8 \times P_2 \times P_2 \times P_2 \leq P_n \ P \leq P_n
\]
if $n \equiv 0 \pmod{4}$ and $n_s > 4$, and $P = P_n$ if $n \equiv 2 \pmod{4}$. In any case, $P$ contains a $P_n$-conjugate of the elementary abelian group $E_n = ((2r - 1, 2r) \mid r = 1, \ldots, n/2)$. If $P$ were a proper subgroup of $P_n$ then we would have a maximal subgroup $R$ of $P_n$ such that $E_n \leq R \leq P_n$, and $Res^n_{P_n}(D)$ would have to be indecomposable. But this is a contradiction to Green’s Indecomposability Theorem [7]. Hence $P = P_n$ when $n$ is even. Moreover, $Res^n_{P_n}(D)$ then also has to be a source of $D$.

(4) This shows that it now remains to prove the following:

(a) If $5 < n \equiv 1 \pmod{4}$ with 2-adic expansion $n = 1 + \sum_{j=1}^{s} 2^{i_j}$, for some $s \geq 1$ and some $i_1 > \cdots > i_s = 2$ then $D$ is not relatively $R_i$-projective, for $i = 1, 2, 3$. Again $R_1, R_2$ and $R_3$ denote the maximal subgroups of $P_n$ appearing in paragraph (2) above.

(b) If $5 < n \equiv 1 \pmod{4}$ with 2-adic expansion $n = 1 + \sum_{j=1}^{s} 2^{i_j}$, for some $s \geq 1$ and some $i_1 > \cdots > i_s > 2$ then $D$ is not relatively $H$-projective. Here $H := \mathfrak{S}_{n-1-2s} \times \mathfrak{S}_{2s-1} \times \mathfrak{S}_{2s-1}.$

(c) If $n$ is even then $D$ restricts indecomposably to the elementary abelian subgroup $E_n = \langle (2r - 1, 2r) \mid r = 1, \ldots, n/2 \rangle$ of $\mathfrak{S}_n$.

In the next subsections we will prove (a), (b) and (c).

4.2 The case when $n$ is odd

In this subsection let $n > 5$ such that $n \equiv 1 \pmod{4}$. Moreover, let $n = 1 + \sum_{j=1}^{s} 2^{i_j}$ be the 2-adic expansion of $n$, for appropriate $s \geq 1$ and $i_1 > \ldots > i_s \geq 2$. Again we set $n_j := 2^{i_j}$, for $j = 1, \ldots, s$. We will prove parts (a) and (b) from the previous subsection which will then complete the proof of Theorem 1.2 in the case where $n$ is odd. In order to show (b) where $n_s > 4$, it will be important to know how $D$ restricts to Young subgroups of the form
\[
\mathfrak{S}_a \times \mathfrak{S}_b \times \mathfrak{S}_c,
\]
where $a + b + c = n - 1$ and $a \equiv b \equiv c \equiv 0 \pmod{4}$, with the possibility that $c = 0$. Furthermore, if $n_s = 4$ then we will determine the restriction of $D$ to
\[
\mathfrak{S}_a \times \mathfrak{S}_2 \times \mathfrak{S}_2
\]
where $a + 4 = n - 1$. From this we will then also derive (a).

We begin by describing the restriction of $D$ to $\mathfrak{S}_{n-1}$, and we use the notation introduced in Subsection 2.2.
Lemma 4.1. With the above notation the following hold:

(i) Let $\Omega$ denote the permutation basis of the the permutation $F \mathfrak{S}_{n-1}$-module $M^{(n-3,2)}$ consisting of the $(n-3,2)$-tabloids. Then the restriction of the Young module $Y^{(n-2,2)}$ to $\mathfrak{S}_{n-1}$ is isomorphic to $M^{(n-3,2)} = F\Omega$.

(ii) Let $U$ be the kernel $U_{F \Omega}$ of the augmentation map $F \Omega \to F$, and let $V$ be the trivial submodule of $F \Omega$ spanned by $\Omega^+$. Then the restriction of $V$ to $\mathfrak{S}_{n-1}$ is isomorphic to $U/V$. In particular, $U/V$ is indecomposable.

Proof. We have $\text{Res}_{\mathfrak{S}_{n-1}}^\mathfrak{S}_n (M^{(n-2,2)}) \cong M^{(n-2,1)} \oplus M^{(n-3,2)}$, by Mackey’s Decomposition Theorem, and

$$M^{(n-2,2)} \cong Y^{(n-1,1)} \oplus Y^{(n-2,2)} \cong S^{(n-1,1)} \oplus Y^{(n-2,2)},$$

by [8]. By the Krull-Schmidt Theorem, it suffices to show that $\text{Res}_{\mathfrak{S}_{n-1}}^\mathfrak{S}_n (S^{(n-1,1)}) \cong M^{(n-2,1)}$. But this is clear from the fact that $S^{(n-1,1)}$ has $F$-basis $\{\omega_1 - \omega_i \mid 1 \leq i \leq n-1\}$ where $\{\omega_1, \ldots, \omega_n\}$ is the permutation basis of the natural permutation $F \mathfrak{S}_n$-module $M^{(n-1,1)}$.

By [17], the Young module $Y^{(n-2,2)}$ is uniserial and

$$Y^{(n-2,2)} = U(F, D^{(n-2,2)}, F).$$

On the other hand, by [8], $F \Omega = M^{(n-3,2)} \cong Y^{(n-3,2)}$, and has thus a unique trivial submodule and a unique trivial quotient module, by [17]. This trivial submodule is then $V$, and the trivial quotient module is $F\Omega/U$. By [13] Th. 11.2.7, we also know that $\text{Res}_{\mathfrak{S}_{n-1}}^\mathfrak{S}_n (D^{(n-2,2)})$ is indecomposable. From this assertion (ii) follows, and the lemma is proved.

Remark 4.2. (a) We now set $N := n - 1$ and write $N = a + b + c$ with $a \equiv b \equiv c \equiv 0 \pmod{4}$ where we also allow $c = 0$. Moreover, we consider the (standard) Young subgroup $H := \mathfrak{S}_a \times \mathfrak{S}_b \times \mathfrak{S}_c$ of $\mathfrak{S}_n$. If $c = 0$ then

$$\text{Res}_H^\mathfrak{S}_n (M^{(N-2,2)}) = A \oplus B \oplus \Delta,$$

where $A \cong M^{(a-2,2)} \boxtimes F$, $B \cong F \boxtimes M^{(b-2,2)}$ and $\Delta \cong M^{(a-1,1)} \boxtimes M^{(b-1,1)}$. More precisely, if $\Omega := \{\{i, j\} \mid 1 \leq i < j \leq N\}$ is the permutation basis of $M^{(N-2,2)} = F\Omega$ consisting of the $(N-2,2)$-tabloids then $\Omega_A := \{\{i, j\} \mid 1 \leq i < j < a\}$ is a basis of $A$, $\Omega_B := \{\{i, j\} \mid a + 1 < i < j \leq a + b\}$ is a basis of $B$, and $\Omega_{\Delta} := \{\{i, j\} \mid 1 \leq i < a < j \leq a + b\}$ is a basis of $\Delta$. Again set $U := U_{F \Omega}$ and $V := V_{F \Omega}$.

(b) Similarly, if $c \neq 0$ then the restriction of $M^{(N-2,2)}$ to $H$ is of the form

$$A \oplus B \oplus C \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_3,$$

where $A \cong M^{(a-2,2)} \boxtimes F \boxtimes F$, $B \cong F \boxtimes M^{(b-2,2)} \boxtimes F$ and $C \cong F \boxtimes F \boxtimes M^{(c-2,2)}$, and where $\Delta_1 \cong M^{(a-1,1)} \boxtimes M^{(b-1,1)} \boxtimes F$, $\Delta_2 \cong M^{(a-1,1)} \boxtimes F \boxtimes M^{(c-1,1)}$ and $\Delta_3 \cong F \boxtimes M^{(b-1,1)} \boxtimes M^{(c-1,1)}$. Using the permutation basis $\Omega$ of $F \Omega = M^{(N-2,2)}$ above, we obtain the following permutation bases for $A, B, C$ and $\Delta_i$:

$$\Omega_A := \{\{i, j\} \mid 1 \leq i < j \leq a\}, \quad \Omega_B := \{\{i, j\} \mid a < i < j \leq a + b\},$$

$$\Omega_C := \{\{i, j\} \mid a + b < i < j \leq a + b + c\}, \quad \Omega_{\Delta_1} := \{\{i, j\} \mid 1 \leq i < a < j \leq a + b\},$$

$$\Omega_{\Delta_2} := \{\{i, j\} \mid 1 \leq i < a, a + b < j \leq a + b + c\}, \quad \Omega_{\Delta_3} := \{\{i, j\} \mid a < i < a + b < j \leq a + b + c\}.$$
Moreover, we then set $\Delta := \Delta_1 \oplus \Delta_2 \oplus \Delta_3$ which has permutation basis $\Omega_\Delta := \Omega_{\Delta_1} \cup \Omega_{\Delta_2} \cup \Omega_{\Delta_3}$. With this notation, the following holds:

**Lemma 4.3.** There is a decomposition of $FH$-modules $\text{Res}_H^S(F\Omega) = A \oplus B' \oplus C' \oplus \Delta'$, where

(i) $\Delta' \cong \Delta$, $\Delta' \subseteq U$ and $\Delta' \cap V = 0$.

(ii) $B' \cong B$, $C' \cong C$, and $V \subseteq A \oplus B' \oplus C'$.

In particular, $\text{Res}_H^S(U/V) \cong (U_{A \oplus B \oplus C}/V_{A \oplus B \oplus C}) \oplus \Delta$.

**Proof.** For each tabloid $\{i, j\} \in \Omega$, we define an element $\{i, j\}' \in F\Omega$ as follows:

\[
\{i, j\}' := \begin{cases} 
\{i, j\}, & \text{if } 1 \leq i < j \leq a, \\
\{i, j\} + \sum_{t=1}^{a} \{t, i\} + \{t, j\}, & \text{if } a < i < j \leq a + b, \\
\{i, j\} + \sum_{t=1}^{a} \{t, i\} + \{t, j\} + \sum_{s=b+1}^{a+b} \{s, i\} + \{s, j\}, & \text{if } a + b < i < j \leq a + b + c, \\
\{i, j\} + \sum_{t \neq t=1} \{i, t\}, & \text{if } a < i \leq a + b < j \leq a + b + c, \\
\{i, j\} + \sum_{t \neq t=1} \{i, t\}, & \text{if } 1 \leq i \leq a < j. 
\end{cases}
\]

With this we define an $F$-linear map

\[
\psi : A \oplus B \oplus C \oplus \Delta = F\Omega \longrightarrow F\Omega, \ {i, j} \longmapsto \{i, j\}'.
\]

After an appropriate re-labelling of the basis elements in $\Omega$, the corresponding matrix has block shape

\[
\Psi = \begin{pmatrix}
I & 0 & * \\
0 & \Phi & 0 \\
0 & * & I
\end{pmatrix}.
\]

If $C = 0$ then the diagonal blocks of $\Psi$ are identity matrices of size $\dim(A)$, $\dim(B)$ and $\dim(\Delta)$, respectively. If $C \neq 0$ then the diagonal blocks of $\Psi$ are $\Phi$ of size $\dim(B) + \dim(C) + \dim(\Delta)$, and identity matrices of size $\dim(A)$ and $\dim(\Delta_1) + \dim(\Delta_2)$, respectively. Moreover, $\Phi$ has also block shape:

\[
\Phi = \begin{pmatrix}
I & 0 & * \\
0 & I & 0 \\
0 & * & I
\end{pmatrix}
\]

with identity matrices of size $\dim(B)$, $\dim(C)$ and $\dim(\Delta_3)$, respectively, on the diagonal.

Thus, in either case, $\Psi$ is invertible, and $\psi$ is bijective. It is easily checked that, in fact, $\psi$ is an $FH$-isomorphism. Next we define the following $FH$-submodules of $F\Omega$: $B' := \psi(B) \cong B$ which has permutation basis $\Omega_{B'} := \psi(\Omega_B)$, $C' := \psi(C) \cong C$ which has permutation basis $\Omega_{C'} := \psi(\Omega_C)$, and $\Delta' := \psi(\Delta) \cong \Delta$ which has permutation basis $\Omega_{\Delta'} := \psi(\Omega_\Delta)$. Then we get

\[
A \oplus B \oplus C \oplus \Delta = \text{Res}_H^S(F\Omega) = \psi(F\Omega) = A \oplus B' \oplus C' \oplus \Delta'
\]

as $FH$-modules. By construction, $\Delta' \subseteq U$, and $\Omega^+ \notin \Delta'$, hence $\Delta' \cap V = 0$. We also claim that $\Omega^+ = \Omega_{A \oplus B' \oplus C'}^+ = (\Omega_A \cup \Omega_{B'} \cup \Omega_{C'})^+$. To see this, we note that

\[
\Omega_{A \oplus B' \oplus C'}^+ = \sum_{1 \leq i < j \leq a} \{i, j\} + \sum_{a < r < s \leq a + b} \{r, s\}' + \sum_{a + b < t < u \leq a + b + c} \{t, u\}'.
\]
and, in the sum on the right hand side, a tabloid \( \{ i, r \} \) with \( i \leq a < r \leq a + b \) occurs with multiplicity \(|\{ a < s \leq a + b \mid s \neq r \}|\), a tabloid \( \{ i, r \} \) with \( i \leq a \) and \( a + b < r \leq a + b + c \) occurs with multiplicity \(|\{ a + b < s \leq a + b + c \mid s \neq r \}|\), and a tabloid \( \{ i, r \} \) with \( a < i \leq a + b < r \leq a + b + c \) occurs with multiplicity \(|\{ a + b < s \leq a + b + c \mid s \neq r \}|\). All of these numbers are odd. So this shows \( V \subseteq A \oplus B' \oplus C' \).

With \( X := A \oplus B' \oplus C' \) we therefore now obtain \( \text{Res}_{H}^{\mathbb{E}_n}(U) = (U \cap X) \oplus \Delta' \), and \( \text{Res}_{H}^{\mathbb{E}_n}(U/V) \cong (U \cap X)/V \oplus \Delta' \). Using the permutation basis \( \Omega_{A \oplus B' \oplus C'} \) of \( A \oplus B' \oplus C' \) again, we deduce that \( U \cap X = U_{A \oplus B' \oplus C'} \), and, as we have just mentioned, \( \Omega^+ = \Omega_{A \oplus B' \oplus C'}^+ \).

Consequently,

\[
(X \cap U)/V = U_{A \oplus B' \oplus C'}/V_{A \oplus B' \oplus C'} \cong U_{A \oplus B' \oplus C'}/V_{A \oplus B' \oplus C'},
\]

by Lemma 2.6, and the assertion of the lemma is proved.

\[ \square \]

**Proposition 4.4.** With the notation as before, let \( W := U_{A \oplus B' \oplus C'}/V_{A \oplus B' \oplus C'} \). Then \( W \) is indecomposable as \( FH \)-module.

**Proof.** We apply Proposition 2.13 with \( \lambda = (a, b, c) \) and \( A = M^{(a-2,2)} \ltimes F \ltimes F \ltimes F \), \( B = F \ltimes M^{(b-2,2)} \ltimes F \ltimes F \) and \( C = F \ltimes F \ltimes M^{(c-2,2)} \). We only have to show that the hypothesis in Remark 2.7 are satisfied. This is the case for \( a, b, c \equiv 0 \pmod{4} \). Namely, applying Lemma 4.1 with \( n + 1 \) in place of \( n \), we deduce that \( \text{Res}_{H}^{\mathbb{E}_n}(U_{A}/V_{A}) \cong \text{Res}_{\mathbb{E}_n}^{\mathbb{E}_{n+1}}(D^{(a-1,2)}) \) is indecomposable. Moreover, it does not have any trivial composition factor, by [17]. Consequently, the same applies to the \( FH \)-module \( U_{A}/V_{A} \). Analogously, the \( FH \)-modules \( U_{B}/V_{B} \) and \( U_{C}/V_{C} \) are indecomposable, and neither of these modules has trivial composition factors.

\[ \square \]

By Lemma 4.3 and Proposition 4.4, we know now an indecomposable direct sum decomposition of the \( FH \)-module \( \text{Res}_{H}^{\mathbb{E}_n}(U/V) \cong \text{Res}_{H}^{\mathbb{E}_n}(D) \). This enables us to prove part (b) of Section 4.1 (4). That is, we obtain

**Proposition 4.5.** Let \( n = 1 + \sum_{j=1}^{s} 2^{i_j} \), for some \( s \geq 1 \) and some \( i_1 > \ldots > i_s > 2 \). Then the simple \( FH \) module \( D \) is not relatively projective with respect to the Young subgroup \( \mathbb{E}_{n-1-2^{i_s}} \times \mathbb{E}_{2^{i_{s-1}}} \times \mathbb{E}_{2^{i_1}} \).

**Proof.** Let \( H := \mathbb{E}_{n-1-2^{i_s}} \times \mathbb{E}_{2^{i_{s-1}}} \times \mathbb{E}_{2^{i_1}} \), let \( a := n - 1 - 2^{i_s} \), and let \( b := c := 2^{i_{s-1}} \). If \( s = 1 \) then, by Lemma 4.3 and Proposition 4.4, \( \text{Res}_{H}^{\mathbb{E}_n}(D) \) splits into the direct sum of two indecomposable modules of dimension \( 2a^2 \) and \( 2a^2 - 2a - 2 \neq 2a^2 \), respectively. If \( s \geq 2 \) then, by Lemma 4.3 and Proposition 4.4, \( \text{Res}_{H}^{\mathbb{E}_n}(D) \) splits into the direct sum of two indecomposable modules of dimension \( ab \) each, an indecomposable module of dimension \( b^2 \), and an indecomposable module of dimension \( (a^2 - a + 2b^2 - 2b - 4)/2 \).

Set \( K := \mathbb{E}_{n-1-2^{i_s}} \times (\mathbb{E}_{2^{i_{s-1}}} \rtimes \mathbb{E}_2) \). Then \( |\mathbb{E}_n : K| \) is odd, so that \( D \) is clearly relatively \( K \)-projective. Assume that \( D \) is also relatively \( H \)-projective. Then there is an indecomposable direct summand \( X \) of \( \text{Res}_{K}^{\mathbb{E}_n}(D) \) which is relatively \( H \)-projective and has common vertices with \( D \). Since \( |K : H| = 2 \), the restriction of this summand \( X \) to \( H \) in turn has to be the direct sum of two indecomposable and in \( K \) conjugate \( FH \)-modules, by [10], Thm. VII.9.3, each of which has common vertices with \( X \) and thus with \( D \). This immediately leads to a contradiction in the case where \( s = 1 \). If \( s \geq 2 \) then we can only have

\[
\text{Res}_{K}^{\mathbb{E}_n}(X) \cong (M^{(a-1,1)} \ltimes M^{(b-1,1)} \ltimes F) \oplus (M^{(a-1,1)} \ltimes F \ltimes M^{(b-1,1)}).
\]

19
But, by Grabmeier’s Theorem [6], [5], the $FH$-module $M^{(a-1,1)} \boxtimes M^{(b-1,1)} \boxtimes F \cong Y^{(a-1,1)} \boxtimes Y^{(b-1,1)} \boxtimes F$ has vertex $P_{a-2} \times P_{b-2} \times P_b \not\subseteq \mathfrak{S}_n \mathfrak{S}_n$. By Knörr’s Theorem [14], the module $D$ cannot be relatively $\mathfrak{S}_{n-2}$-projective, and we have a contradiction also in this case. This finishes the proof of the proposition. \hfill $\square$

**Remark 4.6.** Finally, we prove part (a) of Section 4.1. That is, we suppose that $n > 5$ with $2$-adic expansion $n = 1 + \sum_{j=1}^{s} 2^{i_j}$, for some $s \geq 1$ and some $i_1 > \ldots > i_s = 2$. We further set $n_j := 2^{i_j}$, for $j = 1, \ldots, s$, and show that $D$ is not relatively projective with respect to any of the following maximal subgroups of $P_n$:

$$
R_1 := P_{n_1} \times \cdots \times P_{n_{s-1}} \times B_{n_s},
R_2 := P_{n_1} \times \cdots \times P_{n_{s-1}} \times Q_{n_s},
R_3 := P_{n_1} \times \cdots \times P_{n_{s-1}} \times ((n-4, n-2, n-3, n-1)).
$$

First of all, in analogy to Lemma 4.3, we have the following more general result:

**Lemma 4.7.** Suppose that $5 < n \equiv 1 \pmod{4}$, and let $N := n - 1$. Let further $a := N - 4$ and $H := \mathfrak{S}_a \times \mathfrak{S}_2 \times \mathfrak{S}_2$. Then

(i) $\text{Res}_H^S(M^{(N-2,2)}) \cong (M^{(a-2,2)} \boxtimes F \boxtimes F) \oplus F \oplus (F \boxtimes M^{(12)} \boxtimes M^{(12)}) \oplus (M^{(a-1,1)} \boxtimes M^{(12)} \boxtimes F) \oplus (M^{(a-1,1)} \boxtimes F \boxtimes M^{(12)})$ is an indecomposable direct sum decomposition, and

(ii) $\text{Res}_H^S(D) \cong (M^{(a-2,2)} \boxtimes F \boxtimes F) \oplus (F \boxtimes M^{(12)} \boxtimes M^{(12)}) \oplus (M^{(a-1,1)} \boxtimes M^{(12)} \boxtimes F) \oplus (M^{(a-1,1)} \boxtimes F \boxtimes M^{(12)})$ is an indecomposable direct sum decomposition.

**Proof.** Let $\Omega$ be the permutation basis of $M^{(N-2,2)}$ consisting of the $(N-2,2)$-tableaux, and define

$$
\begin{align*}
\Omega_A & := \{ (i, j) \mid 1 \leq i < j \leq a \}, \\
\Omega_B & := \{ (a+1, a+2) \}, \\
\Omega_C & := \{ (a+3, a+4) \}, \\
\Omega_{\Delta_1} & := \{ (i, a+1) \mid 1 \leq i \leq a \}, \\
\Omega_{\Delta_2} & := \{ (i, a+3) \mid 1 \leq i \leq a \}, \\
\Omega_{\Delta_3} & := \{ (a+1, a+3), (a+1, a+4), (a+2, a+3), (a+2, a+4) \}.
\end{align*}
$$

Setting $A := F \Omega_A$, $B := F \Omega_B$, $C := F \Omega_C$, and $\Delta_k := F \Omega_{\Delta_k}$ for $k = 1, 2, 3$, we obtain

$$
M^{(N-2,2)} = F \Omega = A \oplus B \oplus C \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_3,
$$

(9)

as $F$-vector spaces. Moreover, all of these subspaces of $F \Omega$ are easily checked to be $FH$-submodules of $F \Omega$ such that

$$
\begin{align*}
A & \cong M^{(a-2,2)} \boxtimes F \boxtimes F, & B & \cong F \cong C, & \Delta_3 & \cong F \boxtimes M^{(12)} \boxtimes M^{(12)}, \\
\Delta_1 & \cong M^{(a-1,1)} \boxtimes M^{(12)} \boxtimes F, & \Delta_2 & \cong M^{(a-1,1)} \boxtimes F \boxtimes M^{(12)}. 
\end{align*}
$$

Since $a \equiv 0 \pmod{4}$, (9) is, by [8], in fact a decomposition of $\text{Res}_H^S(M^{(N-2,2)})$ into indecomposable direct summands, proving (i).
By Lemma 4.1, \( \text{Res}_{H}^{S_{n}}(D) \cong U_{F \Omega}/V_{F \Omega} \). For each tabloid \( \{i, j\} \in \Omega \setminus \{N - 1, N\} \), we define
\[
\xi(i, j) := \{i, j\} + \{N - 1, N\},
\]
and we set \( \Xi := \{\xi(i, j) \mid \{i, j\} \neq (a + 1, a + 2)\} \). Here \( \sim : U_{F \Omega} \to U_{F \Omega}/V_{F \Omega} \) denotes the canonical epimorphism. Thus, as mentioned in Remark 2.5, \( \Xi \) is an \( F \)-basis of \( U_{F \Omega}/V_{F \Omega} \).

With this notation, we obtain the following \( F \)-subspaces of \( U_{F \Omega}/V_{F \Omega} \):
\[
A' := F\{\xi(i, j) \mid 1 \leq i < j \leq a\},
\]
\[
\Delta'_1 := F\{\xi(i, a + 1), \xi(i, a + 2) \mid 1 \leq i \leq a\},
\]
\[
\Delta'_2 := F\{\xi(i, a + 3), \xi(i, a + 4) \mid 1 \leq i \leq a\},
\]
\[
\Delta'_3 := F\{\xi(a + 1, a + 3), \xi(a + 1, a + 4), \xi(a + 2, a + 3), \xi(a + 2, a + 4)\}
\]

and a vector space decomposition \( U_{F \Omega}/V_{F \Omega} = A' \oplus \Delta'_1 \oplus \Delta'_2 \oplus \Delta'_3 \). Furthermore, \( A', \Delta'_1, \Delta'_2 \)
and \( \Delta'_3 \) are actually \( FH \)-modules, and the \( F \)-linear map
\[
U_{F \Omega}/V_{F \Omega} \to M^{(N - 2, 2)}, \quad \xi(i, j) \mapsto \{i, j\}
\]
induces isomorphisms of \( FH \)-modules \( A' \cong A, \Delta'_1 \cong \Delta_1, \Delta'_2 \cong \Delta_2 \) and \( \Delta'_3 \cong \Delta_3 \). This proves (ii), and the assertion of the lemma follows. \( \square \)

**Corollary 4.8.** With the notation of Remark 4.6, the simple \( F \mathfrak{S}_{n} \)-module \( D \) is not relatively \( R_{i} \)-projective, for \( i = 1, 2, 3 \).

**Proof.** We set \( N := n - 1, a := n - 1 - 4 = n_{1} + \cdots + n_{s-1} \) and \( H := \mathfrak{S}_{a} \times \mathfrak{S}_{2} \times \mathfrak{S}_{2} \leq \mathfrak{S}_{n} \).

Then \( R_{1} \leq H \leq \mathfrak{S}_{a} \times (\mathfrak{S}_{2} \times \mathfrak{S}_{2}) =: K \).

Assume first that \( D \) is relatively \( R_{1} \)-projective. Since \( |\mathfrak{S}_{n} : K| \) is odd, \( D \) is relatively \( K \)-projective. Hence there is some indecomposable direct summand \( X \) of \( \text{Res}_{H}^{\mathfrak{S}_{n}}(D) \) which has common vertices with \( D \) and is also relatively \( H \)-projective. Since \( |K : H| = 2 \), from [10] Thm. VII.9.3 we deduce that \( \text{Res}_{H}^{K}(X) \) has to be the direct sum of two conjugate indecomposable modules both of which have common vertices with \( X \) and thus with \( D \). By Lemma 4.7, we know that
\[
\text{Res}_{H}^{\mathfrak{S}_{n}}(D) \cong A \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_3
\]
is an indecomposable direct sum decomposition, and \( \dim(A) = \binom{a}{2} \), \( \dim(\Delta_3) = 4 \) and \( \dim(\Delta_1) = 2a = \dim(\Delta_2) \). This forces \( \text{Res}_{H}^{K}(X) \cong \Delta_1 \oplus \Delta_2 \). But \( \Delta_1 \cong M^{(a-1,1)} \oplus M^{(1,2)} \oplus F \) and has thus vertex \( P_{a-2} \times 1 \times P_{2} \leq \mathfrak{S}_{n} \), \( \mathfrak{S}_{n-4} \), by Grabmeier’s Theorem [6], [5].

On the other hand, in consequence of Knörr’s Theorem [14], \( D \) cannot be relatively \( \mathfrak{S}_{n-2} \)-projective, a contradiction. Thus \( D \) is not relatively \( H \)-projective and then not relatively \( R_{1} \)-projective either.

Furthermore, the decomposition (1) above also shows that \( A \mid \text{Res}_{H}^{\mathfrak{S}_{n}}(D) \). Since \( a \equiv 0 \) (mod 4), we have \( A \cong M^{(a-2,2)} \oplus F \oplus F \cong Y^{(a-2,2)} \oplus F \oplus F \), by [8], and \( A \) has vertex \( P_{a-4} \times P_{2} \times P_{2} \times P_{2} \times P_{2} \), by Grabmeier’s Theorem [6], [5]. Hence if \( P \leq P_{n} \) is a vertex of \( D \) then also
\[
E_{n-1} := \langle (2r - 1, 2r) \mid r = 1, \ldots, (n - 1)/2 \rangle \leq P_{a-4} \times P_{2} \times P_{2} \times P_{2} \times P_{2} \leq \mathfrak{S}_{n}, \quad P \leq P_{n}.
\]

But neither \( R_{2} \) nor \( R_{3} \) contains a subgroup conjugate to \( E_{n-1} \). Consequently, \( D \) is not relatively \( R_{i} \)-projective, for \( i = 2, 3 \), and the corollary is proved. \( \square \)

Corollary 4.8 completes the proof of Theorem 1.2 in the case where \( n \) is odd.
4.3 The case when \( n \) is even

From now on, let \( n \geq 6 \) be even. In order to complete our proof of Theorem 1.2, we need to show that \( \text{Res}^{S_n}_{E_n}(D) \) is indecomposable where \( E_n := \langle (2r - 1, 2r) \mid r = 1, \ldots, n/2 \rangle \). We begin our investigation of \( D = S/Rad(S) \) and its restriction to \( E_n \) by giving a convenient \( F \)-basis of \( D \). This will be done in the following two propositions.

**Proposition 4.9.** Let \( n \geq 6 \) be even. For \( a \in \{1, \ldots, n\} \), we define \( \gamma_a := \sum_{i \neq a \neq j} \{i, j\} \in M^{(n-2,2)} \).

(i) If \( n \equiv 2 \pmod{4} \) then \( \text{Rad}(S) = \left\{ \sum_{a=1}^{n} c_a \gamma_a \mid c_a \in F \right\} \).

(ii) If \( n \equiv 0 \pmod{4} \) then \( \text{Rad}(S) = \left\{ \sum_{a=1}^{n} c_a \gamma_a \mid c_a \in F, \sum_{a=1}^{n} c_a = 0 \right\} \).

**Proof.** Suppose first that \( n \equiv 2 \pmod{4} \). From [8], we deduce that \( M^{(n-2,2)} \cong F \oplus Y^{(n-2,2)} \). Moreover, by [17], the Young module \( Y^{(n-2,2)} \) is uniserial with composition series

\[ Y^{(n-2,2)} = U(D^{(n-1,1)}), F, D^{(n-2,2)}), F, D^{(n-1,1)} \, \cdots \]

In particular, \( Y^{(n-2,2)} \) is isomorphic to the kernel of the augmentation map \( \nu : M^{(n-2,2)} \to F \), mapping each tabloid to 1. We now define

\[ \rho : M^{(n-1,1)} \to M^{(n-2,2)}, \{a\} \mapsto \sum_{i \neq a \neq j} \{i, j\} = \gamma_a, \]

for \( a \in \{1, \ldots, n\} \). This is a non-zero homomorphism of \( F \mathcal{S}_n \)-modules. For each \( a \in \{1, \ldots, n\} \), the sum \( \sum_{i \neq a \neq j} \{i, j\} \) has \( \binom{n-1}{2} \equiv 0 \pmod{2} \) terms so that \( \text{im}(\rho) \subseteq \text{ker}(\nu) = Y^{(n-2,2)} \). It remains to show that, in fact, \( \text{im}(\rho) = \text{Rad}(S) \). Since \( n \) is even, the permutation module \( M^{(n-1,1)} \) is uniserial, with composition series \( M^{(n-1,1)} \supset S^{(n-1,1)} \supset F \supset 0 \), by [11], Ex. 5.1. Thus either \( \text{ker}(\rho) \cong F \) or \( \text{ker}(\rho) \cong S^{(n-1,1)} \). In the latter case, we would have \( \rho(\{a\} + \{b\}) = 0 \), for all \( a, b \in \{1, \ldots, n\} \), which is obviously not the case. Therefore, \( \text{ker}(\rho) \cong F \), so that \( \text{im}(\rho) \) has composition factors \( D^{(n-1,1)} \) and \( F \). More precisely, \( \text{im}(\rho) \) is indecomposable of composition length 2, with head isomorphic to \( F \) and socle isomorphic to \( D^{(n-1,1)} \). Since \( Y^{(n-2,2)} \) is uniserial, this forces \( \text{im}(\rho) = \text{Rad}(S) \), and (i) follows.

Next, let \( n \equiv 0 \pmod{4} \). Then from [8] we get \( M^{(n-2,2)} = Y^{(n-2,2)} \). As above, we again consider the augmentation map \( \nu : M^{(n-2,2)} \to F \) and the homomorphism \( \rho : M^{(n-1,1)} \to M^{(n-2,2)}, \{a\} \mapsto \gamma_a := \sum_{i \neq a \neq j} \{i, j\} \). Since each \((n-2,2)\)-polytabloid is the sum of four \((n-2,2)\)-tabloids, we have \( S \subseteq \text{ker}(\nu) \). As in the previous case, also here we deduce that \( \text{im}(\rho) \) is indecomposable of composition length 2, with socle isomorphic to \( D^{(n-1,1)} \) and head isomorphic to \( F \). Furthermore, the definitions of \( \rho \) and \( \nu \) immediately yield \( 0 \neq \text{im}(\rho) \cap \text{ker}(\nu) \neq \text{im}(\rho) \). Hence \( \text{im}(\rho) \cap \text{ker}(\nu) \cong D^{(n-1,1)} \). By [17], \( M^{(n-2,2)} \) has a unique submodule isomorphic to \( D^{(n-1,1)} \), namely \( \text{Rad}(S) \). Thus it now remains to prove

\[ \text{ker}(\nu) \cap \text{im}(\rho) = \left\{ \sum_{a=1}^{n} c_a \gamma_a \mid c_a \in F, \sum_{a=1}^{n} c_a = 0 \right\}. \]

For this let \( x \in \text{ker}(\nu) \cap \text{im}(\rho) \), that is \( x = \sum_{a=1}^{n} c_a \gamma_a \) where \( c_a \in F \) for \( a = 1, \ldots, n \), and \( \sum_{a=1}^{n} c_a \nu(\gamma_a) = 0 \). Since, for each \( a = 1, \ldots, n \), the sum \( \gamma_a \) has \( \binom{n-1}{2} \equiv 1 \pmod{2} \) terms, we get \( 0 = \sum_{a=1}^{n} c_a \nu(\gamma_a) = \sum_{a=1}^{n} c_a \). Conversely, the set on the right hand side of (11) is of course contained in \( \text{ker}(\nu) \cap \text{im}(\rho) \). This completes the proof of (ii). \( \square \)
**Proposition 4.10.** Let \( n \geq 6 \) be even, and let \(- : S \rightarrow S/\text{Rad}(S) = D\) be the canonical epimorphism. For \( j \in \{4, \ldots, n\} \) and \( i \in \{2, \ldots, j-1\} \), we set \( e(ij) =: f(ij) \).

(i) If \( n \equiv 0 \pmod{4} \) then

\[
\mathfrak{B}_n := \{ f(ij) \mid 4 \leq j \leq n - 1, 2 \leq i \leq j - 1 \}
\]

is an \( F \)-basis of \( D \).

(ii) If \( n \equiv 2 \pmod{4} \) then

\[
\mathfrak{B}_n := \{ f(ij) \mid 4 \leq j \leq n - 1, 2 \leq i \leq \min\{j - 1, n - 3\} \}
\]

is an \( F \)-basis of \( D \).

**Proof.** Let \( n \equiv 0 \pmod{4} \). Then \( \dim(D) = \dim(S) - \dim(D^{(n-1,1)}) = \binom{n}{2} - n - (n - 2) = |\mathfrak{B}_n| \). It thus suffices to prove that \( \mathfrak{B}_n \) is \( F \)-linearly independent. For this, suppose that

\[
0 = \sum_{i,j} r(ij) f(ij)
\]

where \( r(ij) \in F \) for \( j = 4, \ldots, n - 1, i = 2, \ldots, j - 1 \). By Proposition 4.9, there are \( r_1, \ldots, r_n \in F \) such that

\[
\sum_{i,j} r(ij) e(ij) = \sum_{a=1}^{n} r_a \gamma_a \in \text{Rad}(S), \tag{12}
\]

that is \( \sum_{a=1}^{n} r_a = 0 \). We show that this implies \( r(ij) = 0 \), for all admissible \( i \) and \( j \). For this, let \( l \in \{2, \ldots, n\} \) and \( k \in \{1, \ldots, l-1\} \). We compare the coefficients of the tabloid \( \{k, l\} \) in both sums of (12). Consider the cases where \( j \in \{n-1, n\} \) first. Then we have

<table>
<thead>
<tr>
<th>tabloid</th>
<th>LHS</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>{k, n} for 1 \leq k \leq n - 1</td>
<td>0</td>
<td>\sum_{a \neq k \neq n} r_a</td>
</tr>
<tr>
<td>{1, n - 1}</td>
<td>\sum_{i=2}^{n-2} r(i, n - 1)</td>
<td>\sum_{a \neq n-1} r_a</td>
</tr>
<tr>
<td>{k, n - 1} for 2 \leq k \leq n - 2</td>
<td>r(k, n - 1)</td>
<td>\sum_{a \neq n-1} r_a</td>
</tr>
</tbody>
</table>

Since \( \sum_{a=1}^{n} r_a = 0 \), this implies \( r := r_1 = \ldots = r_n \), and \( r(k, n - 1) = (n - 2)r = 0 \) for \( k = 2, \ldots, n - 2 \). We may now suppose that \( 4 \leq l < n - 1 \) and argue with reverse induction on \( l \), in order to show that \( r(k, l) = 0 \), for all \( k = 2, \ldots, l - 1 \). Again we compare the coefficients of the tabloids on both sides of (12). This yields:

<table>
<thead>
<tr>
<th>tabloid</th>
<th>LHS</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>{k, l} for 3 \leq k \leq l - 1</td>
<td>r(k, l)</td>
<td>(n - 2)r = 0</td>
</tr>
<tr>
<td>{2, l}</td>
<td>r(2, l) + \sum_{j=l+1}^{n-1} r(l, j)</td>
<td>(n - 2)r = 0</td>
</tr>
<tr>
<td>{1, l}</td>
<td>\sum_{i=2}^{l-1} r(i, l)</td>
<td>(n - 2)r = 0</td>
</tr>
</tbody>
</table>

23
Since, by induction, \( r(l, l + 1) = \ldots = r(l, n - 1) = 0 \), we then also get \( r(k, l) = 0 \), for \( k = 2, \ldots, l - 1 \). This proves (i).

Now let \( n \equiv 2 \pmod{4} \). Then \( \text{dim}(D) = \text{dim}(S) - \text{dim}(D^{(n-1, 1)}) - 1 = \binom{n}{2} - n - (n - 2) - 1 = |\mathfrak{B}_n| \) so that also in this case we only need to show that \( \mathfrak{B}_n \) is \( F \)-linearly independent. For this, let \( 0 = \sum_{i,j} r(ij)f(ij) \) where \( r(ij) \in F \) for \( j = 4, \ldots, n - 1 \) and \( i = 2, \ldots, \min\{n - 3, j - 1\} \). Hence, by Proposition 4.9, there are \( r_1, \ldots, r_n \in F \) such that

\[
\sum_{i,j} r(ij)f(ij) = \sum_{a=1}^{n} r_a \gamma_a \in \text{Rad}(S). \tag{13}
\]

As above, for \( l = 2, \ldots, n \) and \( k = 1, \ldots, l - 1 \), we compare the coefficients of the tabloid \( \{k, l\} \) on both sides of (13), in order to show \( r(ij) = 0 \), for \( j = 4, \ldots, n - 1 \) and \( i = 2, \ldots, \min\{n - 3, j - 1\} \). Again we argue with reverse induction on \( l \). If \( l \geq n - 2 \) then we have

<table>
<thead>
<tr>
<th>tabloid</th>
<th>LHS</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>{k, n} for ( 1 \leq k \leq n - 1 )</td>
<td>0</td>
<td>( \sum_{k \neq a \neq n} r_a )</td>
</tr>
<tr>
<td>{1, n - 1}</td>
<td>( \sum_{i=2}^{n-3} r(i, n - 1) )</td>
<td>( \sum_{1 \neq a \neq n-1} r_a )</td>
</tr>
<tr>
<td>{k, n - 1} for ( 2 \leq k \leq n - 3 )</td>
<td>( r(k, n - 1) )</td>
<td>( \sum_{k \neq a \neq n-1} r_a )</td>
</tr>
<tr>
<td>{n - 2, n - 1}</td>
<td>0</td>
<td>( \sum_{n-2 \neq a \neq n-1} r_a )</td>
</tr>
<tr>
<td>{1, n - 2}</td>
<td>( \sum_{i=2}^{n-3} r(i, n - 2) )</td>
<td>( \sum_{1 \neq a \neq n-2} r_a )</td>
</tr>
<tr>
<td>{k, n - 2} for ( 2 \leq k \leq n - 3 )</td>
<td>( r(k, n - 2) )</td>
<td>( \sum_{k \neq a \neq n-2} r_a )</td>
</tr>
</tbody>
</table>

This implies \( r := r_1 = r_2 = \ldots = r_n \) and \( r(k, n - 2) = r(k, n - 1) = 0 \), for \( k = 2, \ldots, n - 3 \). We may now suppose that \( 4 \leq l < n - 2 \), and that we have already proved \( r(k, j) = 0 \), for \( j = l + 1, \ldots, n - 1 \) and \( k = 2, \ldots, \min\{j - 1, n - 3\} \). Comparing coefficients on both sides of (13) we obtain:

<table>
<thead>
<tr>
<th>tabloid</th>
<th>LHS</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>{k, l} for ( 3 \leq k \leq l - 1 )</td>
<td>( r(k, l) )</td>
<td>0</td>
</tr>
<tr>
<td>{2, l}</td>
<td>( r(2, l) + \sum_{j=l+1}^{n-1} r(l, j) )</td>
<td>0</td>
</tr>
<tr>
<td>{1, l}</td>
<td>( \sum_{i=2}^{l-1} r(i, l) )</td>
<td>0</td>
</tr>
</tbody>
</table>

Therefore, we also get \( 0 = r(2, l) = r(3, l) = \ldots = r(l - 1, l) \), and assertion (ii) follows.

Using the \( F \)-basis \( \mathfrak{B}_n \) of \( D \) given in Proposition 4.10, we can describe the action of the elementary abelian group \( E_n \) on \( D \).
Proposition 4.11. Let $n \geq 6$ be even, and let $E_n := \langle (2r-1, 2r) \mid r = 1, \ldots, n/2 \rangle$. If $n \equiv 0 \pmod{4}$ then let $i \in \{2, \ldots, n-2\}$, and if $n \equiv 2 \pmod{4}$ then let $i \in \{2, \ldots, n-3\}$. Furthermore, set $\Sigma'_i := \sum_{i=0}^{n-1} f(i) + \sum_{i=3}^{n-3} f(i)$. Then,\[ \{ {\text{is an}} \quad \}
\]

(i) If $n \equiv 2 \pmod{4}$ and $r = (n-2)/2$ then
\[ (2r-1, 2r) f(n-3, n-1) = \sum_{j=4}^{n-1} f(2j) + \sum_{i=3}^{n-3} f(i) + \sum_{i=3}^{n-3} f(i, n-1). \]

(ii) If $r = n/2$ then
\[ (2r-1, 2r) f(i, n-1) = \begin{cases} \sum_{i=4}^{n-1} f(2l) + \Sigma', & \text{if $i$ is odd,} \\ \sum_{i=4}^{n-1} f(2l) + \Sigma', & \text{if $i$ is even.} \end{cases} \]

(iii) Otherwise, suppose $(2r-1, 2r) e(ij) = \sum_{a,b} r(ab)e(ab)$. Then $r(ab) = 0$ whenever $f(ab) \notin \mathcal{B}_n$. In particular, $(2r-1, 2r) f(ij) = \sum_{r(ab)\neq 0} r(ab)f(ab)$.

**Proof.** Assertion (iii) follows immediately from Proposition 3.1. Now let $n \equiv 2 \pmod{4}$. Then $(n-3, n-2)e(n-3, n-1) = e(n-2, n-1)$, by Proposition 3.1. Moreover, one checks that
\[ e(n-2, n-1) + \gamma_n = \sum_{j=4}^{n-1} e(2j) + \sum_{i=3}^{n-3} e(ij) + \sum_{i=3}^{n-3} e(i, n-1). \]

From this and Proposition 4.10 we deduce (i).

In order to prove (ii), let $n \geq 6$ be even. Furthermore, if $n \equiv 0 \pmod{4}$ then let $i \in \{2, \ldots, n-2\}$, and if $n \equiv 2 \pmod{4}$ then let $i \in \{2, \ldots, n-3\}$, and set
\[ \Sigma_i := \sum_{l=i+1}^{n-1} e(il) + \sum_{l=3}^{n-3} e(il). \]

Then,
\[ \Sigma_i = \begin{cases} \gamma_1 + \gamma_i, & \text{if $i \in \{2, 3, 4\}$,} \\ \gamma_1 + \gamma_i + \sum_{j=4}^{i-1} e(2j), & \text{if $i \geq 5$ odd,} \\ \gamma_1 + \gamma_i + \sum_{j=4}^{i-1} e(2j), & \text{if $i \geq 6$ even.} \end{cases} \]

Together with Proposition 3.1 and Proposition 4.10 this gives (ii).

**Proposition 4.12.** Let $n \geq 6$ be even. In the notation of Proposition 4.10,
\[ \mathcal{B} := \{ f(24) \} \cup \{ f(2, 2l-1) + f(2, 2l) \mid 3 \leq l \leq (n-2)/2 \} \]
\[ \cup \{ \sum_{i=2k-1}^{2k} \sum_{j=2l-1}^{2l} f(ij) \mid 2 \leq k \leq (n-4)/2, k + 1 \leq l \leq (n-2)/2 \} \]

is an $F$-basis of $\text{Soc}(\text{Res}_{E_n}^D)$.
Proof. By Proposition 4.11, all the elements in \( \mathfrak{B} \) are fixed under \( E_n \), and hence belong to \( \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D)) \). Moreover, \( \mathfrak{B} \) is linearly independent. Conversely, let \( x = \sum_{i,j} r(ij) f(ij) \in \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D)) \) with \( r(ij) \in F \), for all admissible \( i \) and \( j \). We show that we can write \( x \) as an \( F \)-linear combination of elements in \( \mathfrak{B} \). For this, we will distinguish between the cases \( n \equiv 2 \) (mod 4) and \( n \equiv 0 \) (mod 4).

Let \( l \in \{4,\ldots,n-2\} \) and \( k \in \{2,\ldots,l-1\} \). We write

\[
\varphi(f(kl)) = \sum_{i,j} r(ij) f(ij),
\]

with \( r(ij) \in F \) for all admissible \( i \) and \( j \). Consider the subspace \( V := F\{f(ij) \mid j = 4,\ldots,n-2, i = 2,\ldots,j-1\} \) of \( D \). By Propositions 3.1 and 4.11, this is an \( FE_{n-2} \)-submodule of \( D \), and is isomorphic to \( \text{Res}_{E_{n-2}}^{\mathfrak{S}_{n-2}}(S(n-4,2)) \). An explicit isomorphism is given by mapping \( f(ij) \) to the standard \( (n-4,2) \)-polytabloid \( e(ij) \), for \( j = 4,\ldots,n-2, i = 2,\ldots,j-1 \). Hence, by Proposition 4.12, \( \mathfrak{B} \) is an \( F \)-basis of the socle of the \( FE_{n-2} \)-module \( V \). Notice that the transposition \( (n-1,n) \) fixes every element in \( V \) so that \( V \) is, in fact, an \( FE_n \)-submodule of \( D \).

Now suppose that \( n \equiv 2 \) (mod 4), and let \( x = \sum_{i,j} r(ij) f(ij) \in \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D)) \), with \( r(ij) \) for \( j = 4,\ldots,n-1 \) and \( i = 2,\ldots,\min(j-1,n-3) \). Since \( (n-1,n)x = x \), for \( i = 2,\ldots,n-3 \), the coefficients of \( f(i,n-2) \) in \( x \) and \( (n-1,n)x \) must coincide. This yields \( r(i,n-2) = r(i,n-2) + r(i,n-1) \), for \( i = 2,\ldots,n-3 \), and thus

\[
0 = r(2,n-1) = r(3,n-1) = \ldots = r(n-3,n-1).
\]

This shows that \( x \in V \) so that also \( x \in \text{Soc}(V) = F\mathfrak{B} \). This proves the assertion of the proposition in the case \( n \equiv 2 \) (mod 4).

Now suppose that \( n \equiv 0 \) (mod 4). Let \( V \) be as above, and let further \( W := F\{f(i,n-1) \mid i = 2,\ldots,n-2\} \). Then, by Proposition 4.11, \( W \) is also an \( FE_{n-2} \)-submodule of \( D \), and we have

\[
\text{Res}_{E_{n-2}}^{\mathfrak{S}_n}(D) = V \oplus W.
\]

As we have seen above, \( V \) is already an \( FE_n \)-submodule of \( D \). The \( FE_n \)-socle of \( V \) equals the \( FE_{n-2} \)-socle, and has thus basis \( \mathfrak{B} \). Therefore it suffices to show that the socle of \( \text{Res}_{E_n}^{\mathfrak{S}_n}(D) \) is contained in the socle of the \( FE_n \)-module \( V \). For this, let \( x \in \text{Soc}(\text{Res}_{E_n}^{\mathfrak{S}_n}(D)) \). We may write \( x = x_0 + x_1 \), for some \( x_0 \) in the socle of the \( FE_{n-2} \)-module \( V \) and some \( x_1 \) in the socle of the \( FE_{n-2} \)-module \( W \). By our previous considerations, both \( x \) and \( x_0 \) are fixed by \( E_n \), and so also \( x_1 \) is fixed by \( E_n \). We show that this implies \( x_1 = 0 \). We may write \( x_1 = \sum_{i=2}^{n-2} r_i f(i,n-1) \), for some \( r_i \in F \) and \( i = 2,\ldots,n-2 \). Then

\[
0 = (1 + (1,2))x_1 = (\sum_{i=3}^{n-2} r_i)f(2,n-1) + \sum_{j=4}^{n-2} r_j f(2j).
\]

It follows that \( \sum_{i=3}^{n-2} r_i = 0 = r_4 = r_5 = \ldots = r_{n-2} \). Thus also \( r_3 = 0 \), and we are left with \( x_1 = r_2 f(2,n-1) \). Consequently,

\[
r_2 f(2,n-1) = x_1 = (n-1,n)x_1 = r_2 \sum_{j=4}^{n-1} f(2,j).
\]
This now shows that \( r_2 = 0 \) so that \( x_1 = 0 \), proving the assertion also in the case that \( n \equiv 0 \) (mod 4).

**Lemma 4.13.** Let \( n \geq 6 \) be even, and let \( E_n := \langle (2r - 1, 2r) \mid r = 1, \ldots, n/2 \rangle \). Then \( \text{Res}_{E_n}^{E_n}(D) \) is indecomposable.

**Proof.** In the case that \( n = 6 \) the assertion trivially holds, since then \( \text{Soc}(\text{Res}_{E_n}^{E_n}(D)) \cong F \), by Proposition 4.12. Therefore, for the remainder of the proof we may suppose that \( n \geq 8 \). We show that the endomorphism algebra \( \text{End}_{F_{E_n}}(D) \) is local. For this, let \( \varphi^2 = \varphi \in \text{End}_{F_{E_n}}(D) \).

We need to show that \( \varphi \in \{0, 1\} \).

As in the proof of Proposition 4.12, consider the \( FE_n \)-submodule \( V := F\{f(ij) \mid j = 4, \ldots, n-2, i = 2, \ldots, j-1\} \) of \( D \) which, as an \( FE_{n-2} \)-module is isomorphic to \( \text{Res}_{E_{n-2}}^{E_n}(S^{(n-4,2)}) \). Since \((n-1,n)\) acts trivially on \( V \), it acts trivially on \( \varphi(V) \) as well. Comparing the coefficients of \( f(i,n-2) \), for \( i = 2, \ldots, n-3 \), in \( \varphi(f(kl)) \) and in \((n-1,n)\varphi(f(kl)) \), we obtain:

\[
\begin{align*}
  r(i,n-2) &= r(i,n-2) + r(i,n-1) + r(n-2,n-1), \quad \text{if } n \equiv 0 \pmod{4}, \\
  r(i,n-2) &= r(i,n-2) + r(i,n-1), \quad \text{if } n \equiv 2 \pmod{4}.
\end{align*}
\]

Hence, \( r(2,n-1) = r(3,n-1) = \ldots = r(n-3,n-1) = 0 \) if \( n \equiv 2 \pmod{4} \), and \( r_{kl} := r(2,n-1) = r(3,n-1) = \ldots = r(n-2,n-1) \) if \( n \equiv 0 \pmod{4} \). In particular, for \( n \equiv 2 \pmod{4} \) this means \( \varphi(V) \subseteq V \). That is, in this case \( \psi := \varphi|_V \) is an idempotent in \( \text{End}_{F_{E_{n-2}}}(V) \cong \text{End}_{F_{E_{n-2}}}(S^{(n-4,2)}) \). Thus either \( \psi = 0 \) or \( \psi = 1 \), by Lemma 3.5. If \( \psi = 0 \) then, since \( \text{Soc}(\text{Res}_{E_{n-2}}^{E_n}(V)) = \text{Soc}(V) = \text{Soc}(\text{Res}_{E_n}^{E_n}(D)) \), we have \( 0 = \varphi(\text{Soc}(V)) = \varphi(\text{Soc}(\text{Res}_{E_n}^{E_n}(D))) \), and hence \( \varphi = 0 \). If \( \psi \) is the identity map on \( V \) then we analogously get \( 1 - \varphi = 0 \). This proves the lemma in the case when \( n \equiv 2 \pmod{4} \).

Therefore, from now on we may suppose that \( n \equiv 0 \pmod{4} \). We suppose further that \( \varphi \neq 1 \), and we show that \( \varphi(\text{Soc}(\text{Res}_{E_n}^{E_n}(D))) \neq 0 \). Our calculations will be similar to those done in the proof of Lemma 3.5, so we will omit some details here. By Proposition 4.12, it suffices to show that

\[
\begin{align*}
  (i) \quad &\varphi(f(24)) = 0, \\
  (ii) \quad &\varphi(f(2,2l-1)) + \varphi(f(2,2l)) = 0, \text{ for } l = 3, \ldots, (n-2)/2, \\
  (iii) \quad &\varphi(f(2k-1,2l-1) + f(2k-1,2l) + f(2k,2l-1) + f(2k,2l)) = 0, \text{ for } k = 2, \ldots, (n-4)/2 \text{ and } l = k+1, \ldots, (n-2)/2.
\end{align*}
\]

By the first part of the proof, we already know that, for all \( 2 \leq k < l \leq n - 2 \), we have

\[
\varphi(f(kl)) = v_{kl} + r_{kl} \sum_{j=2}^{n-2} f(j,n-1),
\]

where \( v_{kl} \in V \) and \( r_{kl} \in F \) is as above. We claim that

\[
(*) \quad r_{2l} = 0,
\]

for all \( 4 \leq l \leq n - 2 \). Namely, \( (1 + (1,2))f(3l) = f(2l) \), and therefore \( \varphi(f(2,l)) = (1 + (1,2))\varphi(f(3l)) \) lies in \( F\{f(2j) \mid 4 \leq j \leq n - 1\} \). So the coefficient of \( f(3(3n-1)) \) is
0, and hence \( r_{2l} = 0 \).

We now list the properties we use in proving (i)-(iii).

(A) \((1 + (1, 2))f(3j) = f(2j)\) for \(4 \leq j \leq n - 1\), and \((1 + (1, 2))f(i, j) = f(2, i) + f(2, j)\) for \(3 < i < j \leq n - 1\).

(B) Suppose that \(3 \leq l \leq n/2\). Then \((2l - 1, 2l)\) fixes \(f(2, j)\) for \(2l - 1 \neq j \neq 2l\).

(C) For \(3 \leq l \) and \(2l - 1 \leq n - 1\), we have \((1 + (3, 4))f(2, 2l - 1) = f(24)\).

(D) For \(6 \leq 2l \leq n - 2\), the permutation \((2l - 1, 2l)\) swaps \(f(2, 2l - 1)\) and \(f(2, 2l)\).

We fix some \(l \in \{3, \ldots, (n - 2)/2\}\), and consider \(\varphi(f(2, 2l - 1))\). Since \(f(2, 2l - 1)\), by (A), is contained in \((1 + (1, 2))D \subseteq F\{f(j) \mid 4 \leq j \leq n - 1\}\), we can write

\[
\varphi(f(2, 2l - 1)) = \sum_{j=4}^{n-1} \delta_j f(2, j),
\]

for some \(\delta_j \in F\). First note that \(\delta_{n-1} = 0\), by (*). Furthermore, property (B) shows that, for \(r \notin \{1, 2, l\}\), the permutation \((2r - 1, 2r)\) fixes \(f(2, 2l - 1)\). Therefore it fixes \(\varphi(f(2, 2l - 1))\) which means that the coefficients of \(f(2, 2r - 1)\) and \(f(2, 2r)\) in (**) are equal. So

\[
\delta_{2r-1} = \delta_{2r}, \quad \text{for } r \notin \{1, 2, l\}.
\]

Using this and applying \((1 + (3, 4))\) to (**) we get, by (C), that

\[
\varphi(f(2, 4)) = f(2, 4) \sum_{j=5}^{n-2} \delta_j = f(2, 4)(\delta_{2l-1} + \delta_{2l}).
\]

This shows that \(f(2, 4)\) is an eigenvector of \(\varphi\). Since \(\varphi\) is a projection, the corresponding eigenvalue is either 0 or 1. We continue with the case \(\varphi(f(2, 4)) = 0\), and will show that then \(\varphi = 0\). This then gives free that in the case \(\varphi(f(2, 4)) = f(2, 4)\) we must have \(\varphi = 1\) which was excluded.

So part (i) of our statement above holds. Furthermore, we also have \(\delta_{2l-1} = \delta_{2l}\), and hence \(\delta_{2r-1} = \delta_{2r}\), for all \(3 \leq r \leq (n - 2)/2\). Now using the fact that \((2l - 1, 2l)f(2, 2l - 1) = f(2, 2l)\) we get

\[
\varphi(f(2, 2l - 1)) = \varphi(f(2, 2l)).
\]

This proves part (ii). Actually, we have something stronger, namely \(\varphi(f(2, 2l - 1)) = \varphi(f(2, 2l)) = 0\). To see this, recall that \(\delta_{2r-1} = \delta_{2r}\), for \(3 \leq r \leq (n - 2)/2\) so that (**) implies

\[
\varphi(f(2, 2l - 1)) = \varphi^2(f(2, 2l - 1)) = \delta_4 \varphi(f(2, 4)) = 0.
\]

Lastly, we prove (iii). For this, let \(2 \leq k < l \leq (n - 2)/2\), and write

\[
\varphi(f(2k - 1, 2l - 1)) = \sum_{i,j} r(ij)f(ij),
\]

28
for some $r(ij) \in F$ and all admissible $i$ and $j$. We know, by (A), that $(1 + (1, 2))f(2k - 1, 2l - 1) = f(2, 2k - 1) + f(2, 2l - 1)$ if $k > 2$, and $(1 + (1, 2))f(2k - 1, 2l - 1) = f(2, 2l - 1)$ if $k = 2$. Hence, by (ii),

\[ 0 = (1 + (1, 2))\varphi(f(2k - 1, 2l - 1)) = \sum_{j=4}^{n-2} \gamma_j f(2j). \]

Here, for each $4 \leq j \leq n - 2$, we have $0 = \gamma_j = \sum_{2<i<j} r(ij) + \sum_{j<i\leq n-1} r(ji)$. Furthermore, from Proposition 4.11, we get

\[ \varphi(\sum_{i=2k-1}^{2k} \sum_{j=2l-1}^{2l} f(ij)) = (1 + (2k - 1, 2k))(1 + (2l - 1, 2l))\varphi(f(2k - 1, 2l - 1)) \]

\[ = \lambda(\sum_{i=2k-1}^{2k} \sum_{j=2l-1}^{2l} f(ij)), \]

where $\lambda := r(2k - 1, 2l - 1)+r(2k, 2l-1)+r(2k-1, 2l)+r(2k, 2l)$. It remains to show that $\lambda = 0$. For this let $r \in \{2, \ldots, (n-2)/2\} \setminus \{k, l\}$ so that $(2r - 1, 2r)(1+(2l-1, 2l))\varphi(f(2k - 1, 2l - 1)) = (1 + (2l - 1, 2l))\varphi(f(2k - 1, 2l - 1))$, by (B). This implies

\[ 0 = r(2r - 1, 2l - 1) + r(2r - 1, 2l) + r(2r, 2l - 1) + r(2r, 2l), \text{ for } 4 < 2r < 2l - 1, \]

\[ 0 = r(2l - 1, 2r - 1) + r(2l, 2r - 1) + r(2l - 1, 2r) + r(2l, 2r), \text{ for } 2l < 2r - 1 < n - 2. \]

Hence $0 = \gamma_{2l-1} + \gamma_{2l} = \lambda$, and (iii) follows. This completes the proof of the Proposition. \( \square \)

**Acknowledgement.** This article has been written while the second author enjoyed the hospitality of the Mathematical Institute at the University of Oxford. During that time, her research was supported by the Deutsche Forschungsgemeinschaft (DFG grant #DA 1115-1/1).

**References**


Susanne Danz and Karin Erdmann, Mathematical Institute, University of Oxford, 24-29 St Giles’, OX1 3LB, Oxford, UK
email: danz@maths.ox.ac.uk; erdmann@maths.ox.ac.uk