2-Blocks with minimal nonabelian defect groups III

Benjamin Sambale*

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Abstract

We prove that two 2-blocks of (possibly different) finite groups with a common minimal nonabelian defect group and the same fusion system are isotypic (and therefore perfectly isometric) in the sense of Broué. This continues former work by [Cabanes-Picaronny, 1992], [Sambale, 2011] and [Eaton-Külshammer-Sambale, 2012].

Keywords: minimal nonabelian defect groups, perfect isometries, isotypies

AMS classification: 20C15, 20C20

1 Introduction

Since its appearance in 1990, Broué’s Abelian Defect Conjecture gained much attention among representation theorists. On the level of characters it predicts the existence of a perfect isometry between a block with abelian defect group and its Brauer correspondent. These blocks have a common defect group and the same fusion system. Although Broué’s Conjecture is false for nonabelian defect groups (see [4]), one can still ask if perfect isometries or even isotypies exist. We affirmatively answer this question for \( p = 2 \) and minimal nonabelian defect groups (see Theorem 9 below). These are the nonabelian defect groups such that any proper subgroup is abelian. Doing so, we verify the character-theoretic version of Rouquier’s Conjecture [17, A.2] in this special case (see Corollary 10 below). At the same time we provide a new infinite family of defect groups supporting a blockwise \( Z^* \)-Theorem.

By Rédei’s classification of minimal nonabelian \( p \)-groups, one has to consider three distinct families of defect groups. For two of these families the result already appeared in the literature (see [3, 19, 5]). Hence, it suffices to handle the remaining family which we will do in the next section. The proof of the main result is an application of Horimoto-Watanabe [10, Theorem 2]. The last section of the present paper also contains a related result for the nonabelian defect group of order 27 and exponent 9.

Our notation is fairly standard. We consider blocks \( B \) of finite groups with respect to a \( p \)-modular system \((K, O, F)\) where \( O \) is a complete discrete valuation ring with quotient field \( K \) of characteristic 0 and field of fractions \( F \) of characteristic \( p \). As usual, we assume that \( K \) is “large” enough and \( F \) is algebraically closed. The number of irreducible ordinary characters (resp. Brauer characters) of \( B \) is denoted by \( k(B) \) (resp. \( l(B) \)). Moreover, \( k_i(B) \) is the number of those irreducible characters of \( B \) which have height \( i \geq 0 \). For other results on block invariants and fusion systems we often refer to [20]. Moreover, for the definition and construction of perfect isometries we follow [11, 8]. A cyclic group of order \( n \in \mathbb{N} \) is denoted by \( C_n \).

*Institut für Mathematik, Friedrich-Schiller-Universität, 07743 Jena, Germany, benjamin.sambale@uni-jena.de
2 A class of minimal nonabelian defect groups

Let $B$ be a non-nilpotent 2-block of a finite group $G$ with defect group

$$D = \langle x, y \mid x^{2^r} = y^2 = [x, y] = [x, x, y] = [y, x, y] = 1 \rangle \cong C_2^2 \times C_{2^r}$$

where $r \geq 2$, $[x, y] := xyx^{-1}y^{-1}$ and $[x, x, y] := [x, [x, y]]$.

We have already investigated some properties of $B$ in [19], and later gave simplified proofs in [20, Chapter 12]. For the convenience of the reader we restate some of these results.

Lemma 1 ([20 Lemma 12.3]). Let $z := [x, y]$. Then the following holds:

(i) $\Phi(D) = Z(D) = \langle x^2, z \rangle \cong C_{2^{r-1}} \times C_2$.

(ii) $D' = \langle z \rangle \cong C_2$.

(iii) $|\text{Irr}(D)| = 5 \cdot 2^{r-1}$.

Recall that a (saturated) fusion system $\mathcal{F}$ on a $p$-group $P$ determines the following subgroups:

$$Z(\mathcal{F}) := \{x \in P : x \text{ is fixed by every morphism in } \mathcal{F}\}$$

$$\text{fac}(\mathcal{F}) := \{f(x)x^{-1} : x \in Q, f \in \text{Aut}_F(Q)\}$$

$$\text{hyp}(\mathcal{F}) := \{f(x)x^{-1} : x \in Q, f \in O^p(\text{Aut}_F(Q))\}$$

Lemma 2. The fusion system $\mathcal{F}$ of $B$ is the constrained fusion system of the finite group $A_4 \times C_{2^r}$ where $C_{2^r}$ acts as a transposition in $\text{Aut}(A_4) \cong S_4$. In particular, $B$ has inertial index 1 and $Q := \langle x^2, y, z \rangle \cong C_{2^{r-1}} \times C_2$ is the only $\mathcal{F}$-essential subgroup of $D$. Moreover, $\text{Aut}_F(Q) \cong S_3$. Without loss of generality, $Z(\mathcal{F}) = \langle x^2 \rangle$ and $\text{hyp}(B) = \text{fac}(B) = \text{fac}(\mathcal{F}) = \langle y, z \rangle$.

Proof. We have seen in [20, Proposition 12.7] that $\mathcal{F}$ is constrained and coincides with the fusion system of $A_4 \times C_{2^r}$. The construction of the semidirect product $A_4 \rtimes C_{2^r}$ is slightly different in [20], but it is easy to see that both constructions give isomorphic groups. The remaining claims follow from the proof of [20, Proposition 12.7].

By a result of Watanabe [25, Theorem 3 and Lemma 3], the hyperfocal subgroup of a 2-block is trivial or non-cyclic. Hence, our situation with a Klein-four (hyper)focal subgroup represents the first non-trivial example in some sense. Recall that a $B$-subsection is a pair $(u, b_u)$ such that $u \in D$ and $b_u$ is a Brauer correspondent of $B$ in $C_G(u)$.

Lemma 3. The set $R := Z(D) \cup \{x^iy^j : i, j \in \mathbb{Z}, i \text{ odd}\}$ is a set of representatives for the $\mathcal{F}$-conjugacy classes of $D$ with $|R| = 2^{r+1}$. For $u \in R$ let $(u, b_u)$ be a $B$-subsection. Then $b_u$ has defect group $C_D(u)$. Moreover, $l(b_u) = 1$ whenever $u \in R \setminus \{x^2\}$.

Proof. By Lemma 2 it is easy to see that $R$ is in fact a set of representatives for the $\mathcal{F}$-conjugacy classes of $D$. Observe that $\langle u \rangle$ is fully $\mathcal{F}$-normalized for all $u \in R$. Hence, by [20, Lemma 1.34], $b_u$ has defect group $C_D(u)$ and fusion system $\mathcal{F} \langle \langle u \rangle \rangle$. It is easy to see that $C_D(\langle u \rangle)$ is trivial unless $u \in Z(\mathcal{F}) = \langle x^2 \rangle$. This shows $l(b_u) = 1$ for $u \in R \setminus \{x^2\}$.

Theorem 4 ([20, Theorem 12.4]). We have $k(B) = 5 \cdot 2^{r-1}$, $k_0(B) = 2^{r+1}$, $k_1(B) = 2^{r-1}$ and $l(B) = 2$.

Proof. By Lemma 2, we have $|D : \text{fac}(B)| = 2^r$. In particular, $2^r \mid k_0(B)$ by [16, Theorem 1]. Moreover, [11, Theorem 1.1] implies $2^{r+1} \leq k_0(B)$. By Lemma 3 we have $l(b_2) = 1$. Thus, we obtain $k_0(B) = 2^{r+1}$ by a result of Robinson (see [23, Theorem 4.12]). In order to determine $l(B)$, we use induction on $r$. Let $u := x^2$. Then $b_u$ dominates a block $b_u$ of $C_G(u)/\langle u \rangle$ with defect group $\overline{D} := D/\langle u \rangle \cong D_8$ and fusion system $\overline{\mathcal{F}} := \mathcal{F}/\langle u \rangle$. By [13, Theorem 6.3], $(x^2, y, z)/\langle u \rangle \cong C_2^2$ is the only $\overline{\mathcal{F}}$-essential subgroup of $\overline{D}$. Therefore, a result of Brauer (see [20, Theorem 8.1]) shows that $l(b_u) = l(b_{u}) = 2$. By Lemma 5 and [20, Theorem 1.35] it follows that
Now let \( |Z(D) : Z(D) \cap \text{foc}(B)| = 2^{r-1} \), we have \( 2^{r-1} \mid k_i(B) \) for \( i \geq 1 \) by [10, Theorem 2]. Thus, by [13, Theorem 3.4] we obtain
\[
2^{r+2} \leq k_0(B) + 4(k(B) - k_0(B)) \leq \sum_{i=0}^{\infty} k_i(B)2^{2i} \leq |D| = 2^{r+2}.
\]
This gives \( k_1(B) = 2^{r-1} \) and \( k(B) = k_0(B) + k_1(B) = 5 \cdot 2^{r-1} \). In case \( r = 2 \), [20, Theorem 1.35] implies
\[
l(b) = k(B) - \sum_{b \neq u \in R} l(b_u) = 10 - 8 = 2.
\]
Now let \( r \geq 3 \) and \( 1 \neq \langle u \rangle < \langle x^2 \rangle \). Then \( b_u \) as above has the same type of defect group as \( B \) except that \( r \) is smaller. Hence, induction gives \( l(b_u) = l(b_\overline{u}) = 2 \). Now the claim \( l(B) = 2 \) follows again by [20, Theorem 1.35].

In the following results we denote the set of irreducible characters of \( B \) of height \( i \) by \( \text{Irr}_i(B) \).

**Proposition 5** ([20, Proposition 12.9]). The set \( \text{Irr}_0(B) \) contains four 2-rational characters and two families of 2-conjugate characters of size \( 2^i \) for every \( i = 1, \ldots, r - 1 \). The characters of height 1 split into two 2-rational characters and one family of 2-conjugate characters of size \( 2^i \) for every \( i = 2, \ldots, r - 2 \).

**Proposition 6.** There are 2-rational characters \( \chi_i \in \text{Irr}(B) \) for \( i = 1, 2, 3 \) such that
\[
\text{Irr}_1(B) = \{ \chi_i \cdot \lambda : i = 1, 2, \lambda \in \text{Irr}(D/\text{foc}(B)) \},
\]
\[
\text{Irr}_1(B) = \{ \chi_3 \cdot \lambda : \lambda \in \text{Irr}(Z(D)/\text{foc}(B)) \}.
\]
In particular, the characters of height 1 have the same degree and \( \mid \{ \chi(1) : \chi \in \text{Irr}_0(B) \} \mid \leq 2 \).

**Proof.** We have already seen in the proof of Theorem 4 that the action of \( D/\text{foc}(B) \) on \( \text{Irr}_0(B) \) via the *-construction has two orbits, and the action of \( Z(D)/\text{foc}(B) \) on \( \text{Irr}_1(B) \) is regular. By Proposition 5 we can choose 2-rational representatives for these orbits. Notice that we identify the sets \( \text{Irr}(D/\text{foc}(B)) \) and \( \text{Irr}(Z(D)/\text{foc}(B)) \) with subsets of \( \text{Irr}(D) \) in an obvious manner.

In the situation of Proposition 6 it is conjectured that \( \chi_1(1) \neq \chi_2(1) \) (see [14]).

**Proposition 7** ([20, Proposition 12.8]). The Cartan matrix of \( B \) is given by
\[
2^{r-1} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}
\]
up to basic sets.

Observe that Proposition 7 also gives the Cartan matrix for the defect group \( D_8 \) and the corresponding fusion system (this would be the case \( r = 1 \)).

Now we are in a position to obtain the generalized decomposition matrix of \( B \). This completes partial results in [19, Section 3.3].

**Proposition 8.** Let \( R \) and \( \chi_i \) be as in Lemma 3 and Proposition 7 respectively. Then there are basic sets for \( b_u \ (u \in R) \) and signs \( \epsilon, \sigma \in \{ \pm 1 \} \) such that the generalized decomposition numbers of \( B \) have the following form
\[
\begin{array}{c|cccc}
\chi & x^{2i} & x^{2i} & x^{2i+1} & x^{2i+1} \\
\hline
\chi_{11} & (1, 0) & 1 & 1 & 1 \\
\chi_{12} & (0, \epsilon) & \epsilon & \epsilon & -\epsilon \\
\chi_{13} & (\sigma, \sigma) & -2\sigma & 0 & 0 \\
\end{array}
\]
Proof. Since the Galois group of $\mathbb{Q}(e^{2\pi i/2^n})$ over $\mathbb{Q}$ acts on the columns of the generalized decomposition matrix (cf. Proposition 5), we only need to determine the numbers $d_{u,\varphi}^e$ for $u \in \{x, xy, x^2, z\}$ ($i = 1, 2, 3, j = 1, \ldots, r$). First let $u = x$. Then the orthogonality relations show that

$$2^r|d_{x,\varphi}^e|^2 + 2^r|d_{x,\varphi}^{e-1}|^2 = 2^{r+1}.$$

Since $\chi_1$ and $\chi_2$ have height 0, we have $d_{x,\varphi}^e \neq 0 \neq d_{x,\varphi}^{e-1}$ (see [20 Proposition 1.36]). It follows that $d_{x,\varphi}^e = \pm 1$ for $i = 1, 2$ and $d_{x,\varphi}^e = 0$, because $\chi_1$ is 2-rational. By replacing $\varphi$ with $-\varphi$ if necessary (i.e. changing the basic set for $b_u$), we may assume that $d_{x,\varphi}^e = 1$. We set $d_{x,\varphi}^{e-1} = \epsilon_0$. Similarly, we obtain $d_{x,\varphi}^e = 1$, $d_{x,\varphi}^e = \pm 1$ and $d_{x,\varphi}^{e-1} = 0$. Now since the columns $d_x^e$ and $d_x^{e-1}$ of the generalized decomposition matrix are orthogonal, we obtain $d_{x,\varphi}^{e-1} = -\epsilon_0$.

Now let $u = x^2$ for some $j \in \{1, \ldots, r\}$. Let $\text{IBr}(b_u) = \{\varphi_1, \varphi_2\}$ (see proof of Theorem 4). Then by Proposition 7 we get

$$2^r|d_{x^2,\varphi_1}^1|^2 + 2^r|d_{x^2,\varphi_2}^1|^2 + 2^r|d_{x^2,\varphi_1}^{e-1}|^2 = 2^{r+1},$$
$$2^r|d_{x^2,\varphi_1}^1|^2 + 2^r|d_{x^2,\varphi_2}^1|^2 + 2^r|d_{x^2,\varphi_1}^{e-1}|^2 = 3 \cdot 2^{r-1},$$
$$2^r|d_{x^2,\varphi_1}^1|^2 + 2^r|d_{x^2,\varphi_2}^1|^2 + 2^r|d_{x^2,\varphi_1}^{e-1}|^2 = 2^{r-1}.$$

Obviously, $d_{x^2,\varphi_1}^1 d_{x^2,\varphi_2}^1 = 0$ and we may assume that $(d_{x^2,\varphi_1}^1, d_{x^2,\varphi_2}^1) = (1, 0)$ and $(d_{x^2,\varphi_1}^1, d_{x^2,\varphi_2}^1) = (0, \epsilon_1)$ for a sign $\epsilon_j \in \{\pm 1\}$. Moreover, $d_{x^2,\varphi_1}^1 = d_{x^2,\varphi_2}^1 = \sigma_j \in \{\pm 1\}$. Now let $u = x^2$. Then we have

$$2^r|d_{x^2,\varphi_1}^1|^2 + 2^r|d_{x^2,\varphi_2}^1|^2 + 2^r|d_{x^2,\varphi_1}^{e-1}|^2 = 2^{r+2}.$$

It is known that $2 \mid d_{x^2,\varphi}^1 \neq 0$, since $b_u$ is major (see [20 Proposition 1.36]). This gives $d_{x^2,\varphi_1}^1 = 1$, $d_{x^2,\varphi_2}^1 = 1$ and $d_{x^2,\varphi_1}^{e-1} = -\epsilon_2$. By the orthogonality to $x^{2i}$ we obtain that $d_{x^2,\varphi}^1 = -2\sigma_j$ and $d_{x^2,\varphi_1}^{e-1} = \epsilon_j$.

It remains to show that the signs $\epsilon_j$ and $\sigma_j$ do not depend on $j$. For this we consider characters $\lambda, \psi \in \text{Irr}(D)$ whose values are given as follows

<table>
<thead>
<tr>
<th>$x^2$</th>
<th>$x^2 z$</th>
<th>$x$</th>
<th>$xy$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Observe that $\psi$ is the inflation of the irreducible character of $D/\langle x^2 \rangle \cong D_4$ of degree 2. It is easy to see that $(\lambda + \psi)(x^{2k}y) = -1 = -1 - 2 = (\lambda + \psi)(x^{2k}z)$ for every $k \in \mathbb{Z}$. It follows that $\lambda + \psi$ is $\mathcal{F}$-stable, i.e. $(\lambda + \psi)(u) = (\lambda + \psi)(v)$ whenever $u$ and $v$ are $\mathcal{F}$-conjugate. By Brönn-Leu, $\chi_1 * (\lambda + \psi)$ is a generalized character of $B$. In particular, the scalar product $(\chi_1 * (\lambda + \psi), \chi_3) = \alpha$ is an integer. This number can be computed by using the so-called contribution numbers $m_{u,\chi_3}^u := d_{u,\chi_3}^1 C_u^{-1} d_{x^2,\varphi}^T$, where $C_u$ is the Cartan matrix of $b_u$ and $d_{x^2,\varphi}$ is the row of the generalized decomposition matrix corresponding to $(u, b_u)$ and $x^2$. In case $u = x^2$, we have

$$C_u^{-1} = 2^{r-2} \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

by Proposition 7. This gives $m_{x^2,\chi_3}^u = 2^{-r-1} \sigma_j$. Similarly, $m_{x^2,\chi_3}^u = -2^{-r-1} \sigma_j$ for $u = x^{2i}$. Thus, we obtain

$$(\chi_1 * (\lambda + \psi), \chi_3) = \sum_{u \in R} (\lambda + \psi)(u) m_{x^2,\chi_3}^u = \sum_{u \in \text{Irr}(D)} (\lambda + \psi)(u) m_{x^2,\chi_3}^u = \sum_{j=1}^{r-1} \sigma_j 2^{-r-j-1} = 2^{-r+1} \sigma_j + \sum_{j=1}^{r-1} \sigma_j 2^{-r-j}.$$
3 The main result

Theorem 9. Let $B$ and $\bar{B}$ be 2-blocks of (possibly different) finite groups with a common minimal nonabelian defect group and the same fusion system. Then $B$ and $\bar{B}$ are isotypic (and therefore perfectly isometric).

Proof. We may assume that $B$ is not nilpotent by Broué-Puig [2]. Let $D$ be a defect group of $B$ and $\bar{B}$. If $|D| = 8$, then the claim follows from [3]. Now suppose that $D$ is given as in [1]. We will attach a tilde to everything associated with $\bar{B}$. By Proposition [3] and [10] Theorem 2 there is a perfect isometry $I : CF(G, B) \to CF(G, \bar{B})$ where $CF(G, B)$ denotes the space of class functions with basis $Irr(B)$ over $K$. It remains to show that $I$ is also an isotypy. In order to do so, we follow [3, Section V.2]. For each $u \in D$ let $CF(C_G(u)_2, b_u)$ be the space of class functions on $C_G(u)$ which vanish on the $p$-singular classes and are spanned by $IBr(b_u)$. The decomposition map $d_G^u : CF(G, B) \to CF(C_G(u)_2, b_u)$ is defined by

$$d_G^u(\chi)(s) := \chi(e_{b_u}us) = \sum_{\varphi \in IBr(b_u)} d_G^u(\varphi)(s)$$

for $\chi \in Irr(B)$ and $s \in C_G(u)_2$ where $e_{b_u}$ is the block idempotent of $b_u$ over $O$. Then $I$ determines isometries

$$I^u : CF(C_G(u)_2, b_u) \to CF(C_G(u)_2, \bar{b}_u)$$

by the equation $d_G^u \circ I = I^u \circ d_G^u$. Note that $I^1$ is the restriction of $I$. We need to show that $I^u$ can be extended to a perfect isometry $\tilde{I}^u : CF(C_G(u), b_u) \to CF(C_G(u), \bar{b}_u)$. Suppose first that $b_u$ is nilpotent. Then by Proposition [3] $d_G^u(\chi_1) = \epsilon \varphi$ and $d_G^u(I(\chi_1)) = \tilde{\epsilon} \tilde{\varphi}$ where $IBr(b_u) = \{ \varphi \}$ and $IBr(\bar{b}_u) = \{ \tilde{\varphi} \}$ for some signs $\epsilon, \tilde{\epsilon} \in \{ \pm 1 \}$. It follows that $I^u(\varphi) = \epsilon \tilde{\epsilon} \tilde{\varphi}$. Let $\psi \in Irr_0(b_u)$ and $\tilde{\psi} \in Irr_0(\bar{b}_u)$ be 2-rational characters. Then it is well-known that $\varphi = d_{C_G(u)}^u(\psi)$ and $Irr(b_u) = \{ \psi \lambda : \lambda \in Irr(D) \}$ (see [2]). Therefore, we may define $\tilde{I}^u$ by $\tilde{I}^u(\psi \lambda) := \epsilon \tilde{\epsilon} \tilde{\varphi}$ for $\lambda \in Irr(D)$. Then $\tilde{I}^u$ is a perfect isometry and

$$\tilde{I}^u(\varphi) = \tilde{I}^u(d_{C_G(u)}^u(\psi)) = d_{C_G(u)}^1(I^u(\psi)) = \epsilon \tilde{\epsilon} d_{C_G(u)}^1(\tilde{\psi}) = \epsilon \tilde{\epsilon} \tilde{\varphi} = I^u(\varphi).$$

Hence, $\tilde{I}^u$ extends $I^u$. Moreover, $\tilde{I}^u$ does not depend on the generator of $\langle u \rangle$, since the signs $\epsilon$ and $\tilde{\epsilon}$ were defined by means of 2-rational characters.

Assume next that $b_u$ is non-nilpotent. Then $u \in \langle x^2 \rangle$ and $b_u$ has defect group $D$. By Proposition [3] we can choose basic sets $\varphi_1, \varphi_2$ (resp. $\tilde{\varphi}_1, \tilde{\varphi}_2$) for $b_u$ (resp. $\bar{b}_u$) such that $\varphi_i = d_{C_G(u)}^u(\chi_i)$ and $\tilde{\varphi}_i = d_{C_G(u)}^u(I(\chi_i))$ for $i = 1, 2$. Then $I^u(\varphi_i) = \tilde{\varphi}_i$ for $i = 1, 2$. Since the Cartan matrix of $b_u$ with respect to the basic set $\varphi_1, \varphi_2$ is already fixed (and given by Proposition [7]), we find 2-rational characters $\psi_i \in Irr_0(b_u)$ such that $d_{C_G(u)}^u(\psi_i) = \epsilon_i \varphi_i$ with $\epsilon_i \in \{ \pm 1 \}$ for $i = 1, 2$ (see proof of Proposition [8]). Similarly, one has $\tilde{\psi}_i \in Irr_0(\bar{b}_u)$ such that $d_{C_G(u)}^u(\tilde{\psi}_i) = \tilde{\epsilon}/\tilde{\psi}_i$. Then, by what we have already shown, there exists a perfect isometry $\tilde{I}^u : CF(C_G(u), b_u) \to CF(C_G(u), \bar{b}_u)$ sending $\psi_i$ to $\epsilon_i \tilde{\psi}_i$ for $i = 1, 2$. We have

$$\tilde{I}^u(\varphi_i) = \epsilon_i \tilde{I}^u(d_{C_G(u)}^u(\psi_i)) = \epsilon_i d_{C_G(u)}^1(I^u(\psi_i)) = \tilde{\epsilon}_i d_{C_G(u)}^1(\tilde{\psi}_i) = \tilde{\epsilon}_i = I^u(\varphi_i)$$

for $i = 1, 2$. This shows that $\tilde{I}^u$ extends $I^u$. Moreover, it is easy to see that $\tilde{I}^u$ does not depend on the generator of $\langle u \rangle$.

Altogether we have proved the theorem if $D$ is given as in [1]. By [20] Theorem 12.4 it remains to handle the case

$$D \cong \langle x, y | x^{2r} = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle$$

where $r \geq 2$. Here $B$ and $\bar{B}$ are Morita equivalent and therefore perfectly isometric. However, a Morita equivalence does not automatically provide an isotypy. Nevertheless, in this special case the Morita equivalence is a composition of various “natural” equivalences (namely Fong reductions, Külshammer-Puig reduction and Külshammer’s reduction for blocks with normal defect groups, see [5] proof of Theorem 1]). In particular, the generalized decomposition matrices of $B$ and $\bar{B}$ coincide up to signs (see [21]). Now we can use the same methods as above in order to construct an isotypy. In fact, for every $\bar{B}$-subsection $(u, b_u)$ one has that $b_u$ is nilpotent or $u = [x, y]$ and $b_u$ Morita equivalent to $B$ (see proof of [19] Proposition 4.3]). We omit the details. □
Corollary 10. Let $B$ be a $2$-block of a finite group $G$ with minimal nonabelian defect group $D \not\cong D_8$. Then $B$ is isotypic to a Brauer correspondent in $\text{NG}(\text{hyp}(B))$.

Proof. Let $b_B$ be a Brauer correspondent of $B$ in $D \text{C}_G(D)$. Since $D \text{C}_G(D) \subseteq \text{NG}(\text{hyp}(B))$, the Brauer correspondent $b := b_B^{\text{NG}(\text{hyp}(B))}$ of $B$ has defect group $D$. By Theorem 9, it suffices to show that $B$ and $b$ have the same inertial quotient. Furthermore, $b$ has defect group as given in (1), then the fusion system is constrained and the automorphisms of the essential subgroup (if it exists) also act on $\text{hyp}(B)$. Hence, $B$ is nilpotent if and only if $b$ is nilpotent. Again the claim follows from Theorem 9.

We remark that Corollary 10 would be false in case $D \cong D_8$. The principal $2$-block of $\text{GL}(3,2)$ gives a counterexample. If $B$ is a block of a finite group $G$ with defect group as given in (1), then $B$ is also isotypic to a Brauer correspondent in $\text{C}_G(u)$ where $u \in \mathbb{Z}(F)$. This resembles Glauberman’s $Z^*$-Theorem.

In the situation of Theorem 9 (or Corollary 10) it is desirable to extend the isotypies to Morita equivalences (as we did in [2]). This is not always possible if $|D| = 8$, since for example the principal $2$-blocks of the symmetric groups $S_4$ and $S_5$ are not Morita equivalent. Nevertheless, the possible Morita equivalence classes in case $|D| = 8$ are known by Erdmann’s classification of tame algebras [6] (at least over $F$, cf. [9]). In view of [5] one may still ask if two non-nilpotent $2$-blocks with isomorphic defect groups as in Section 2 are Morita equivalent. We will see that the answer is again negative.

Consider the groups $G_1 := A_4 \times C_2$ and $G_2 := A_5 \times C_2$ constructed similarly as in Lemma 2. Then $G_1 / Z(G_1) \cong S_4$ and $G_2 / Z(G_2) \cong S_5$. Let $B_i$ be the principal $2$-block of $G_i$, and let $\mathcal{B}_i$ be the principal $2$-block of $G_i / Z(G_i)$ for $i = 1, 2$. Then the Cartan matrix of $B_i$ is just the Cartan matrix of $\mathcal{B}_i$ multiplied by $|Z(G_i)| = 2^{r-1}$. It is known that the Cartan matrices of $\mathcal{B}_1$ and $\mathcal{B}_2$ do not coincide (regardless of the labeling of the simple modules). Therefore, $B_1$ and $B_2$ are not Morita equivalent.

Nevertheless, the structure of a finite group $G$ with a minimal nonabelian Sylow $2$-subgroup $P$ as given in (1) is fairly restricted. More precisely, Glauberman’s $Z^*$-Theorem implies $x^2 \in Z^*(G)$, and the structure of $G / Z^*(G)$ follows from the Gorenstein-Walter Theorem [7]. In particular, $G$ has at most one nonabelian composition factor by Feit-Thompson.

We use the opportunity to present a related result for $p = 3$ which extends [20] Theorem 8.15.

Theorem 11. Let $B$ and $\tilde{B}$ be non-nilpotent blocks of (possibly different) finite groups both with defect group $C_9 \rtimes C_3$. Then $B$ and $\tilde{B}$ are isotypic.

Proof. As in the proof of Theorem 9 we will make use of Theorem 2. Let

$$D := \langle x, y \mid x^9 = y^3 = 1, yxy^{-1} = x^4 \rangle$$

be a defect group of $B$, and let $F$ be the fusion system of $B$. By Stancu [21], $B$ is controlled with inertial index $2$, and we may assume that $x$ and $x^{-1}$ are $F$-conjugate (see proof of [20] Theorem 8.8). Then $\mathcal{R} := \{1, x, x^3, y, y^2, xy, xy^2\}$ is a set of representatives for the $F$-conjugacy classes of $D$ (see proof of [20] Theorem 8.15). It suffices to show that the generalized decomposition numbers of $B$ are essentially unique (up to basic sets and signs and permutations of rows). Since the Galois group of $\mathbb{Q}(e^{2\pi i/9})$ over $\mathbb{Q}$ acts on the columns of the generalized decomposition matrix, we only need to determine the numbers $d_{\chi \varphi}^u$ for $u \in \{x, x^3, y, xy\}$. By [20] Theorem 8.15 there are four $3$-rational characters $\chi_i \in \text{Irr}(B)$ $(i = 1, \ldots, 4)$ such that $\chi_1, \chi_2, \chi_3$ have height 0 and $\chi_4$ has height 1. Since $\text{soc}(B) = \langle x \rangle$, we see that

$$\text{Irr}(B) = \{\chi_i \ast \lambda : i = 1, 2, 3, \lambda \in \text{Irr}(D/\text{soc}(B))\} \cup \{\chi_4\}.$$ 

Let $u := x^3$. Then $\text{IBr}(b_u) = \{\varphi\}$ and $d_{\chi \varphi}^u$ are non-zero (rational) integers. Moreover, $d_{\chi \varphi}^u \equiv 0 \pmod{3}$. After permuting $\chi_1, \chi_2$ and $\chi_3$ and changing the basic set for $b_u$ if necessary, we may assume that $d_{\chi \varphi}^u = 2,$
$d_{\chi \varphi}^u := \epsilon_1 \in \{ \pm 1 \}, d_{\chi \varphi}^u := \epsilon_2 \in \{ \pm 1 \}$ and $d_{\chi \varphi}^u = 3 \epsilon_3 \in \{ \pm 3 \}$. Now let $u := x$. Then $d_{\chi \varphi}^u = \pm 1$ for $i = 1, 2, 3$ and $d_{\chi \varphi}^u = 0$. We may choose a basic set for $b_u$ such that $d_{\chi \varphi}^u = 1$. Then by the orthogonality relations, $d_{\chi \varphi}^u = -\epsilon_1$ and $d_{\chi \varphi}^u = -\epsilon_2$. Next let $u := y$. Then $b_u$ dominates a block of $C_G(u)/\langle u \rangle$ with cyclic defect group $C_D(u)/\langle u \rangle \cong C_3$ and inertial index 2. This yields $\text{IBr}(b_u) = \{ \varphi_1, \varphi_2 \}$ and the Cartan matrix of $b_u$ is given by

$$
\begin{pmatrix}
3 & 2 & 1 \\
2 & 1 & 2
\end{pmatrix}
$$

(not only up to basic sets, but this is not important here). We can choose a basic set such that $(d_{\chi \varphi}^u, d_{\chi \varphi}^u) = (1, 1), (d_{\chi \varphi}^u, d_{\chi \varphi}^u) = (\sigma_1, 0), (d_{\chi \varphi}^u, d_{\chi \varphi}^u) = (0, \sigma_2)$ and $(d_{\chi \varphi}^u, d_{\chi \varphi}^u) = (0, 0)$ for some signs $\sigma_1, \sigma_2 \in \{ \pm 1 \}$. Finally for $u := xy$ we obtain $d_{\chi \varphi}^u = 1, d_{\chi \varphi}^u = -\sigma_1, d_{\chi \varphi}^u = -\sigma_2$ and $d_{\chi \varphi}^u = 0$ after changing the basic set if necessary. The following table summarizes the results

<table>
<thead>
<tr>
<th>$u$</th>
<th>$x^3$</th>
<th>$x$</th>
<th>$y$</th>
<th>$xy$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{\chi \varphi}^u$</td>
<td>$\epsilon_1$</td>
<td>$(\epsilon_1, 0)$</td>
<td>$(\epsilon_1, 0)$</td>
<td>$\epsilon_1$</td>
</tr>
<tr>
<td>$d_{\chi \varphi}^u$</td>
<td>$\epsilon_2$</td>
<td>$(0, \epsilon_2)$</td>
<td>$(0, \epsilon_2)$</td>
<td>$\epsilon_2$</td>
</tr>
<tr>
<td>$d_{\chi \varphi}^u$</td>
<td>$3 \epsilon_3$</td>
<td>$(\epsilon_3, \epsilon_3)$</td>
<td>$-\epsilon_3$</td>
<td>$-\epsilon_3$</td>
</tr>
</tbody>
</table>

It suffices to show that $\epsilon_1 = \sigma_1$ for $i = 1, 2$ (observe that we do not need the ordinary decomposition numbers in order to apply [10 Theorem 2]). For this, let $\lambda \in \text{Irr}(D/(x^3))$ such that $\lambda(x) = e^{2\pi i/3}$ and $\lambda(y) = 1$. Then the generalized character $\psi := \lambda + \overline{\lambda} \cdot 2 \cdot 1_D$ of $D$ is constant on $\langle x \rangle \setminus \langle x^3 \rangle$ and thus $F$-stable. By [11], $\chi_1 * \psi$ is a generalized character of $B$ and $(\chi_1 * \psi, \chi_2)_G \in \mathbb{Z}$. As in the proof of Theorem [9] we compute

$$(\chi_1 * \psi, \chi_2)_G = \sum_{u \in \mathcal{R}} \psi(u) m_{\chi_1 \chi_2}^u = \psi(x) m_{\chi_1 \chi_x}^x + \psi(xy) m_{\chi_1 \chi_x}^{xy} + \psi(xy^2) m_{\chi_1 \chi_x}^{xy^2} = \frac{1}{3} \epsilon_1 \epsilon_2.$$

This shows $\epsilon_1 = \sigma_1$. Similarly, one gets $\epsilon_2 = \sigma_2$ by computing $(\chi_1 * \psi, \chi_3)_G$. Hence, [10 Theorem 2] gives a perfect isometry $I : \text{CF}(G, B) \to \text{CF}(\hat{G}, \hat{B})$. In order to show that $I$ is also an isotypy, we make use of the notation introduced in the proof of Theorem [9]. Let $u \in D$ such that $b_u$ is nilpotent. Then by the table above, we have $\text{IBr}(b_u) = \{ \pm d_2^u(\chi_2) \}$. Thus, one can extend $I^u$ just as in Theorem [9]. Now suppose that $b_u$ is non-nilpotent and thus $u = y$ (up to inversion). We choose a basic set $\varphi_1, \varphi_2$ for $b_u$ as above such that $d_{\varphi_1}^u(\chi_1) = \varphi_{-1}$ for $i = 2, 3$. Now we have to determine the ordinary decomposition numbers of $b_u$ with respect to $\varphi_1, \varphi_2$. The defect group of $b_u$ is $C_D(y) = \langle x^3, y \rangle \cong C_3 \times C_3$ and $\text{soc}(b_u) = \langle x^3 \rangle$. By Kiyota [12], $k(b_u) = 9$. Therefore, there are 3-rational characters $\psi_i \in \text{Irr}(b_u)$ such that

$$\text{Irr}(b_u) = \{ \psi_i \cdot \lambda : i = 1, 2, 3, \lambda \in \text{Irr}(\langle x^3, y \rangle/\langle x^3 \rangle) \}.$$

By the Cartan matrix of $b_u$ given above (with respect to $\varphi_1, \varphi_2$), it follows immediately that $d_{\varphi_1}^u(\psi_i) = \epsilon_i \varphi_i$ with $\epsilon_i \in \{ \pm 1 \}$ for $i = 1, 2$ after a suitable permutation of $\psi_1, \psi_2, \psi_3$. Similarly, $d_{\varphi_2}^u(\psi_i) = \epsilon_i \varphi_i$. By a result of Usami [22], there is a perfect isometry $\text{CF}(C_G(u), b_u) \to \text{CF}(C_{\hat{G}}(u), \hat{b}_u)$. However, we need the additional information that $\psi_i$ is mapped to $\pm \psi_i$. In order to show this, we use [10 Theorem 2] again. Observe that $d_{\psi_1}^u(\psi_i) = \zeta d_{\psi_1}^u(\psi_i) = \zeta \epsilon_i \varphi_i$ for a cube root of unity $\zeta$. But since $d_{\psi_1}^u$ is rational, we have $\zeta = 1$. Now an elementary application of the orthogonality relations shows that the generalized decomposition matrix of $b_u$ (in $C_G(u)$) is determined by

<table>
<thead>
<tr>
<th>$v$</th>
<th>$1$</th>
<th>$y$</th>
<th>$x^3$</th>
<th>$x^3y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{\psi_1}^u$</td>
<td>$(\epsilon_1, 0)$</td>
<td>$(\epsilon_1, 0)$</td>
<td>$\epsilon_1$</td>
<td>$\epsilon_1$</td>
</tr>
<tr>
<td>$d_{\psi_2}^u$</td>
<td>$(0, \epsilon_2)$</td>
<td>$(0, \epsilon_2)$</td>
<td>$\epsilon_2$</td>
<td>$\epsilon_2$</td>
</tr>
<tr>
<td>$d_{\psi_3}^u$</td>
<td>$(\epsilon_3, \epsilon_3)$</td>
<td>$(\epsilon_3, \epsilon_3)$</td>
<td>$-\epsilon_3$</td>
<td>$-\epsilon_3$</td>
</tr>
</tbody>
</table>

It follows that there is a perfect isometry $\tilde{I}^u : \text{CF}(C_G(u), b_u) \to \text{CF}(C_{\hat{G}}(u), \hat{b}_u)$ such that $\tilde{I}^u(\psi_i) = \epsilon_i \epsilon_i \tilde{\psi_i}$ for $i = 1, 2$. Therefore $\tilde{I}^u$ extends $I^u$. In the proof of Theorem [9] it is also clear that $\tilde{I}^u$ is independent of the choice of the generator of $\langle u \rangle$. This finishes the proof.
The proof method of Theorem 11 also works for other defect groups. In fact, Watanabe [23] showed independently (using more complicated methods) that two $p$-blocks ($p > 2$) with a common metacyclic, minimal nonabelian defect group and the same fusion system are perfectly isometric. Again, this gives evidence for the character-theoretic version of Rouquier’s Conjecture (see [25, Theorem 2]). As another remark, Holloway-Koshitani-Kunugi [8, Example 4.3] constructed a perfect isometry between the principal $3$-block of $G := \text{Aut}(\text{SL}(2,8)) \cong 2G_2(3)$ and its Brauer correspondent. Since $G$ has a Sylow $3$-subgroup isomorphic to $C_9 \times C_3$, this is a special case of Theorem 11. Note that in the introduction of Ruengrot [18] it is erroneously stated that these blocks are not perfectly isometric.

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References


[23] A. Watanabe, *On blocks of finite groups with metacyclic, minimal non-abelian defect groups*, manuscript.
