Solution of Brauer’s $k(B)$-Conjecture for $\pi$-blocks of $\pi$-separable groups

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October 9, 2018

Abstract

Answering a question of Pálfy and Pyber, we first prove the following extension of the $k(GV)$-Problem: Let $G$ be a finite group and let $A$ be a coprime automorphism group of $G$. Then the number of conjugacy classes of the semidirect product $G \rtimes A$ is at most $|G|$. As a consequence we verify Brauer’s $k(B)$-Conjecture for $\pi$-blocks of $\pi$-separable groups which was proposed by Y. Liu. This generalizes the corresponding result for blocks of $p$-solvable groups. We also discuss equality in Brauer’s Conjecture. On the other hand, we construct a counterexample to a version of Olsson’s Conjecture for $\pi$-blocks which was also introduced by Liu.

Keywords: $\pi$-blocks, Brauer’s $k(B)$-Conjecture, $k(GV)$-Problem

AMS classification: 20C15

1 Introduction

One of the oldest outstanding problems in the representation theory of finite groups is Brauer’s $k(B)$-Conjecture [1]. It asserts that the number $k(B)$ of ordinary irreducible characters in a $p$-block $B$ of a finite group $G$ is bounded by the order of a defect group of $B$. For $p$-solvable groups $G$, Nagao [12] has reduced Brauer’s $k(B)$-Conjecture to the so-called $k(GV)$-Problem: If a $p'$-group $G$ acts faithfully and irreducibly on a finite vector space $V$ in characteristic $p$, then the number $k(GV)$ of conjugacy classes of the semidirect product $G \rtimes V$ is at most $|V|$. Eventually, the $k(GV)$-Problem has been solved in 2004 by the combined effort of several mathematicians invoking the classification of the finite simple groups. A complete proof appeared in [15].

Brauer himself already tried to replace the prime $p$ in his theory by a set of primes $\pi$. Different approaches have been given later by Iizuka, Isaacs, Reynolds and others (see the references in [16]). Finally, Slattery developed in a series of papers [16, 17, 18] a nice theory of $\pi$-blocks in $\pi$-separable groups (precise definitions are given in the third section below). This theory was later complemented by Laradji [5, 9] and Y. Zhu [20]. The success of this approach is emphasized by the verification of Brauer’s Height Zero Conjecture and the Alperin–McKay Conjecture for $\pi$-blocks of $\pi$-separable groups by Manz–Staszewski [11] Theorem 3.3] and Wolf [19] Theorem 2.2] respectively. In 2011, Y. Liu [10] put forward a variant of Brauer’s $k(B)$-Conjecture for $\pi$-blocks in $\pi$-separable groups. Since $\{p\}$-separable groups are $p$-solvable and $\{p\}$-blocks are $p$-blocks, this generalizes the results mentioned in the first paragraph. Liu verified his conjecture in the special case where $G$ has a nilpotent normal

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Hall π-subgroup. The aim of the present paper is to give a full proof of Brauer’s $k(B)$-Conjecture for π-blocks in π-separable groups (see Theorem 3 below). In order to do so, we need to solve a generalization of the $k(GV)$-Problem (see Theorem 1 below). In this way we answer a question raised by Pálfy and Pyber at the end of [13] (see also [6]). The proof relies on the classification of the finite simple groups. Motivated by Robinson’s theorem [14] for blocks of $p$-solvable groups, we also show that equality in Brauer’s Conjecture can only occur for π-blocks with abelian defect groups. Finally, we construct a counterexample to a version of Olsson’s Conjecture which was also proposed by Liu [10].

2 A generalized $k(GV)$-Problem

In the following we use the well-known formula $k(G) ≤ k(N)k(G/N)$ where $N ≤ G$ (see [12] Lemma 1]).

Theorem 1. Let $G$ be a finite group, and let $A ≤ \text{Aut}(G)$ such that $(|G|, |A|) = 1$. Then $k(G\times A) ≤ |G|$. 

Proof. We argue by induction on $|G|$. The case $G = 1$ is trivial and we may assume that $G ≠ 1$. Suppose first that $G$ contains an $A$-invariant normal subgroup $N ≤ G$ such that $1 < N < G$. Let $B := C_A(G/N) ≤ A$. Then $B$ acts faithfully on $N$ and by induction we obtain $k(NB) ≤ |N|$. Similarly we have $k((G/N) \times (A/B)) ≤ |G/N|$. It follows that

$$k(GA) ≤ k(NB)k(GA/NB) ≤ |N|k((G/N)(A/B)) ≤ |N||G/N| = |G|.$$ 

Hence, we may assume that $G$ has no proper non-trivial $A$-invariant normal subgroups. In particular, $G$ is characteristically simple, i.e. $G = S_1 \times \ldots \times S_n$ with simple groups $S := S_1 \cong \ldots \cong S_n$. If $S$ has prime order, then $G$ is elementary abelian and the claim follows from the solution of the $k(GV)$-Problem (see [15]). Therefore, we assume in the following that $S$ is non-abelian.

We discuss the case $n = 1$ (that is $G$ is simple) first. Since $(|A|, |G|) = 1$, $A$ is isomorphic to a subgroup $\text{Out}(G)$. If $G$ is an alternating group or a sporadic group, then $\text{Out}(G)$ divides 4 and $A = 1$ as is well-known. In this case the claim follows since $k(GA) = k(G) ≤ |G|$. Hence, we may assume that $S$ is a group of Lie type over a field of size $p^l$ for a prime $p$. According to the Atlas [2, Table 5], the order of $\text{Out}(G)$ has the form $dfg$. Here $d$ divides the order of the Schur multiplier of $G$ and therefore every prime divisor of $d$ divides $|G|$. Moreover, $g \mid 6$ and in all cases $g$ divides $|G|$. Consequently, $|A| ≤ f ≤ \log_2 p^l ≤ \log_2 |G|$. On the other hand, [3, Theorem 9] shows that $k(G) ≤ \sqrt{|G|}$. Altogether, we obtain

$$k(GA) ≤ k(G)\|A| ≤ \sqrt{|G|}\log_2 |G| ≤ |G|$$

(note that $|G| ≥ |\mathfrak{A}_5| = 60$ where $\mathfrak{A}_5$ denotes the alternating group of degree 5).

It remains to handle the case $n > 1$. Here $\text{Aut}(G) \cong \text{Aut}(S) \rtimes \mathfrak{S}_n$ where $\mathfrak{S}_n$ is the symmetric group of degree $n$. Let $B := N_A(S_1) \cap \ldots \cap N_A(S_n) ≤ A$. Then $B ≤ \text{Out}(S_1) \times \ldots \times \text{Out}(S_n)$ and the arguments from the $n = 1$ case yield

$$k(GB) ≤ k(G)|B| = k(S)^n|B| ≤ \left(\sqrt{|S|}\log_2 |S|\right)^n.$$  \hspace{1cm} (2.1)

By Feit–Thompson, $|G|$ has even order and $A/B ≤ \mathfrak{S}_n$ has odd order since $(|G|, |A|) = 1$. A theorem of Dixon [3] implies that $|A/B| ≤ 3^n$. If $|G| = 60$, then $G \cong \mathfrak{A}_5$, $B = 1$ and

$$k(GA) ≤ k(\mathfrak{A}_5)^n|A| ≤ (5\sqrt{3})^n ≤ 60^n = |G|.$$ 

Therefore, we may assume that $|G| ≥ |\text{PSL}(3,2)| = 168$. Then (2.1) gives

$$k(GA) ≤ k(GB)|A/B| ≤ (\sqrt{3}|S|\log_2 |S|)^n ≤ |S|^n = |G|.$$

\hfill \Box
3 \( \pi \)-Blocks of \( \pi \)-separable groups

Let \( \pi \) be a set of primes. Recall that a finite group \( G \) is called \( \pi \)-separable if \( G \) has a normal series

\[
1 = N_0 \leq \ldots \leq N_k = G
\]
such that each quotient \( N_i/N_{i-1} \) is a \( \pi \)-group or a \( \pi' \)-group. The following consequence of Theorem 1 generalizes and proves the conjecture made in [6].

**Corollary 2.** For every \( \pi \)-separable group \( G \) we have \( k(G/O_{\pi'}(G)) \leq |G|_\pi \).

**Proof.** We may assume that \( O_{\pi'}(G) = 1 \) and \( N := O_{\pi}(G) \neq 1 \). We argue by induction on \( |N| \). By the Schur–Zassenhaus Theorem, \( N \) has a complement in \( O_{\pi'}(G) \) and Theorem 1 implies \( k(O_{\pi'}(G)) \leq |N| \).

Now induction yields

\[
k(G) \leq k(O_{\pi'}(G))k(G/O_{\pi'}(G)) \leq |N||G/N|_\pi = |G|_\pi. \]

A \( \pi \)-block of a \( \pi \)-separable group \( G \) is a minimal non-empty subset \( B \subseteq \text{Irr}(G) \) such that \( B \) is a union of \( p \)-blocks for every \( p \in \pi \) (see [12, Definition 1.12 and Theorem 2.15]). In particular, the \( \{p\} \)-blocks of \( G \) are the \( p \)-blocks of \( G \). In accordance with the notation for \( p \)-blocks we set \( k(B) := |B| \) for every \( \pi \)-block \( B \).

A **defect group** \( D \) of a \( \pi \)-block \( B \) of \( G \) is defined inductively as follows. Let \( \chi \in B \) and let \( \lambda \in \text{Irr}(O_{\pi'}(G)) \) be a constituent of the restriction \( \chi|_{O_{\pi'}(G)} \) (we say that \( B \) lies over \( \lambda \)). Let \( G_{\lambda} \) be the inertial group of \( \lambda \) in \( G \). If \( G_{\lambda} = G \), then \( D \) is a Hall \( \pi \)-subgroup of \( G \) (such subgroups always exist in \( \pi \)-separable groups). Otherwise we take a \( \pi \)-block \( b \) of \( G_{\lambda} \) lying over \( \lambda \). Then \( D \) is a defect group of \( b \) up to \( G \)-conjugation (see [17, Definition 2.2]). It was shown in [17, Theorem 2.1] that this definition agrees with the usual definition for \( p \)-blocks.

The following theorem verifies Brauer’s \( k(B) \)-Conjecture for \( \pi \)-blocks of \( \pi \)-separable groups (see [10]).

**Theorem 3.** Let \( B \) be a \( \pi \)-block of a \( \pi \)-separable group \( G \) with defect group \( D \). Then \( k(B) \leq |D| \).

**Proof.** We mimic Nagao’s reduction [12] of Brauer’s \( k(B) \)-Conjecture for \( p \)-solvable groups. Let \( N := O_{\pi'}(G) \), and let \( \lambda \in \text{Irr}(N) \) lying under \( B \). By [16, Theorem 2.10] and [17, Corollary 2.8], the Fong–Reynolds Theorem holds for \( \pi \)-blocks. Hence, we may assume that \( \lambda \) is \( G \)-stable and \( B \) is the set of irreducible characters of \( G \) lying over \( \lambda \) (see [16, Theorem 2.8]). Then \( D \) is a Hall \( \pi \)-subgroup of \( G \) by the definition of defect groups. By [7, Problem 11.10] and Corollary 2 it follows that \( k(B) \leq k(G/N) \leq |G|_\pi = |D| \).

In the situation of Theorem 1, it is known that \( GA \) contains only one \( \pi \)-block where \( \pi \) is the set of prime divisors of \( |G| \) (see [16, Corollary 2.9]). Thus, in the proof of Theorem 3 one really needs to full strength of Theorem 1.

Liu [10] has also proposed the following conjecture (cf. [17, Definition 2.13]):

**Conjecture 4** (Olsson’s Conjecture for \( \pi \)-blocks). Let \( B \) be a \( \pi \)-block of a \( \pi \)-separable group \( G \) with defect group \( D \). Let \( k_0(B) \) be the number of characters \( \chi \in B \) such that \( \chi(1)_\pi |D| = |G|_\pi \). Then \( k_0(B) \leq |D : D'| \).

This conjecture however is false. A counterexample is given by \( G = \text{PSL}(2, 2^5) \times C_5 \) where \( C_5 \) acts as a field automorphism on \( \text{PSL}(2, 2^5) \). Here \( |G| = 2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 31 \) and we choose \( \pi = \{2, 3, 11, 31\} \). Then \( O_\pi(G) = \text{PSL}(2, 2^5) \) and [16, Corollary 2.9] implies that \( G \) has only one \( \pi \)-block \( B \) which must contain the five linear characters of \( G \). Moreover, \( B \) has defect group \( D = O_\pi(G) \) by [17, Lemma 2.3]. Hence, \( k_0(B) \geq 5 > 1 = |D : D'| \) since \( D \) is simple.
4 Abelian defect groups

In this section we prove that the equality $k(B) = |D|$ in Theorem 3 can only hold if $D$ is abelian. We begin with Gallagher’s observation [4] that $k(G) = k(N) k(G/N)$ for $N \leq G$ implies $G = C_G(x)N$ for all $x \in N$. Next we analyze equality in our three results above.

**Lemma 5.** Let $G$ be a finite group and $A \leq \text{Aut}(G)$ such that $(|G|, |A|) = 1$. If $k(G \rtimes A) = |G|$, then $G$ is abelian.

**Proof.** We assume that $k(GA) = |G|$ and argue by induction on $|G|$. Suppose first that there is an $A$-invariant normal subgroup $N \leq G$ such that $1 < N < G$. As in the proof of Theorem 1 we set $B := C_A(G/N)$ and obtain $k(GA) = k(NB) k(GA/NB)$. By induction, $N$ and $G/N$ are abelian and $GA = C_{GA}(x)NB = C_{GA}(x)B$ for every $x \in N$. Hence $G \leq C_{GA}(x)$ and $N \leq Z(G)$. Therefore, $G$ is nilpotent (of class at most 2). Then every Sylow subgroup of $G$ is $A$-invariant and we may assume that $G$ is a $p$-group. In this case the claim follows from [14, Theorem 1'].

Hence, we may assume that $G$ is characteristically simple. If $G$ is non-abelian, then we easily get a contradiction by following the arguments in the proof of Theorem 1. \hfill \Box

**Lemma 6.** Let $G$ be a $\pi$-separable group such that $O_{\pi'}(G) = 1$ and $k(G) = |G|_\pi$. Then $G = O_{\pi\pi'}(G)$.

**Proof.** Let $N := O_{\pi'}(G)$. Since $O_{\pi'}(N) \leq O_{\pi'}(G) = 1$, we have $k(N) \leq |N|_\pi$ by Corollary 2. Moreover, $O_{\pi'}(G/N) = 1$, $k(G/N) \leq |G/N|_\pi$ and $k(G) = k(N) k(G/N)$. In particular, $G = C_G(x)N$ for every $x \in N$. Let $g \in G$ be a $\pi$-element. Then $g$ is a class-preserving automorphism of $N$ and also of $N/O_{\pi}(G)$. Since $N/O_{\pi}(G) = O_{\pi'}(G/O_{\pi}(G))$ is a $\pi'$-group, it follows that $g$ acts trivially on $N/O_{\pi}(G)$. By the Hall–Higman Lemma 1.2.3, $N/O_{\pi}(G)$ is self-centralizing and therefore $g \in N$. Thus, $G/N$ is a $\pi'$-group and $N = G$. \hfill \Box

**Theorem 7.** Let $B$ be a $\pi$-block of a $\pi$-separable group with non-abelian defect group $D$. Then $k(B) < |D|$.

**Proof.** We assume that $k(B) = |D|$. Following the proof of Theorem 3, we end up with a $\pi$-separable group $G$ such that $D \leq G$, $O_{\pi'}(G) = 1$ and $k(G) = |G|_\pi = |D|$. By Lemma 6, $D \leq G$ and by Lemma 5, $D$ is abelian. \hfill \Box

Similar arguments imply the following $\pi$-version of [14, Theorem 3] which also extends Corollary 2.

**Theorem 8.** Let $G$ be a $\pi$-separable group such that $O_{\pi'}(G) = 1$ and $H \leq G$. Then $k(H) \leq |G|_\pi$ and equality can only hold if $|H|_\pi = |G|_\pi$.

The proof is left to the reader.

**Acknowledgment**

This work is supported by the German Research Foundation (projects SA 2864/1-1 and SA 2864/3-1).
References


