Abstract

Let $Z_B$ be the center of a $p$-block $B$ of a finite group with defect group $D$. We show that the Loewy length $LL(Z_B)$ of $Z_B$ is bounded by $|D| + p - 1$ provided $D$ is not cyclic. If $D$ is non-abelian, we prove the stronger bound $LL(Z_B) < \min\{p^{d-1}, 4p^{d-2}\}$ where $|D| = p^d$. Conversely, we classify the blocks $B$ with $LL(Z_B) \geq \min\{p^{d-1}, 4p^{d-2}\}$. This extends some results previously obtained by the present authors. Moreover, we characterize blocks with uniserial center.

Keywords: center of blocks, Loewy length

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1 Introduction

The aim of this paper is to extend some results on Loewy lengths of centers of blocks obtained in [8, 11]. In the following we will reuse some of the notation introduced in [8]. In particular, $B$ is a block of a finite group $G$ with respect to an algebraically closed field $F$ of characteristic $p > 0$. Moreover, let $D$ be a defect group of $B$. The second author has shown in [11, Corollary 3.3] that the Loewy length of the center of $B$ is bounded by

$$LL(Z_B) \leq |D| - \frac{|D|}{\exp(D)} + 1$$

where $\exp(D)$ is the exponent of $D$. It was already known to Okuyama [9] that this bound is best possible if $D$ is cyclic. The first and the third author have given in [8, Theorem 1] the optimal bound $LL(Z_B) \leq LL(FD)$ for blocks with abelian defect groups. Our main result of the present paper establishes the following bound for blocks with non-abelian defect groups:

$$LL(Z_B) \leq \min\{p^{d-1}, 4p^{d-2}\}$$

where $|D| = p^d$. As a consequence we obtain

$$LL(Z_B) \leq p^{d-1} + p - 1$$

for all blocks with non-cyclic defect groups. It can be seen that this bound is optimal whenever $B$ is nilpotent and $D \cong C_{p^{d-1}} \times C_p$.

In the second part of the paper we show that $LL(Z_B)$ depends more on $\exp(D)$ than on $|D|$. We prove for instance that $LL(Z_B) \leq d^2 \exp(D)$ unless $d = 0$. Finally, we use the opportunity to improve a result of Willems [14] about blocks with uniserial center.

In addition to the notation used in the papers cited above, we introduce the following objects. Let $\Cl(G)$ be the set of conjugacy classes of $G$. A $p$-subgroup $P \leq G$ is called a defect group of $K \in \Cl(G)$ if $P$ is a Sylow
Let $\nu$ be a $p$-subgroup of $C_G(x)$ for some $x \in K$. Let $\text{Cl}_P(G)$ be the set of conjugacy classes with defect group $P$. Let $K^+ := \sum_{x \in K} x \in FG$ and $K := K \cap \text{Cl}_P(G) \subseteq ZFG$.

Note that $\text{Lemma 3}$. Recall from \cite[Lemma 9]{8} the following group $\text{Lemma 2}$. Let $\text{Lemma 1}$ begin by restating a lemma of Passman \cite[Lemma 2]{12}. For the convenience of the reader we provide a slightly easier proof.

**Lemma 1** (Passman). Let $P$ be a central $p$-subgroup of $G$. Then $I_{<P}(G) \cdot JZF_G = I_{<P}(G) \cdot JFP$.

**Proof.** Let $K$ be a conjugacy class of $G$ with defect group $P$, and let $x \in K$. Then $P$ is the only Sylow $p$-subgroup of $C_G(x)$, and the $p$-factor $u$ of $x$ centralizes $x$. Thus $u \in P$. Hence $u$ is the $p$-factor of every element in $K$, and $K = uK'$ where $K'$ is a $p$-regular conjugacy class of $G$ with defect group $P$. This shows that $I := I_{<P}(G)$ is a free $FP$-module with the $p$-regular class sums with defect group $P$ as an $FP$-basis. The canonical epimorphism $\nu : FG \rightarrow F[G/P]$ maps $I$ into $I_1(G/P) \subseteq SF[G/P]$. Thus $\nu(I \cdot JZF_G) \subseteq SF[G/P] \cdot JZF[G/P] = 0$. Hence $I \cdot JZF_G \subseteq I \cdot JFP$. The other inclusion is trivial.

**Lemma 2.** Let $P \leq G$ be a $p$-subgroup of order $p^n$. Then

(i) $I_{<P}(G) \cdot JZF_G^{LL(FZ(P))} \subseteq I_{<P}(G)$,

(ii) $I_{<P}(G) \cdot JZF_G^{(p^n-1)/(p-1)} = 0$.

**Proof.**

(i) Let $\text{Br}_P : ZFG \rightarrow ZFC_G(P)$ be the Brauer homomorphism. Since $\text{Ker}(\text{Br}_P) \cap I_{<P}(G) = I_{<P}(G)$, we need to show that $\text{Br}_P(I_{<P}(G) \cdot JZF_G^{LL(FZ(P))}) = 0$. By \textbf{Lemma 1} we have

\[
\text{Br}_P(I_{<P}(G) \cdot JZF_G^{LL(FZ(P))}) \subseteq I_{<Z(P)}(C_G(P)) \cdot JZF_G^{LL(FZ(P))}
= I_{<Z(P)}(C_G(P)) \cdot JZF_G^{LL(FZ(P))} = 0.
\]

(ii) We argue by induction on $n$. The case $n = 1$ follows from $I_1(G) \subseteq SFG$. Now suppose that the claim holds for $n - 1$. Since $\text{LL}(FZ(P)) \leq |P| = p^n$, (i) implies

\[
I_{<P}(G) \cdot JZF_G^{(p^n-1)/(p-1)} = I_{<P}(G) \cdot JZF_G^{p^n} \cdot JZF_G^{(p^n-1)/(p-1)}
\leq I_{<P}(G) \cdot JZF_G^{(p^n-1)/(p-1)}
= \sum_{Q < P} I_{<Q}(G) \cdot JZF_G^{(p^n-1)/(p-1)} = 0.
\]

Recall from \cite[Lemma 9]{8} the following group

\[
W_{p^d} := \langle x, y, z \mid x^{p^{d-2}} = y^p = z^p = [x, y] = [x, z] = 1, [y, z] = x^{p^{d-3}} \rangle.
\]

Note that $W_{p^d}$ is a central product of $C_{p^d}$ and an extraspecial group of order $p^d$. Now we prove our main theorem which improves \cite[Theorem 12]{8}.

**Theorem 3.** Let $B$ be a block of $FG$ with non-abelian defect group $D$ of order $p^d$. Then one of the following holds
Proof. By [8, Proposition 15], we may assume that $p > 2$. Since $D$ is non-abelian, $|D : Z(D)| \geq p^2$ and $LL(FZ(D)) \leq p^{d-2}$. Let $Q$ be a maximal subgroup of $D$. If $Q$ is cyclic, then $D \cong M_{p^n}$ and the claim follows from [8, Proposition 10]. Hence, suppose that $Q$ is not cyclic. Then $LL(FZ(Q)) \leq p^{d-2} + p - 1$. Now setting $\lambda := \frac{p^{d-1} - 1}{p-1}$ it follows from Lemma 2 that

\[
JZB2^{p^{d-2} + p - 1 + \lambda} \subseteq 1_B JZFG2^{p^{d-2} + p - 1 + \lambda} \subseteq I_{\leq D}(G) \cdot JZFG2^{p^{d-2} + p - 1 + \lambda} \\
\subseteq I_{< D}(G) \cdot JZFG2^{p^{d-2} + p - 1 + \lambda} = \sum_{Q < D} I_{< Q}(G) \cdot JZFG^{2^{p^{d-2} + p - 1 + \lambda}} \\
\subseteq \sum_{Q < D} I_{< Q}(G) \cdot JZFG^{\lambda} = 0.
\]

Since $2p^{d-2} + p - 1 + \lambda \leq 4p^{d-2}$, we are done in case $p \geq 5$ and $D \cong W_{p^a}$. If $p = 3$ and $D \cong W_{p^a}$, then the claim follows from [8, Lemma 11]. Now suppose that $D \not\cong W_{p^a}$. If $Z(D)$ is cyclic of order $p^{d-2}$, then the claim follows from [8, Lemma 9 and Proposition 10]. Hence, suppose that $Z(D)$ is non-cyclic or $|Z(D)| < p^{d-2}$. Then $d \geq 4$ and $LL(FZ(D)) \leq p^{d-3} + p - 1$. The arguments above give $LL(ZB) \leq p^{d-2} + p^{d-3} + 2p - 2 + \lambda$, hence we are done whenever $p > 3$.

In the following we assume that $p = 3$. Here we have $LL(ZB) \leq 3^{d-2} + 3^{d-3} + 4 + \frac{1}{2}(3^{d-1} - 1)$ and it suffices to handle the case $d = 4$. By [11, Theorem 3.2], there exists a non-trivial $B$-subsection $(u, b)$ such that

\[
LL(ZB) \leq (|\langle u \rangle| - 1)LL(ZB) + 1
\]

where $\bar{b}$ is the unique block of $FC_G(u)/\langle u \rangle$ dominated by $b$. We may assume that $\bar{b}$ has defect group $C_{D}(u)/\langle u \rangle$ (see [13, Lemma 1.34]). If $u \not\in Z(D)$, we obtain $LL(ZB) < |C_D(u)| \leq 27$ as desired. Hence, let $u \in Z(D)$. Then $D/\langle u \rangle$ is not cyclic. Moreover, by our assumption on $Z(D)$, we have $|\langle u \rangle| = 3$. Now it follows from [8, Theorem 1, Proposition 10 and Lemma 11] applied to $\bar{b}$ that

\[
LL(ZB) \leq 2LL(ZB) + 1 \leq 23 < 27.
\]

We do not expect that the bounds in Theorem 3 are sharp. In fact, we do not know if there are $p$-blocks $B$ with non-abelian defect groups of order $p^d$ such that $p > 2$ and $LL(ZB) > p^{d-2}$. See also Proposition 7 below.

Corollary 4. Let $B$ be a block of $FG$ with non-cyclic defect group of order $p^d$. Then

\[
LL(ZB) \leq p^{d-1} + p - 1.
\]

Proof. By Theorem 3, we may assume that $B$ has abelian defect group $D$. Then [8, Theorem 1] implies $LL(ZB) \leq LL(FD) \leq p^{d-1} + p - 1$.

We are now in a position to generalize [8, Corollary 16].

Corollary 5. Let $B$ be a block of $FG$ with defect group $D$ of order $p^d$ such that $LL(ZB) \geq \min\{p^{d-1}, 4p^{d-2}\}$. Then one of the following holds

(i) $D$ is cyclic.
(ii) $D \cong C_{p^d} \times C_p$.
(iii) $D \cong C_2 \times C_2 \times C_2$ and $B$ is nilpotent.
Proof. Again by [Theorem 3] we may assume that $D$ is abelian. By [8, Corollary 16], we may assume that $p > 2$. Suppose that $D$ is of type $(p^{a_1}, \ldots, p^{a_e})$ such that $s \geq 3$. Then
\[
\min\{p^{d-1}, 4p^{d-2}\} \leq LL(ZB) = p^{a_1} + \ldots + p^{a_s} - s + 1 \leq p^{a_1} + p^{a_2} + p^{a_3} + \ldots + p^{a_s} - 2 \leq p^{d-2} + 2(p - 1).
\]
This clearly leads to a contradiction. Therefore, $s \leq 2$ and the claim follows.

In case (i) of [Corollary 5] it is known conversely that $LL(ZB) = \frac{p^{d-1}}{|B|} + 1 > p^{d-1}$ (see [8, Corollary 2.8]).

Our next result gives a more precise bound by invoking the exponent of a defect group.

**Theorem 6.** Let $B$ be a block of $FG$ with defect group $D$ of order $p^d > 1$ and exponent $p^r$. Then
\[
LL(ZB) \leq \left(\frac{d}{e} + 1\right)\left(\frac{d}{2} + \frac{1}{p - 1}\right)(p^r - 1).
\]
In particular, $LL(ZB) \leq d^2p^r$.

**Proof.** Let $\alpha := \lfloor d/e \rfloor$. Let $P \leq D$ be abelian of order $p^{\alpha+1}$ with $0 \leq \alpha$ and $0 \leq j < e$. If $P$ has type $(p^{a_1}, \ldots, p^{a_r})$, then $a_i \leq e$ for $i = 1, \ldots, r$ and
\[
LL(FP) = (p^{a_1} - 1) + \ldots + (p^{\alpha} - 1) + 1 \leq i(p^e - 1) + p^j.
\]
Arguing as in [Theorem 3] we obtain
\[
LL(ZB) \leq \sum_{i=0}^{\alpha} \sum_{j=0}^{e-1} i(p^e - 1) + p^j = e(p^e - 1)\left(\sum_{i=0}^{\alpha} i\right) + (\alpha + 1)p^e - 1 \leq \left(\frac{d}{e} + 1\right)\left(\frac{d}{2} + \frac{1}{p - 1}\right)(p^r - 1).
\]
This proves the first claim. For the second claim we note that
\[
\left(\frac{d}{e} + 1\right)\left(\frac{d}{2} + \frac{1}{p - 1}\right) \leq (d + 1)\left(\frac{d}{2} + 1\right) \leq d^2
\]
unless $d \leq 3$. In these small cases the claim follows from [Theorem 3] and [Corollary 4].

If $2e > d$ and $p$ is large, then the bound in [Theorem 6] is approximately $dp^r$. The groups of the form $G = D = C_{p^r} \times \ldots \times C_{p^r}$ show that there is no bound of the form $LL(ZB) \leq C p^r$ where $C$ is an absolute constant. A more careful argumentation in the proof above gives the stronger (but opaque) bound
\[
LL(ZB) \leq \alpha(p^e - 1)\left(\frac{e(\alpha - 1)}{2} + \frac{1}{p - 1} + d - \alpha e\right) + \beta(p^e - 1) + \frac{p^{d-\alpha e} - 1}{p - 1} + p^{d-2-\beta e}
\]
for non-abelian defect groups where $\alpha := \lfloor \frac{d}{e} \rfloor$ and $\beta := \lfloor \frac{d - 2}{2} \rfloor$. We omit the details.

In the next result we compute the Loewy length for $d = e + 1$.

**Proposition 7.** Let $B$ be a block of $FG$ with non-abelian defect group of order $p^d$ and exponent $p^{d-1}$. Then
\[
LL(ZB) \leq \begin{cases} 
2^{d-2} + 1 & \text{if } p = 2, \\
2^{d-2} & \text{if } p > 2
\end{cases}
\]
and both bounds are optimal for every $d \geq 3$. 

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Proof. Let \( D \) be a defect group of \( B \). If \( p > 2 \), then \( D \cong M_{p^d} \) and we have shown \( LL(ZB) \leq p^{d-2} \) in [8 Proposition 10]. Equality holds if and only if \( B \) is nilpotent.

Therefore, we may assume \( p = 2 \) in the following. The modular groups \( M_{p^d} \) are still handled by [8 Proposition 10]. Hence, it remains to consider the defect groups of maximal nilpotency class, i.e., \( D \in \{D_{2d}, Q_{2d}, SD_{2d}\} \).

By [8 Proposition 10], we may assume that \( d \geq 4 \). The isomorphism type of \( ZB \) is uniquely determined by \( D \) and the fusion system of \( B \) (see [2]). The possible cases are listed in [13 Theorem 8.1]. If \( B \) is nilpotent, [8 Proposition 8] gives \( LL(ZB) = LL(ZFD) \leq LL(FD') = 2^{d-2} \). Moreover, in the case \( D \cong D_{2d} \) and \( l(B) = 3 \) we have \( LL(ZB) \leq k(B) - l(B) + 1 = 2^{d-2} + 1 \) by [11 Proposition 2.2]. In the remaining cases we present \( B \) by quivers with relations which were constructed originally by Erdmann [9]. We refer to [4 Appendix B].

(i) \( D \cong D_{2d}, l(B) = 2 \):

\[
\begin{align*}
\alpha & \xrightarrow{\beta} \gamma & & \beta \eta = \eta \gamma = \gamma \beta = \alpha^2 = 0, \\
\beta & \xrightarrow{\eta} & & \eta^2 = \gamma \alpha \beta.
\end{align*}
\]

By [4 Lemma 2.3.3], we have

\[
ZB = \langle 1, \beta \gamma, \alpha \beta \gamma, \eta^i : i = 1, \ldots, 2^{d-2} \rangle.
\]

It follows that \( JZB^2 = \langle \eta^2 \rangle \) and \( LL(ZB) = 2^{d-2} + 1 \).

(ii) \( D \cong Q_{2d}, l(B) = 2 \): Here [15 Lemma 6] gives the isomorphism type of \( ZB \) directly as a quotient of a polynomial ring

\[
ZB \cong F[U, Y, S, T]\langle Y^{2d-2+1}, U^2 - Y^{2d-2}, S^2, T^2, SY, SU, ST, UY, UT, YT \rangle.
\]

It follows that \( JZB^2 = \langle Y^2 \rangle \) and again \( LL(ZB) = 2^{d-2} + 1 \).

(iii) \( D \cong Q_{2d}, l(B) = 3 \):

\[
\begin{align*}
\beta & \xrightarrow{\kappa} \gamma & & \kappa \beta \gamma = \gamma \beta \kappa, \\
\gamma & \xrightarrow{\delta} \beta & & \delta \gamma = \gamma \delta, \\
\kappa & \xrightarrow{\delta} \kappa \beta \gamma & & \kappa \beta \gamma \delta = \gamma \delta \kappa \beta \gamma = \lambda \kappa \eta = 0.
\end{align*}
\]

By [4 Lemma 2.5.15],

\[
ZB = \langle 1, \beta \gamma + \gamma \beta, (\kappa \lambda)^i + (\lambda \kappa)^i, \delta \eta + \eta \delta, (\beta \gamma)^2, (\kappa \lambda)^{2d-2}, (\delta \eta)^2 : i = 1, \ldots, 2^{d-2} - 1 \rangle.
\]

We compute

\[
\begin{align*}
(\beta \gamma + \gamma \beta)^2 & = (\beta \gamma)^2 + (\gamma \beta)^2 + \delta \lambda \beta + (\beta \gamma)^2 + (\delta \eta)^2, \\
(\beta \gamma + \gamma \beta)(\kappa \lambda + \lambda \kappa) & = \beta \gamma \kappa \lambda = \beta \delta \eta \delta \lambda = \beta \delta \eta \gamma \beta \gamma = 0, \\
(\beta \gamma + \gamma \beta)(\delta \eta + \eta \delta) & = \gamma \beta \delta \eta = 0, \\
(\beta \gamma)(\beta \gamma)^2 & = (\beta \gamma)^3 = \beta \gamma \beta \delta \lambda = 0, \\
\beta \gamma & \xrightarrow{\alpha} \gamma \beta \lambda & & (\alpha \beta \gamma)\beta \gamma \lambda = \beta \gamma \lambda, \\
\gamma \beta \lambda & \xrightarrow{\alpha} \gamma \beta \lambda \kappa & & \lambda \kappa \eta \delta = 0, \\
\gamma \beta \lambda \kappa & \xrightarrow{\alpha} & & \kappa \beta \gamma \lambda \kappa \eta \delta = 0.
\end{align*}
\]
By \([5, \text{Section 5.2.2}]\), we have
\[
(\kappa \lambda + \lambda \kappa) (\delta \eta)^2 = 0,
\]
\[
(\delta \eta + \eta \delta)^2 = (\delta \eta)^2 + (\eta \delta)^2 = (\delta \eta)^2 + \lambda \beta \delta = (\delta \eta)^2 + (\lambda \kappa)^{2d-2},
\]
\[
(\delta \eta + \eta \delta)(\beta \gamma)^2 = 0,
\]
\[
(\delta \eta + \eta \delta)(\lambda \kappa)^{2d-2} = \eta \delta (\lambda \kappa)^{2d-2} = \eta \delta \eta \kappa = 0,
\]
\[
(\delta \eta + \eta \delta)(\delta \eta)^2 = \delta \lambda \beta \delta \eta = \gamma \beta \gamma \beta \delta \eta = 0,
\]
\[
(\beta \gamma)^2(\beta \gamma)^2 = (\beta \gamma)^2(\lambda \kappa)^{2d-2} = (\beta \gamma)^2(\delta \eta)^2 = 0,
\]
\[
(\lambda \kappa)^{2d-2}(\lambda \kappa)^{2d-2} = (\lambda \kappa)^{2d-2}(\delta \eta)^2 = 0,
\]
\[
(\delta \eta)^2(\delta \eta)^2 = \gamma \kappa \eta (\delta \eta)^2 = \gamma \beta \gamma (\delta \eta)^2 = 0.
\]

Hence, \(JZB^2 = \langle (\kappa \lambda)^2 + (\kappa \lambda)^2, (\beta \gamma)^2 + (\delta \eta)^2 \rangle\) and \(JZB^3 = \langle (\kappa \lambda)^3 + (\kappa \lambda)^3 \rangle\). This implies \(LL(ZB) = 2^{d-2} + 1\).

(iv) \(D \cong SD_{2d}, \ k(B) = 2^{d-2} + 3\) and \(l(B) = 2\):

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\gamma & \delta & \eta \\
\eta & \alpha & \beta
\end{array}
\]

\[
\begin{array}{ccc}
\gamma \beta = \eta \gamma = \beta \eta = 0,
\alpha^2 = \beta \gamma, \ a \beta \gamma = \beta \alpha \gamma,
\eta^{2d-2} = \gamma \alpha \beta.
\end{array}
\]

By \([5, \text{Section 5.1}]\), we have
\[
ZB = \text{span}\{1, \beta \gamma, a \beta \gamma, \eta^i : i = 1, \ldots, 2^{d-2}\}.
\]

As in (i) we obtain \(JZB^2 = \langle \eta^2 \rangle\) and \(LL(ZB) = 2^{d-2} + 1\).

(v) \(D \cong SD_{2d}, \ k(B) = 2^{d-2} + 4\) and \(l(B) = 2\):

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\gamma & \delta & \eta \\
\eta & \alpha & \beta
\end{array}
\]

\[
\begin{array}{ccc}
\beta \eta = a \beta \gamma \alpha \beta, \ \gamma \beta = \eta^{2d-2-1},
\alpha \beta \gamma = \gamma \alpha \beta \gamma,
\beta \eta^2 = \eta^2 \gamma = \alpha^2 = 0.
\end{array}
\]

By \([5, \text{Section 5.2.2}]\), we have
\[
ZB = \text{span}\{1, a \beta \gamma + \beta \gamma \alpha + \gamma \alpha \beta, \beta \gamma \alpha \beta \gamma, (a \beta \gamma)^2, \eta^i, \eta + a \beta \gamma \alpha : i = 2, \ldots, 2^{d-2}\}.
\]

Since \((a \beta \gamma)^2 = beta \gamma = (\beta \gamma \alpha)^2\) and \((a \beta \gamma)^2 = \eta \gamma \beta = \eta^{2d-2}\), it follows that
\[
(a \beta \gamma + \beta \gamma \alpha + \gamma \alpha \beta)^2 = (a \beta \gamma)^2 + (\beta \gamma \alpha)^2 + (\gamma \alpha \beta)^2 = \eta^{2d-2}.
\]

Similarly,
\[
(a \beta \gamma + \beta \gamma \alpha + \gamma \alpha \beta) \beta \gamma \alpha \beta = 0,
\]
\[
(a \beta \gamma + \beta \gamma \alpha + \gamma \alpha \beta)(\alpha \beta \gamma)^2 = 0,
\]
\[
(a \beta \gamma + \beta \gamma \alpha + \gamma \alpha \beta) \eta^2 = 0,
\]
\[
(a \beta \gamma + \beta \gamma \alpha + \gamma \alpha \beta)(\eta + a \beta \gamma \alpha) = 0,
\]
\[
(\beta \gamma \alpha \beta \gamma)^2 = 0,
\]
\[
\beta \gamma \alpha \beta \gamma (a \beta \gamma)^2 = 0,
\]
\[
\beta \gamma \alpha \beta \gamma \eta^2 = \beta \gamma \alpha \beta \eta^2 \gamma = 0,
\]
\[
\beta \gamma \alpha \beta \gamma (\eta + a \beta \gamma \alpha) = \beta \gamma (a \beta \gamma)^2 \alpha = 0,
\]
\[
(\alpha \beta \gamma)^2(a \beta \gamma)^2 = 0,
\]
\[(\alpha \beta \gamma)^2 \eta^2 = 0,\]
\[(\alpha \beta \gamma)^2 (\eta + \alpha \beta \gamma \alpha) = 0,\]
\[\eta^2 (\eta + \alpha \beta \gamma \alpha) = \eta^3,\]
\[(\eta + \alpha \beta \gamma \alpha)^2 = \eta^2.\]

Consequently, \(JZB^2 = \langle \eta^2 \rangle\) and \(LL(ZB) = 2^{d-2} + 1.\)

(vi) \(D \cong SD_{2d}, \; l(B) = 3:\)

\[\begin{array}{c}
\lambda \\
\delta \\
\end{array} \begin{array}{c}
\kappa \\
\gamma \\
\eta \\
\end{array} \beta
\]

\[\kappa \eta = \eta \gamma = \gamma \kappa = 0, \; \delta \lambda = (\gamma \beta)^{2d-2-1} \gamma, \]
\[\beta \delta = \kappa \lambda \kappa, \; \lambda \beta = \eta.\]

From [4, Lemma 2.4.16] we get

\[ZB = \text{span}\{1, (\beta \gamma)^i + (\gamma \beta)^i, \kappa \lambda + \lambda \kappa, (\beta \gamma)^{2d-2}, (\lambda \kappa)^2, \delta \eta : i = 1, \ldots, 2^{d-2} - 1\}.\]

We compute

\[(\beta \gamma + \gamma \beta)((\beta \gamma)^{2d-2} - 1 + (\gamma \beta)^{2d-2} - 1) = (\beta \gamma)^{2d-2} + \delta \lambda \beta = (\beta \gamma)^{2d-2} + \delta \eta,\]
\[(\beta \gamma + \gamma \beta)(\kappa \lambda + \lambda \kappa) = (\beta \gamma)(\kappa \lambda) = 0,\]
\[(\beta \gamma + \gamma \beta)(\beta \gamma)^{2d-2} = \beta \delta \lambda \beta \gamma = \kappa \lambda \kappa \gamma = 0,\]
\[(\beta \gamma + \gamma \beta)(\lambda \kappa)^2 = 0,\]
\[(\beta \gamma + \gamma \beta)\delta \eta = \beta \delta \eta = \gamma \kappa \lambda \kappa \eta = 0,\]
\[(\kappa \lambda + \lambda \kappa)^2 = \beta \delta \lambda + (\lambda \kappa)^2 = (\beta \gamma)^{2d-2} + (\lambda \kappa)^2,\]
\[(\kappa \lambda + \lambda \kappa)(\beta \gamma)^{2d-2} = \kappa \lambda \beta \gamma (\beta \gamma)^{2d-2} - 1 = \kappa \eta (\beta \gamma)^{2d-2} - 1 = 0,\]
\[(\kappa \lambda + \lambda \kappa)(\lambda \kappa)^2 = \lambda (\beta \gamma)^{2d-2} \kappa \eta (\beta \gamma)^{2d-2} - 1 \kappa = 0,\]
\[(\kappa \lambda + \lambda \kappa)\delta \eta = 0,\]
\[(\beta \gamma)^{2d-2} (\beta \gamma)^{2d-2} = (\beta \gamma)^{2d-2} (\lambda \kappa)^2 = (\beta \gamma)^{2d-2} \delta \eta = 0,\]
\[(\lambda \kappa)^2 (\lambda \kappa)^2 = (\lambda \kappa)^2 \delta \eta = 0,\]
\[(\delta \eta)^2 = \delta \lambda \delta \eta = \delta \lambda \kappa \lambda \kappa \eta = 0.\]

Hence, \(JZB^2 = \langle (\beta \gamma)^2 + (\gamma \beta)^2, (k \lambda)^2 + \delta \eta \rangle\) and \(JZB^3 = \langle (\beta \gamma)^3 + (\gamma \beta)^3 \rangle.\) This implies \(LL(ZB) = 2^{d-2} + 1.\)

It is interesting to note the difference between even and odd primes in Proposition 7. For \(p = 2,\) non-nilpotent blocks gives larger Loewy lengths while for \(p > 2\) the maximal Loewy length is only assumed for nilpotent blocks.

Recall that a lower defect group of a block \(B\) of \(FG\) is a \(p\)-subgroup \(Q \leq G\) such that

\[I_{<Q}(G)1_B \neq I_{\leq Q}(G)1_B.\]

In this case \(Q\) is conjugate to a subgroup of a defect group \(D\) of \(B\) and conversely \(D\) is also a lower defect group since \(1_B \in I_{< D}(G) \setminus I_{< D}(G).\) It is clear that in the proofs of Theorem 3 and Theorem 6 it suffices to sum over the lower defect groups of \(B.\) In particular there exists a chain of lower defect groups \(Q_1 < \ldots < Q_n = D\) such that \(LL(ZB) \leq \sum_{i=1}^n LL(FZ(Q_i)).\) Unfortunately, it is hard to compute the lower defect groups of a given block.

The following proposition generalizes [4, Theorem 1.5].
Proposition 8. Let $B$ be a block of $FG$. Then $ZB$ is uniserial if and only if $B$ is nilpotent with cyclic defect groups.

Proof. Suppose first that $ZB$ is uniserial. Then $ZB \cong F[X]/(X^n)$ for some $n \in \mathbb{N}$; in particular, $ZB$ is a symmetric $F$-algebra. Then [10] Theorems 3 and 5 implies that $B$ is nilpotent with abelian defect group $D$. Thus, by a result of Broué and Puig [1] (see also [7]), $B$ is Morita equivalent to $FD$; in particular, $FD$ is also uniserial. Thus $D$ is cyclic.

Conversely, suppose that $B$ is nilpotent with cyclic defect group $D$. Then the Broué-Puig result mentioned above implies that $B$ is Morita equivalent of $FD$. Thus $ZB \cong ZFD = FD$. Since $FD$ is uniserial, the result follows. □

A similar proof shows that $ZB$ is isomorphic to the group algebra of the Klein four group over an algebraically closed field of characteristic 2 if and only if $B$ is nilpotent with Klein four defect groups.

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References
