The Alperin-McKay Conjecture for metacyclic, minimal non-abelian defect groups

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Abstract

We prove the Alperin-McKay Conjecture for all $p$-blocks of finite groups with metacyclic, minimal non-abelian defect groups. These are precisely the metacyclic groups whose derived subgroup have order $p$. In the special case $p = 3$, we also verify Alperin’s Weight Conjecture for these defect groups. Moreover, in case $p = 5$ we do the same for the non-abelian defect groups $C_{25} \rtimes C_5^n$. The proofs do not rely on the classification of the finite simple groups.

Keywords: Alperin-McKay Conjecture, metacyclic defect groups

AMS classification: 20C15, 20C20

1 Introduction

Let $B$ be a $p$-block of a finite group $G$ with respect to an algebraically closed field of characteristic $p$. Suppose that $B$ has a metacyclic defect group $D$. We are interested in the number $k_i(B)$ (respectively $k(B)$) of irreducible characters of $B$ (of height $i \geq 0$), and the number $l(B)$ of irreducible Brauer characters of $B$. If $p = 2$, these invariants are well understood and the major conjectures are known to be true by work of several authors (see [4, 31, 35, 37, 11, 9]). Thus we will focus on the case $p > 2$ in the present work. Here at least Brauer’s $k(B)$-Conjecture, Olsson’s Conjecture and Brauer’s Height Zero Conjecture are satisfied for $B$ (see [14, 43, 38]).

By a result of Stancu [40], $B$ is a controlled block. Moreover, if $D$ is a non-split extension of two cyclic groups, it is known that $B$ is nilpotent (see [7]). Thus we will focus on the case $p > 2$ in the present work. Here at least Brauer’s $k(B)$-Conjecture, Olsson’s Conjecture and Brauer’s Height Zero Conjecture are satisfied for $B$ (see [14, 43, 38]).

By a result of Stancu [40], $B$ is a controlled block. Moreover, if $D$ is a non-split extension of two cyclic groups, it is known that $B$ is nilpotent (see [7]). Then a result by Puig [33] describes the source algebra of $B$ in full detail. Thus we may assume in the following that $D$ is a split extension of two cyclic groups. A famous theorem by Dade [6] handles the case where $D$ itself is cyclic by making use of Brauer trees. The general situation is much harder – even the case $D \cong C_3 \times C_3$ is still open (see [21, 22, 20, 25]). Now consider the subcase where $D$ is non-abelian. Then a work by An [1] shows that $G$ is not a quasisimple group. On the other hand, the algebra structure of $B$ in the $p$-solvable case can be obtained from Külshammer [27]. If $B$ has maximal defect (i.e. $D \in \text{Syl}_p(G)$), the block invariants of $B$ were determined in [15]. If $B$ is the principal block, Horimoto and Watanabe [20] constructed a perfect isometry between $B$ and its Brauer correspondent in $N_G(D)$.

Let us suppose further that $D$ is a split extension of a cyclic group and a group of order $p$ (i.e. $D$ is the unique non-abelian group with a cyclic subgroup of index $p$). Here the difference $k(B) - l(B)$ is known from [10]. Moreover, under additional assumptions on $G$, Holloway, Koshitani and Kunugi [19] obtained the block invariants precisely. In the special case where $D$ has order $p^3$, incomplete information are given by Hendren [17]. Finally, one has full information in case $|D| = 27$ by work of the present author [38, Theorem 4.5].

In the present work we consider the following class of non-abelian split metacyclic groups

$$ D = \langle x, y \mid x^{p^m} = y^{p^n} = 1, \ yxy^{-1} = x^{1+p^{m-1}} \rangle \cong C_{p^m} \rtimes C_{p^n} \quad (1.1) $$

where $m \geq 2$ and $n \geq 1$. By a result of Rédei (see [24, Aufgabe III.7.22]) these are precisely the metacyclic, minimal non-abelian groups. A result by Knoche (see [21, Aufgabe III.7.24]) implies further that these are exactly the metacyclic groups with derived subgroup of order $p$. In particular the family includes the non-abelian group with a cyclic subgroup of index $p$ mentioned above. The main theorem of the present paper states that $k_0(B)$
is locally determined. In particular the Alperin-McKay Conjecture holds for $B$. Recall that the Alperin-McKay Conjecture asserts that $k_0(B) = k_0(b)$ where $b$ is the Brauer correspondent of $B$ in $\text{N}_G(D)$. This improves some of the results mentioned above. We also prove that every irreducible character of $B$ has height 0 or 1. This is in accordance with the situation in $\text{Irr}(D)$. In the second part of the paper we investigate the special case $p = 3$. Here we are able to determine $k(B)$, $k_i(B)$ and $l(B)$. This gives an example of Alperin’s Weight Conjecture and the Ordinary Weight Conjecture. Finally, we determine the block invariants for $p = 5$ and $D \cong C_{25} \times C_{5^m}$ where $n \geq 1$.

As a new ingredient (compared to [35]) we make use of the focal subgroup of $B$.

## 2 The Alperin-McKay Conjecture

Let $p$ be an odd prime, and let $B$ be a $p$-block with split metacyclic, non-abelian defect group $D$. Then $D$ has a presentation of the form

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1, \ yxy^{-1} = x^{1+p^l} \rangle$$

where $0 < l < m$ and $m - l \leq n$. Elementary properties of $D$ are stated in the following lemma.

**Lemma 2.1.**

(i) $D' = \langle x^p \rangle \cong C_{p^{m-1}}$.

(ii) $Z(D) = \langle x^{p^{m-l}} \rangle \times \langle y^{p^{m-l}} \rangle \cong C_p \times C_{p^{m-1}}$.

**Proof.** Omitted. □

We fix a Sylow subpair $(D, b_D)$ of $B$. Then the conjugation of subpairs $(Q, b_Q) \leq (D, b_D)$ forms a saturated fusion system $\mathcal{F}$ on $D$ (see [2, Proposition IV.3.14]). Here $Q \leq D$ and $b_Q$ is a uniquely determined block of $C_G(Q)$. We also have subsections $(u, b_u)$ where $u \in D$ and $b_u := b_{(u)}$. By Proposition 5.4 in [10], $\mathcal{F}$ is controlled. Moreover by Theorem 2.5 in [14] we may assume that the inertial group of $B$ has the form $\text{N}_G(D, b_D)/C_G(D) = \text{Aut}_\mathcal{F}(D) = \langle \text{lnn}(D), \alpha \rangle$ where $\alpha \in \text{Aut}(D)$ such that $\alpha(x) \in \langle x \rangle$ and $\alpha(y) = y$. By a slight abuse of notation we often write $\text{Out}_\mathcal{F}(D) = \langle \alpha \rangle$. In particular the inertial index $e(B) := |\text{Out}_\mathcal{F}(D)|$ is a divisor of $p - 1$. Let

$$\text{foc}(B) := \langle f(a)a^{-1} : a \in Q \leq D, \ f \in \text{Aut}_\mathcal{F}(Q) \rangle$$

be the focal subgroup of $B$ (or of $\mathcal{F}$). Then it is easy to see that $\text{foc}(B) \subseteq \langle x \rangle$. In case $e(B) = 1$, $B$ is nilpotent and $\text{foc}(B) = D'$. Otherwise $\text{foc}(B) = D$. For the convenience of the reader we collect some estimates on the block invariants of $B$.

**Proposition 2.2.** Let $B$ be as above. Then

$$(p^l + p^{l-1} - p^{2l-m-1} - 1 + e(B))p^n \leq k(B) \leq \left(\frac{p^l - 1}{e(B)} + e(B)\right)(p^{n+m-l-2} + p^n - p^{n-2}),$$

$$2p^n \leq k_0(B) \leq \left(\frac{p^l - 1}{e(B)} + e(B)\right)p^n,$$

$$\sum_{i=0}^{\infty} p^{2i}k_i(B) \leq \left(\frac{p^l - 1}{e(B)} + e(B)\right)p^{n+m-l},$$

$$l(B) \geq e(B) \mid p - 1,$$

$$p^n \mid k_0(B), \ p^n-\mathbb{m}+1 \mid k_i(B) \text{ for } i \geq 1,$$

$$k_i(B) = 0 \text{ for } i > 2(m - l).$$

**Proof.** Most of the inequalities are contained in Proposition 2.1 to Corollary 2.5 in [38]. By Theorem 1 in [36] we have $p^n \mid |D : \text{foc}(B)| \mid k_0(B)$. In particular $p^n \leq k_0(B)$. In case $k_0(B) = p^n$ it follows from [23] that $B$ is nilpotent. However then we would have $k_0(B) = |D : D'| = p^{n+l} > p^n$. Therefore $2p^n \leq k_0(B)$. Theorem 2 in [36] implies $p^{n-\mathbb{m}+1} \mid |Z(D) : Z(D) \cap \text{foc}(B)| \mid k_i(B)$ for $i \geq 1$. □
Now we consider the special case where \(m = l + 1\). As mentioned in the introduction these are precisely the metacyclic, minimal non-abelian groups. We prove the main theorem of this section.

**Theorem 2.3.** Let \(B\) be a \(p\)-block of a finite group with metacyclic, minimal non-abelian defect groups for an odd prime \(p\). Then

\[
k_0(B) = \left( \frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^n
\]

with the notation from \((1.1)\). In particular the Alperin-McKay Conjecture holds for \(B\).

**Proof.** By Proposition 2.2 we have

\[
p^n \mid k_0(B) \leq \left( \frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^n.
\]

Thus, by way of contradiction we may assume that

\[
k_0(B) \leq \left( \frac{p^{m-1} - 1}{e(B)} + e(B) - 1 \right) p^n.
\]

We also have

\[
k(B) \geq \left( \frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B) \right) p^n
\]

from Proposition 2.2. Hence the sum \(\sum_{i=0}^{\infty} p^{2i} k_i(B)\) will be small if \(k_0(B)\) is large and \(k_1(B) = k(B) - k_0(B)\). This implies the following contradiction

\[
\left( \frac{p^{m-1} - 1}{e(B)} + p^2 + e(B) - 1 \right) p^n = \left( \frac{p^{m-1} - 1}{e(B)} + e(B) - 1 \right) p^n + \left( \frac{p^{m-2} - p^{m-3} - 1}{e(B)} + 1 \right) p^{n+2}
\]

\[
\leq \sum_{i=0}^{\infty} p^{2i} k_i(B) \leq \left( \frac{p^{m-1} - 1}{e(B)} + p e(B) \right) p^n < \left( \frac{p^{m-1} - 1}{e(B)} + p^2 \right) p^n.
\]

Since the Brauer correspondent of \(B\) in \(N_G(D)\) has the same fusion system, the Alperin-McKay Conjecture follows.

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Isaacs and Navarro [22, Conjecture D] proposed a refinement of the Alperin-McKay Conjecture by invoking Galois automorphisms. We show (as an improvement of Theorem 4.3 in [38]) that this conjecture holds in the special case \(|D| = p^3\) of Theorem 2.3. We will denote the subset of \(\text{Irr}(B)\) of characters of height 0 by \(\text{Irr}_0(B)\).

**Corollary 2.4.** Let \(B\) be a \(p\)-block of a finite group \(G\) with non-abelian, metacyclic defect group of order \(p^3\). Then Conjecture D in [22] holds for \(B\).

**Proof.** Let \(D\) be a defect group of \(B\). For \(k \in \mathbb{N}\), let \(\mathbb{Q}_k\) be the cyclotomic field of degree \(k\). Let \(|G|_{p'}\) be the \(p'\)-part of the order of \(G\). It is well-known that the Galois group \(G := \text{Gal}(\mathbb{Q}_{|G|} \mid \mathbb{Q}_{|G|_{p'}})\) acts canonically on \(\text{Irr}(B)\). Let \(\gamma \in G\) be a \(p\)-element. Then it suffices to show that \(\gamma\) acts trivially on \(\text{Irr}_0(B)\). By Lemma IV.6.10 in [12] it is enough to prove that \(\gamma\) acts trivially on the \(F\)-conjugacy classes of subsections of \(B\) via \(\gamma(u, b_u) := (u^\tau, b_u)\) where \(u \in D\) and \(\tau \in \mathbb{Z}\). Since \(\gamma\) is a \(p\)-element, this action is certainly trivial unless \(|\langle u \rangle| = p^2\). Here however, the action of \(\gamma\) on \(\langle u \rangle\) is just the \(D\)-conjugation. The result follows.

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In the situation of Corollary 2.4 one can say a bit more: By Proposition 3.3 in [38], \(\text{Irr}(B)\) splits into the following families of \(p\)-conjugate characters:

- \((p - 1)/e(B) + e(B)\) orbits of length \(p - 1\),
- two orbits of length \((p - 1)/e(B)\),
- at least \(e(B)\) \(p\)-rational characters.
Without loss of generality, let $e(B) > 1$. By Theorem 4.1 in [35] we have $k_1(B) \leq (p-1)/e(B) + e(B) - 1$. Moreover, Proposition 4.1 of the same paper implies $k_1(B) < p - 1$. In particular, all orbits of length $p-1$ of $p$-conjugate characters must lie in Irr$_0(B)$. In case $e(B) = p - 1$ the remaining $(p-1)/e(B) + e(B)$ characters in Irr$_0(B)$ must be $p$-rational. Now let $e(B) < \sqrt{p - 1}$. Then it is easy to see that Irr$_0(B)$ contains just one orbit of length $(p-1)/e(B)$ of $p$-conjugate characters. Unfortunately, it is not clear if this also holds for $e(B) \geq \sqrt{p - 1}$.

Next we improve the bound coming from Proposition 2.2 on the heights of characters.

**Proposition 2.5.** Let $B$ be a $p$-block of a finite group with metacyclic, minimal non-abelian defect groups. Then $k_1(B) = k(B) - k_0(B)$. In particular, $B$ satisfies the following conjectures:

- Eaton’s Conjecture [39]
- Eaton-Moretó Conjecture [10]
- Robinson’s Conjecture [23, Conjecture 4.14.7]
- Malle-Navarro Conjecture [29]

**Proof.** By Theorem 2 in [37] we may assume $p > 2$ as before. By way of contradiction suppose that $k_1(B) > 0$ for some $i \geq 2$. Since

$$k(B) \geq \left(\frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B)\right)p^n,$$

we have $k(B) - k_0(B) \geq (p^{m-1} - p^{m-2})p^{n-1}/e(B)$ by Theorem 2.3. By Proposition 2.2 $k_1(B)$ and $k_i(B)$ are divisible by $p^{n-1}$. This shows

$$\left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right)p^n + \left(\frac{p^{m-1} - p^{m-2}}{e(B)} - 1\right)p^{n+1} + p^{n+3} \leq \sum_{i=0}^{\infty} p^i k_i(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right)p^{n+1}.$$

Hence, we derive the following contradiction

$$p^{n+3} - p^{n+1} \leq \left(\frac{1 - p}{e(B)} + e(B)(p - 1)\right)p^n \leq p^{n+2}.$$

This shows $k_1(B) = k(B) - k_0(B)$. Now Eaton’s Conjecture is equivalent to Brauer’s $k(B)$-Conjecture and Olsson’s Conjecture. Both are known to hold for all metacyclic defect groups. Also the Eaton-Moretó Conjecture and Robinson’s Conjecture are trivially satisfied for $B$. The Malle-Navarro Conjecture asserts that $k(B)/k_0(B) \leq k(D) = p$ and $k(B)/l(B) \leq k(D)$, By Theorem 2.3 and Proposition 2.2, the first inequality reduces to $p^{n-1} + p^n - p^{n-2} \leq p^{n+1}$ which is true. For the second inequality we observe that every conjugacy class of $D$ has at most $p$ elements, since $|D : Z(D)| \equiv p^2$. Hence, $k(D) = |Z(D)| + \frac{|D| - |Z(D)|}{p} = p^{n+m-1} + p^{n+m-2} - p^{n+m-3}$.

Now Proposition 2.2 gives

$$\frac{k(B)}{l(B)} \leq k(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right)(p^{n-1} + p^n - p^{n-2}) \leq p^{n+m-1} + p^{n+m-2} - p^{n+m-3} = k(D).$$

We use the opportunity to present a result for $p = 3$ and a different class of metacyclic defect groups (where $l = 1$ with the notation above).

**Theorem 2.6.** Let $B$ be a $3$-block of a finite group $G$ with defect group

$$D = \langle x, y \mid x^{3^m} = y^{3^n} = 1, \ yxy^{-1} = x^4 \rangle,$$

where $2 \leq m \leq n + 1$. Then $k_0(B) = 3^{n+1}$. In particular, the Alperin-McKay Conjecture holds for $B$.

**Proof.** We may assume that $B$ is non-nilpotent. By Proposition 2.2 we have $k_0(B) \in \{2 \cdot 3^n, 3^{n+1}\}$. By way of contradiction, suppose that $k_0(B) = 2 \cdot 3^n$. Let $P \in Syl_p(G)$. Since $D/\ocs(B)$ acts freely on Irr$_0(B)$, there are $3^n$ characters of degree $a|P : D|$, and $3^m$ characters of degree $b|P : D|$ in $B$ for some $a, b \geq 1$ such that $3 \mid a, b.$ Hence,

$$\sum_{\chi \in \text{Irr}(B)} \chi(1)^2 \mid = 3^n |P : D|^2 (a^2 + b^2) = |P : D|^2 |D : \ocs(B)|.$$
A generalization of the argument in the proof shows that in the situation of Proposition 2.2, $k_0(B) = 2p^n$ can only occur if $p \equiv 1 \pmod{4}$.

3 Lower defect groups

In the following we use the theory of lower defect groups in order to estimate $l(B)$. We cite a few results from the literature. Let $B$ be a $p$-block of a finite group $G$ with defect group $D$ and Cartan matrix $C$. We denote the multiplicity of an integer $a$ as elementary divisor of $C$ by $m(a)$. Then $m(a) = 0$ unless $a$ is a $p$-power. It is well-known that $m(|D|) = 1$. Brauer [3] expressed $m(p^n) (n \geq 0)$ in terms of $1$-multiplicities of lower defect groups (see also Corollary V.10.12 in [12]):

$$m(p^n) = \sum_{R \in \mathcal{R}} m^{(1)}_B(R)$$

where $\mathcal{R}$ is a set of representatives for the $G$-conjugacy classes of subgroups $R \leq D$ of order $p^n$. Later (3.1) was refined by Broué and Olsson by invoking the fusion system $\mathcal{F}$ of $B$.

**Proposition 3.1** (Broué-Olsson [5]). For $n \geq 0$ we have

$$m(p^n) = \sum_{R \in \mathcal{R}} m^{(1)}_B(R, b_R)$$

where $\mathcal{R}$ is a set of representatives for the $\mathcal{F}$-conjugacy classes of subgroups $R \leq D$ of order $p^n$.

**Proof.** This is (2S) of [5].

In the present paper we do not need the precise (and complicated) definition of the non-negative numbers $m^{(1)}_B(R)$ and $m^{(1)}_B(R, b_R)$. We say that $R$ is a lower defect group for $B$ if $m^{(1)}_B(R, b_R) > 0$. In particular, $m^{(1)}_B(D, b_D) = m^{(1)}_B(D) = m(|D|) = 1$. A crucial property of lower defect groups is that their multiplicities can usually be determined locally. In the next lemma, $b^{N_G(R, b_R)}_R$ denotes the (unique) Brauer correspondent of $b_R$ in $N_G(R, b_R)$.

**Lemma 3.2.** For $R \leq D$ and $B_R := b^{N_G(R, b_R)}_R$ we have $m^{(1)}_B(R, b_R) = m^{(1)}_B(R)$. If $R$ is fully $\mathcal{F}$-normalized, then $B_R$ has defect group $N_D(R)$ and fusion system $N_{\mathcal{F}}(R)$.

**Proof.** The first claim follows from (2Q) in [5]. For the second claim we refer to Theorem IV.3.19 in [2].

Another important reduction is given by the following lemma.

**Lemma 3.3.** For $R \leq D$ we have $\sum_{Q \in \mathcal{R}} m^{(1)}_B(Q) \leq l(b_R)$ where $\mathcal{R}$ is a set of representatives for the $N_G(R, b_R)$-conjugacy classes of subgroups $Q$ such that $R \leq Q \leq N_D(R)$.

**Proof.** This is implied by Theorem 5.11 in [32] and the remark following it. Notice that in Theorem 5.11 it should read $B \in \text{Bl}(G)$ instead of $B \in \text{Bl}(Q)$.

In the local situation for $B_R$ also the next lemma is useful.

**Lemma 3.4.** If $O_p(Z(G)) \nsubseteq R$, then $m^{(1)}_B(R) = 0$.

**Proof.** See Corollary 3.7 in [32].

Now we apply these results.

**Lemma 3.5.** Let $B$ be a $p$-block of a finite group with metacyclic, minimal non-abelian defect group $D$ for an odd prime $p$. Then every lower defect group of $B$ is $D$-conjugate either to $\langle y \rangle$, $\langle y^p \rangle$, or to $D$. 

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Proof. Let $R < D$ be a lower defect group of $B$. Then $m(|R|) > 0$ by Proposition 3.1 Corollary 5 in [32] shows that $p^{e - 1} | |R|$. Since $F$ is controlled, the subgroup $R$ is fully $F$-centralized and fully $F$-normalized. The fusion system of $b_R$ on $C_D(R)$ is given by $C_F(R)$ (see Theorem IV.3.19 in [2]). Suppose for the moment that $C_F(R)$ is trivial. Then $b_R$ is nilpotent and $l(b_R) = 1$. Let $B_R := b_R^{N_D(R,b_R)}$. Then $B_R$ has defect group $N_D(R)$ and $m^{(1)}_{N_D}(N_D(R)) = 1$. Hence, Lemmas 5.2 and 5.3 imply $m^{(1)}_{B_R}(R,b_R) = m^{(1)}_{B_R}(R) = 0$. This contradiction shows that $C_F(R)$ is non-trivial. In particular $R$ is centralized by a non-trivial $p'$-automorphism $\beta \in \Aut_D(R)$. By the Schur-Zassenhaus Theorem, $\beta$ is $\Inn(D)$-conjugate to a power of $\alpha$. Thus, $R$ is $D$-conjugate to a subgroup of $(y)$. The result follows.

Proposition 3.6. Let $B$ be a $p$-block of a finite group with metacyclic, minimal non-abelian defect groups for an odd prime $p$. Then $e(B) \leq l(B) \leq 2e(B) - 1$.

Proof. Let  

$$D = \langle x, y \mid xp^n = y^{p^n} = 1, xy^{p^n - 1} = x^{1 + p^{m - 1}} \rangle$$

be a defect group of $B$. We argue by induction on $n$. Let $n = 1$. By Proposition 2.2 we have $e(B) \leq l(B)$ and

$$k(B) \leq \left( \frac{p^{m - 1} - 1}{e(B)} + e(B) \right)(1 + p - p^{-1}).$$

Moreover, Theorem 3.2 in [32] gives

$$k(B) - l(B) = \frac{p^{m} + p^{m - 1} - p^{m - 2} - p}{e(B)} + e(B)(p - 1).$$

Hence,

$$l(B) = k(B) - (k(B) - l(B)) \leq \left( \frac{p^{m - 1} - 1}{e(B)} + e(B) \right)(1 + p - p^{-1}) - \frac{p^{m} + p^{m - 1} - p^{m - 2} - p}{e(B)} - e(B)(p - 1)$$

$$= 2e(B) - \frac{1}{p} \left( e(B) - \frac{1}{e(B)} \right) - 1/e(B),$$

and the claim follows in this case.

Now suppose $n \geq 2$. We determine the multiplicities of the lower defect groups by using Lemma 3.5. As usual $m(|D|) = 1$. Consider the subpair $(\langle y \rangle, b_y)$. By Lemmas 3.1 and 3.2 we have $m(p^n) = m^{(1)}_{b_y}(\langle y \rangle, b_y) = m^{(1)}_{b_y}(\langle y \rangle)$ where $B_y := b_y^{N_G(\langle y \rangle, b_y)}$. Since $N_D(\langle y \rangle) = C_D(y)$, it follows easily that $N_G(\langle y \rangle, b_y) = C_G(y)$ and $B_y = b_y$. By Theorem IV.3.19 in [2] the block $b_y$ has defect group $C_D(y)$ and fusion system $C_F(y)$. In particular $e(b_y) = e(B)$. It is well-known that $b_y$ dominates a block $b_y$ of $C_D(y)/\langle y \rangle$ with cyclic defect group $C_D(y)/\langle y \rangle$ and $e(b_y) = e(b_y) = e(B)$ (see [30] Theorem 5.8.11). By Dade’s Theorem [6] on blocks with cyclic defect groups we obtain $l(b_y) = e(B)$. Moreover, the Cartan matrix of $B_y$ has elementary divisors $p^n$ and $|C_D(y)|$ where $p^n$ occurs with multiplicity $e(B) - 1$. Since $\langle y \rangle \subseteq Z(C_G(y))$, Lemma 3.4 implies $m(p^n) = m^{(1)}_{b_y}(\langle y \rangle) = e(B) - 1$.

Now consider $(\langle u \rangle, b_u)$ where $u := y^p \in Z(D)$. Here $b_u$ has defect group $D$. By the first part of the proof (the case $n = 1$) we obtain $l(b_u) = l(b_u) \leq 2e(B) - 1$. As above we have $m(p^{n - 1}) = m^{(1)}_{b_u}(\langle u \rangle, b_u) = m^{(1)}_{b_u}(\langle u \rangle)$. Since $p^n$ occurs as elementary divisor of the Cartan matrix of $b_u$ with multiplicity $e(B) - 1$ (see above), it follows that $m(p^{n - 1}) = m^{(1)}_{b_u}(\langle u \rangle) \leq e(B) - 1$. Now $l(B)$ is the sum over the multiplicities of elementary divisors of the Cartan matrix of $B$ which is at most $m(|D|) + m(\langle y \rangle) + m(\langle u \rangle) \leq 1 + e(B) - 1 + e(B) - 1 = 2e(B) - 1$.

The next proposition gives a reduction method.

Proposition 3.7. Let $p > 2$, $m \geq 2$ and $e | p - 1$ be fixed. Suppose that $l(B) = e$ holds for every block $B$ with defect group  

$$D = \langle x, y \mid xp^n = y^{p^n} = 1, xy^{p^n - 1} = x^{1 + p^{m - 1}} \rangle$$
and \( e(B) = e \). Then every block \( B \) with \( e(B) = e \) and defect group
\[
D = \langle x, y \mid x^{p^m} = y^p = 1, \ yxy^{-1} = x^{1 + p^{m-1}} \rangle
\]
where \( n \geq 1 \) satisfies the following:

\[
k_0(B) = \left( \frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^n, \quad k_1(B) = \frac{p^{m-1} - p^{m-2}}{e(B)} p^{n-1},
\]
\[
k(B) = \left( \frac{p^n + p^{m-1} - p^{m-2} - p}{e(B)} + e(B)p \right) p^{n-1}, \quad l(B) = e(B).
\]

**Proof.** We use induction on \( n \). In case \( n = 1 \) the result follows from Theorem 3.2 in [38], Theorem 2.3 and Proposition 2.5.

Now let \( n \geq 2 \). Let \( R \) be a set of representatives for the \( \mathcal{F} \)-conjugacy classes of elements of \( D \). We are going to use Theorem 5.9.4 in [30]. For \( 1 \neq u \in R \), \( b_u \) has metacyclic defect group \( C_D(u) \) and fusion system \( C_{\mathcal{F}}(\langle u \rangle) \).

If \( C_{\mathcal{F}}(\langle u \rangle) \) is non-trivial, \( \alpha \in \text{Aut}_F(D) \) centralizes a \( D \)-conjugate of \( u \). Hence, we may assume that \( u \in (g) \) in this case. If \( \langle u \rangle = (g) \), then \( b_u \) dominates a block \( b_u \) of \( C_G(u) \) with cyclic defect group \( C_D(u)/\langle u \rangle \). Hence, \( l(b_u) = l(b_u) = e(B) \). Now suppose that \( \langle u \rangle < (g) \). Then by induction we obtain \( l(b_u) = l(b_u) = e(B) \). Finally assume that \( C_{\mathcal{F}}(\langle u \rangle) \) is trivial. Then \( b_u \) is nilpotent and \( l(b_u) = 1 \). It remains to determine \( R \). The powers of \( y \) are pairwise non-conjugate in \( F \). As in the proof of Proposition 2.5 \( D \) has precisely \( p^n + m - 1 \) conjugacy classes. Let \( C \) be one of these classes which do not intersect \( \langle y \rangle \). Assume \( \alpha^i(C) = C \) for some \( i \in \mathbb{Z} \) such that \( \alpha^i \neq 1 \). Then there are elements \( u \in C \) and \( w \in D \) such that \( \alpha^i(u) = wuw^{-1} \). Hence \( \gamma := w^{-1} \alpha^i \in N_G(D, b_p) \cap C_G(u) \). Since \( \gamma \) is not a \( p \)-element, we conclude that \( u \) is conjugate to a power of \( y \) which was excluded. This shows that no nontrivial power of \( \alpha \) can fix \( C \) as a set. Thus, all these conjugacy classes split in
\[
\frac{p^2 + p - p^{m-3} - 1}{e(B)} p^{n+m-3}
\]
orbits of length \( e(B) \) under the action of \( \text{Out}_F(D) \). Now Theorem 5.9.4 in [30] implies
\[
k(B) - l(B) = \left( \frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B) \right) p^n - e(B).
\]
By Proposition 3.6 it follows that
\[
k(B) \leq \left( \frac{p^{m-1} + p^{m-2} - p}{e(B)} + e(B)p \right) p^{n-1} + e(B) - 1. \tag{3.2}
\]
By Proposition 2.2 the left hand side of (3.2) is divisible by \( p^{n-1} \). Since \( e(B) - 1 < p^{n-1} \), we obtain the exact value of \( k(B) \). It follows that \( l(B) = e(B) \). Finally, Theorem 2.3 and Proposition 2.5 give \( k_i(B) \).

For \( p = 3 \), Proposition 3.6 implies \( l(B) \leq 3 \). Here we are able to determine all block invariants.

**Theorem 3.8.** Let \( B \) be a non-nilpotent \( 3 \)-block of a finite group with metacyclic, minimal non-abelian defect groups. Then
\[
k_0(B) = \frac{3^{m-2} + 1}{2} 3^{n+1}, \quad k_1(B) = 3^{m+n-3},
\]
\[
k(B) = \frac{11 \cdot 3^{m-2} + 9}{2} 3^{n-1}, \quad l(B) = e(B) = 2
\]
with the notation from (1.1).

**Proof.** By Proposition 3.7 it suffices to settle the case \( n = 1 \). Here the claim holds for \( m \leq 3 \) by Theorem 3.7 in [38]. We will extend the proof of this result in order to handle the remaining \( m \geq 4 \). Since \( B \) is non-nilpotent, we have \( e(B) = 2 \). By Theorem 2.3 we know \( k_0(B) = (3^m + 9)/2 \). By way of contradiction we may assume that \( l(B) = 3 \) and \( k_1(B) = 3^{m-2} + 1 \) (see Theorem 3.4 in [38]).
We consider the generalized decomposition numbers \( d^2 \chi_{\rho} \), where \( z := x^3 \in \mathbb{Z}(D) \) and \( \rho_z \) is the unique irreducible Brauer character of \( b_z \). Let \( d^2 := (d^2 \chi_{\rho} : \chi \in \text{Irr}(B)) \). By the orthogonality relations we have \( (d^2, d^3) = 3^{m+1} \). As in [18] Section 4 we can write

\[
d^2 = \sum_{i=0}^{2 \cdot 3^{m-2} - 1} a_i \chi_{\rho}^i
\]

for integral vectors \( a_i \) and a primitive \( 3^{m-1} \)-th root of unity \( \zeta_{3^m-1} \in \mathbb{C} \). Since \( z \) is \( F \)-conjugate to \( z^{-1} \), the vector \( d^2 \) is real. Hence, the vectors \( a_i \) are linearly dependent. More precisely, it turns out that the vectors \( a_i \) are spanned by \( \{ a_j : j \in J \} \) for a suitable ordering of \( J \). We remark that Alperin’s Weight Conjecture is also true for the abelian defect groups.

Let \( q \) be the quadratic form corresponding to the Dynkin diagram of type \( A_{3m-2} \). We set \( a(\chi) := (a_j(\chi) : j \in J) \) for \( \chi \in \text{Irr}(B) \). Since the subsection \( (z, b_z) \) gives equality in Theorem 4.10 in [18], we have

\[
k_0(B) + 9k_1(B) = \sum_{\chi \in \text{Irr}(B)} q(a(\chi))
\]

for a suitable ordering of \( J \). This implies \( q(a(\chi)) = 3^{2h(\chi)} \) for \( \chi \in \text{Irr}(B) \) where \( h(\chi) \) is the height of \( \chi \). Moreover, if \( a_0(\chi) \neq 0 \), then \( a_0(\chi) = \pm 3^{h(\chi)} \) by Lemma 3.6 in [38]. By Lemma 4.7 in [18] we have \( (a_0, a_0) = 27 \).

In the next step we determine the number \( \beta \) of \( 3 \)-rational characters of of height 1. Since \( (a_0, a_0) = 27 \), we have \( \beta < 4 \). On the other hand, the Galois group \( G \) of \( \mathbb{Q}(\zeta_{3^m-1}) \cap \mathbb{R} \) over \( \mathbb{Q} \) acts on \( d^2 \) and the length of every non-trivial orbit is divisible by 3 (because \( G \) is a 3-group). This implies \( \beta = 1 \), since \( k_1(B) = 3^{m-2} + 1 \).

In order to derive a contradiction, we repeat the argument with the subsection \((x, b_x)\). Again we get equality in Theorem 4.10, but this time for \( k_0(B) \) instead of \( k_0(B) + 9k_1(B) \). Hence, \( d^2(\chi) = 0 \) for characters \( \chi \in \text{Irr}(B) \) of height 1. Again we can write \( d^2 = \sum_{i=0}^{2 \cdot 3^{m-1} - 1} \pi_i \zeta_{3^m} \) where \( \pi_i \) are integral vectors. Lemma 4.7 in [18] implies \( (\pi_0, \pi_0) = 9 \). Using Lemma 3.6 in [38] we also have \( \pi_0(\chi) \in \{0, \pm 1\} \). By Proposition 3.3 in [38] we have precisely three \( 3 \)-rational characters \( \chi_1, \chi_2, \chi_3 \in \text{Irr}(B) \) of height 0 (note that altogether we have four \( 3 \)-rational characters). Then \( a_0(\chi_i) = \pm \pi_0(\chi_i) = \pm 1 \) for \( i = 1, 2, 3 \). By [36] Section 1 we have \( \lambda \ast \chi_i \in \text{Irr}_0(B) \) and \( (\lambda \ast \chi_i)(u) = \chi_i(u) \) for \( \lambda \in \text{Irr}(D/\text{pc}(B)) \equiv C_3 \) and \( u \in \{x, z\} \). Since this action on \( \text{Irr}(B) \) is free, we have three characters \( \psi \in \text{Irr}(B) \) such that \( a_0(\psi) = \pm \pi_0(\psi) = \pm 1 \). In particular \( (a_0, \pi_0) \equiv 1 \) (mod 2). By the orthogonality relations we have \( (d^2, d^r) = 0 \) for all \( j \in \mathbb{Z} \) such that \( 3 \nmid j \). Using Galois theory we get the final contradiction \( 0 = (d^2, \pi_0) = (a_0, \pi_0) \equiv 1 \) (mod 2).

In the smallest case \( D \cong C_9 \times C_3 \) of Theorem 3.3 even more information on \( B \) were given in Theorem 4.5 in [38].

**Corollary 3.9.** Alperin’s Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every 3-block with metacyclic, minimal non-abelian defect groups.

**Proof.** Let \( D \) be a defect group of \( B \). Since \( B \) is controlled, Alperin’s Weight Conjecture asserts that \( l(B) = l(B_D) \) where \( B_D \) is a Brauer correspondent of \( B \) in \( N_G(D) \). Since both numbers equal \( e(B) \), the conjecture holds.

Now we prove the Ordinary Weight Conjecture in the form of [2] Conjecture IV.5.49. Since \( \text{Out}_F(D) \) is cyclic, all 2-cocycles appearing in this version are trivial. Therefore the conjecture asserts that \( k(B) \) only depends on \( F \) and thus on \( e(B) \). Since the conjecture is known to hold for the principal block of the solvable group \( G = D \rtimes C_e(B) \), the claim follows.

We remark that Alperin’s Weight Conjecture is also true for the abelian defect groups \( D \cong C_{3^n} \times C_{3^m} \) where \( n \neq m \) (see [11] [34]).

We observe another consequence for arbitrary defect groups.

**Corollary 3.10.** Let \( B \) be a 3-block of a finite group with defect group \( D \). Suppose that \( D/\langle z \rangle \) is metacyclic, minimal non-abelian for some \( z \in \mathbb{Z}(D) \). Then Brauer’s \( k(B) \)-Conjecture holds for \( B \), i.e. \( k(B) \leq |D| \).
Proof. Let \((z, b_z)\) be a major subsection of \(B\). Then \(b_z\) dominates a block \(\overline{b_z}\) of \(C_G(z)/\langle z \rangle\) with metacyclic, minimal non-abelian defect group \(D/\langle z \rangle\). Hence, Theorem 3.10 implies \(l(b_z) = l(\overline{b_z}) \leq 2\). Now the claim follows from Theorem 2.1 in \[38\].

In the situation of Theorem 3.8 it is straight-forward to distribute \(\text{Irr}(B)\) into families of \(3\)-conjugate and \(3\)-rational characters (cf. Proposition 3.3 in \[38\]). However, it is not so easy to see which of these families lie in \(\text{Irr}_0(B)\).

Now we turn to \(p = 5\).

**Theorem 3.11.** Let \(B\) be a \(5\)-block of a finite group with non-abelian defect group \(C_{25} \rtimes C_{5^n}\) where \(n \geq 1\). Then

\[
\begin{align*}
k_0(B) &= \left(\frac{4}{e(B)} + e(B)\right)5^n, \\
k_1(B) &= \frac{4}{e(B)}5^{n-1}, \\
l(B) &= e(B).
\end{align*}
\]

Proof. By Proposition 3.7 it suffices to settle the case \(n = 1\). Moreover by Theorem 4.4 in \[38\] we may assume that \(e(B) = 4\). Then by Theorem 2.3 above and Proposition 4.2 in \[38\] we have \(k_0(B) = 25\), \(1 \leq k_1(B) \leq 3, 26 \leq k(B) \leq 28\) and \(4 \leq l(B) \leq 6\). We consider the generalized decomposition numbers \(d_{\psi^n,z}^{x}\), where \(z := x^5 \in \mathbb{Z}(D)\) and \(\psi_z\) is the unique irreducible Brauer character of \(b_z\). Since all non-trivial powers of \(z\) are \(F\)-conjugate, the numbers \(d_{\psi^n,z}^{x}\) are integral. Also, these numbers are non-zero, because \((z, b_z)\) is a major subsection. Moreover, \(d_{\psi^n,z}^{x} \equiv 0 \text{ (mod } p)\) for characters \(\chi \in \text{Irr}(B)\) of height 1 (see Theorem V.9.4 in \[12\]). Let \(d^2 := (d_{\psi^n,z}^{x} : \chi \in \text{Irr}(B))\). By the orthogonality relations we have \((d^2, d^2) = 125\). Suppose by way of contradiction that \(k_1(B) > 1\). Then it is easy to see that \(d_{\psi^n,z}^{x} \equiv \pm 5\) for characters \(\chi \in \text{Irr}(B)\) of height 1. By \[38\] Section 1, the numbers \(d_{\psi^n,z}^{x}\) \((\chi \in \text{Irr}(B))\) split in five orbits of length 5 each. Let \(\alpha\) (respectively \(\beta, \gamma\)) be the number of orbits of entries \(\pm 1\) (respectively \(\pm 2, \pm 3\)) in \(d^2\). Then the orthogonality relations reads

\[
\alpha + 4\beta + 9\gamma + 5k_1(B) = 25.
\]

Since \(\alpha + \beta + \gamma = 5\), we obtain

\[
3\beta + 8\gamma = 20 - 5k_1(B) \in \{5, 10\}.
\]

However, this equation cannot hold for any choice of \(\alpha, \beta, \gamma\). Therefore we have proved that \(k_1(B) = 1\). Now Theorem 4.1 in \[38\] implies \(l(B) = 4\).

**Corollary 3.12.** Alperin’s Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every \(5\)-block with non-abelian defect group \(C_{25} \rtimes C_{5^n}\).

Proof. See Corollary 3.9.

Unfortunately, the proof of Theorem 3.11 does not work for \(p = 7\) and \(e(B) = 6\) (even by invoking the other generalized decomposition numbers). However, we have the following partial result.

**Proposition 3.13.** Let \(p \in \{7, 11, 13, 17, 23, 29\}\) and let \(B\) be a \(p\)-block of a finite group with defect group \(C_{p^2} \rtimes C_{p^n}\) where \(n \geq 1\). If \(e(B) = 2\), then

\[
\begin{align*}
k_0(B) &= \frac{p + 3}{2}p^n, \\
k_1(B) &= \frac{p - 1}{2}2p^{n-1}, \\
l(B) &= 2.
\end{align*}
\]

Proof. We follow the proof of Theorem 4.4 in \[38\] in order to handle the case \(n = 1\). After that the result follows from Proposition 3.7.
In fact the first part of the proof of Theorem 4.4 in [38] applies to any prime $p \geq 7$. Hence, we know that the generalized decomposition numbers $a_{\chi, z} = a_0(\chi)$ for $z := x^p$ and $\chi \in \text{Irr}_0(B)$ are integral. Moreover,

$$\sum_{\chi \in \text{Irr}_0(B)} a_0(\chi)^2 = p^2.$$ 

The action of $D/\text{foc}(B)$ on $\text{Irr}_0(B)$ shows that the values $a(\chi)$ distribute in $(p+3)/2$ parts of $p$ equal numbers each. Therefore, Eq. (4.1) in [38] becomes

$$\sum_{i=2}^{\infty} r_i(i^2 - 1) = \frac{p - 3}{2}$$

for some $r_i \geq 0$. This gives a contradiction. 

\[\square\]

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