Bridg-It – Beating Shannon's Analog Heuristic

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ABSTRACT

In 1951 Shannon provided a simple analog heuristic for the connection game Bridg-It. Although this heuristic is based only on a simple network flow analysis, Shannon reported that it almost always wins against human players when having the first move. In this note, we analyse this heuristic showing examples where the heuristic fails. Furthermore, we consider the question whether the first player always wins if both players use Shannon's heuristic.

1. INTRODUCTION

In Gardner, 1961, p. 84–87, the game was described as Game of Gale. Today it is best known under the name Bridg-It or Bird Cage. Already in 1951, Shannon had constructed a robot playing this game on a small board. Here we consider a straightforward generalisation of this game, the Shannon Switching Game.

The Shannon Switching Game is a zero-sum strategy game with full information for two players, CUT and SHORT. The game is played on a finite connected undirected graph with two special nodes, P and Q. Both players move alternately. On CUT's turn, he removes one uncolored edge from the graph, whereas SHORT in his turn colors one (not yet removed) edge. SHORT wins by establishing a colored path from P to Q. If this is not possible anymore, i. e. P and Q are disconnected, CUT wins. There is no draw possible.

In the view of game theory and perfect play the game was solved by Lehman, 1964. But in 1951, Shannon had introduced a very simple but quite powerful heuristic suitable for implementation on a simple analog machine. The basic idea was to consider the game as an electrical circuit: uncolored edges are resistors of unity conductance, removed edges have zero conductance and colored edges are short-circuit (infinite conductance). Now Shannon's heuristic works as follows:

> **Applying voltage between the nodes P and Q, the heuristic selects the edge with most current flow as next move.**

If there are more than one edge of maximum flow, one of them is selected randomly. Of course, the heuristic is rule of thumb: Edges with high current seem to be important in the network, so it is natural to remove respectively to fix them.

Based on this idea Claude E. Shannon himself built the first robot for the game in 1951. It consists of an electrical circuit containing switches and light bulbs[2] which connect the nodes. By applying voltage to the circuit the most bright bulb indicates the edge with the highest flow.

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[2] Due to the fact, that the resistance of a light bulb highly depends on the applied voltage, Shannon actually used a combination of light bulbs and resistors (Heinz Nixdorf MuseumsForum Paderborn, see http://en.hnf.de/Special_exhibitions/Shannon/Image_gallery.asp).

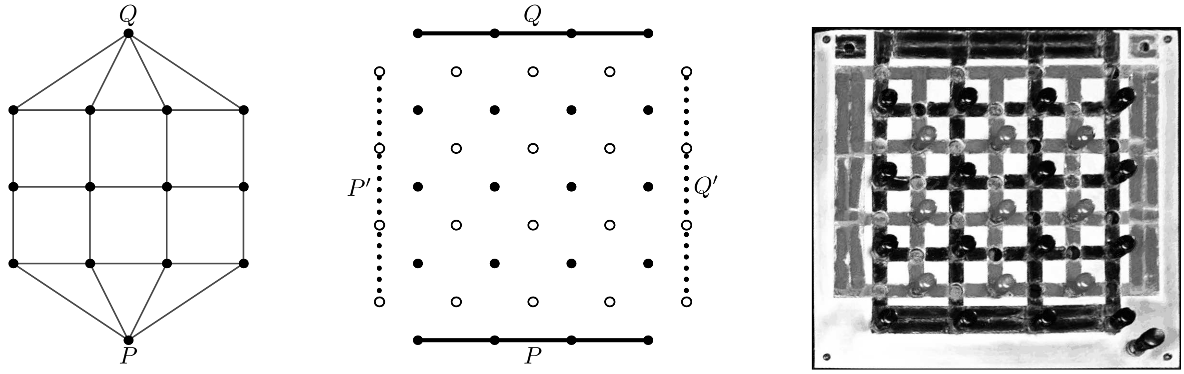


Figure 1: Example for self-dual M-class of Shannon's Switching Game. Left: graph for board size $M = 4$. Center: corresponding Bridg-It board. Right: Shannon's robot from 1951 (Heinz Nixdorf MuseumsForum Paderborn, 2009).

2. PLANAR GRAPHS AND SELF-DUAL GAMES (M-CLASS)

While the objectives of CUT and SHORT appear different, there is a relation between them, at least if we look at planar graphs[3]. Considering the dual game[4], the roles of CUT and SHORT are interchanged, cf. Figure 1.

If the graphs of the game and the dual game are isomorphic, with PQ and $P'Q'$ corresponding, we call the game self-dual. In self-dual games it is obvious that the first player has a winning strategy (by strategy stealing argument). Here we introduce a class of self-dual games: Let $M > 1$ an integer. The graph of the M-class-game is a rectangular $M \times (M-1)$-grid with two additional nodes P and Q connected to the first and respectively to the last layer, see Figure 1.

3. APPLYING SHANNON'S ANALOG HEURISTIC

Before applying Shannon's heuristic, that means selecting the edge with most current in the resistors network, we should mention that there is no difference between applying the heuristic on the original or the dual network. Note that even in a self-dual starting situation SHORT's and CUT's networks aren't isomorphic during the game, but they are dual. From electrical engineering it is known, that every planar network has a corresponding dual network and all physical quantities have corresponding quantities with well-defined values too (Kuriakose, 2005). The correspondence between resistance and conductance in combination with the property that all our resistors are equal leads to following result:[5]

Shannon's heuristic on the dual game selects the same moves as on the original game.

Thus, without loss of generality we can make the following assumptions

1. The player CUT always starts.
2. Shannon's heuristic is applied to the same graph[6], independent whether CUT or SHORT is to to move.

So, in the self-dual M-class of games (as introduced above) the only parameter is the board size M.

[3]Which should have an embedding with P and Q lying on the outer boundary.

[4]The graph of the dual game is constructed as follows: Add the artificial edge PQ to the graph G of the game. Construct the dual graph (regions $\leftrightarrow$ nodes) and remove the edge $P'Q'$ (dual edge to PQ). The points P' and Q' are the special nodes of the dual game.

[5]This explanation will be elaborated in more detail in Fischer, 2010.

[6]It is useful (and necessary in case of iterative solving the Kirchhoff equations) to work with a reduced version of the graph: all edges played by CUT are removed, all nodes connected by colored edges are identified (this can result in multiple edges).

4. RESULTS

4.1 Minimal Example where Shannon's Analog Heuristic Fails ($M = 3$)

Figure 2 shows a game on the board $M = 3$. The first player, CUT, moves according to Shannon's heuristic. The numbers inside the nodes denote their electrical potential, which is calculated by the Kirchhoff's circuit laws. The potential values are scaled to be integers. The flow through each edge is equal to the difference of the potential of its endpoints. In the beginning all horizontal edges have flow of zero, while all other edges have flow equal to 1. So the first move is not uniquely determined.

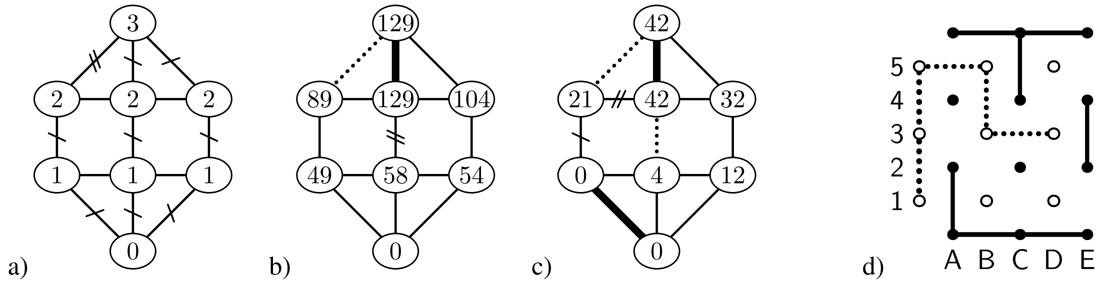


Figure 2: Lost game for $M = 3$. Graphs with potentials on CUT's turn. CUT selects one (marked with two bars) of the edges with maximum potential difference (marked with bars). a) All vertical edges have equal flow of 1. b) CUT's move is uniquely determined (flow of 61). c) Two equal edges (flow of 21). d) Same game after six moves on the Bridg-It diagram. Moves are denoted by the coordinates of the midpoint of the edge. To avoid confusion CUT's moves are written with capital letters and SHORT's moves with lower case letters: 1. A5 2. c5 3. C3 4. a1 5. B4 6. e3. This position is a straightforward winning situation for SHORT, for instance: 7. E1 8. d2 9. C1 10. b2 11. E5 12. d4.

The second player, SHORT, does not use the heuristic, but plays "perfectly"[7]. In Figure 2, part c), we see that Shannon plays imperfectly in its third move. The edges A3 and B4 both have a flow of 21, while E3 only has a flow of 20. But A3 and B4 are forming a chain, so there is no need for CUT to remove one of them before SHORT has marked the other one.

For $M = 3$, the first move A5 together with its symmetric moves A1, E1 and E5 are the only losing moves. So, Shannon wins with probability of 5/9 against a "perfect" player.

4.2 Minimal Example where Shannon's Analog Heuristic Always Fails ($M = 4$)

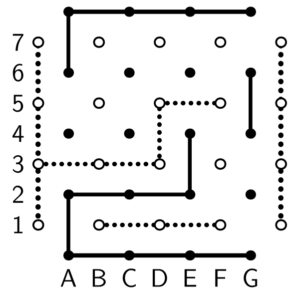
On the board size of $M = 4$, every first move is a losing move for Shannon when playing against an omnipotent adversary. With respect to symmetry there are four different first moves, namely A1, C1, A3 and C3, for the Shannon heuristic. In Table 1, we give an example match for each case.

Aside from the occurrence of chain in the second case and some trivial situations in the endgames, all moves of Shannon's heuristic are uniquely determined, i. e. there are no multiple edges of maximum flow. So, Table 1 is a complete chart, that proves that Shannon's heuristic can lose every game.

However, for a human player it is not so easy to see how to beat the heuristic. According to Gardner, Shannon reported "when his machine has first move, it almost always wins. Out of hundreds of games played, the machine has had only two losses when it had the first move, and they may have been due to circuit failures or improper playing of the game. If the human player has first move, it is not difficult to beat the machine, but the machine wins if a gross error is made" (Gardner, 1961).

[7] Here a perfect player means an omnipotent agent, who also knows that its opponent uses Shannon's analog heuristic.

1. A1	2. c1	1. C1	2. a1	1. A3	2. c3	**1. C3**	**2. e3**
3. C3	4. e3	3. A3	4. c3	3. C1	4. a1	**3. E1**	**4. d2**
5. E5	6. a7	5. B2	6. e1	5. see left [(b)]		**5. C1**	**6. a1**
7. A5	8. d4	7. D2	8. e3			**7. A3**	**8. b2**
9. C5	10. g5	9. E5	10. g5			**9. E5**	**10. a7**
11. G7	12. f6	11. G7	12. d4			**11. D4**	**12. g5** [(c)]
13. E7	14. d6	13. C5	14. a7				
15. F4	16. g3	15. A5 [(a)]	16. f6				
17. G1	18. f2	17. E7	18. d6				
19. C7	20. b6	19. F4	20. g3				
21. E1	22. d2	21. C7	22. b6				
		23. G1	24. f2				



13. G7	14. f6
15. F4	16. g3
17. E7	18. d6
19. C7	20. b6
21. G1	22. f2

[(a)] CUT selects one of two equal edges forming a chain (A5 and B4), see discussion of the case $M = 3$ and Figure 2 c).

[(b)] From here on we have the same sequence of moves as in the second case.

[(c)] Shown in the diagram: A straightforward winning situation for SHORT.

Table 1: Board size $M = 4$. For all possible first moves, Shannon's heuristic always loses.

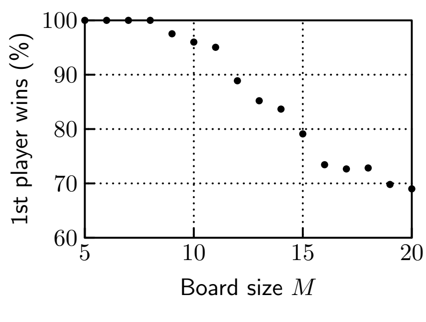
4.3 Shannon vs. Shannon

We have learned that Shannon's analog heuristic is not perfect, as with perfect play the first player has a winning strategy. Now we can ask the question if Shannon at least always wins against Shannon. In case of multiple edges with maximum flow Shannon selects any of these edges randomly. So the results could depend on this selection of the moves, especially in the very first move.

Fortunately, apart the first move, cases of equally evaluated edges do not appear often. So, we can analyse the game tree easily at least until $M = 10$. In the cases $M = 2, \ldots, 8$ always the first player wins, independently of the decisions between equal edges. But, in case of $M = 9$ there is aside from symmetry exactly one first move, namely E9, where the second player wins. More interestingly: apart from some trivial situations the remaining game is uniquely determined. The game is shown in Table 2.

So in case $M = 9$, the winning quota of the first player is exactly $1-2/81$, because there are 81 equal valued first moves. Considering larger boards, the winning quota decreases. For the limit $M \to \infty$, we conjecture that the winning quota will converge to 50 percent.

Remark: A complete analysis of the heuristic has a time complexity of at least $O(M^9)$ (for $O(M^2)$ first moves and a game duration of $O(M)$ up to $O(M^2)$ moves we have to solve a linear system of equations with $O(M^2)$ variables). So it is difficult[8] to go to values higher than $M = 20$.



5. DISCUSSION

The analysis of Shannon's analog heuristic we showed that there exists examples where the heuristic fails. In the case $M = 4$, that is the board size of Shannon's original robot, an adversary knowing about the heuristic can win as second player, while it is known theoretically the first player has a winning strategy. Experimental tests suggest that this result holds also for all boards sizes $M > 4$ of this self-dual layout. We conjecture that the winning quota will converge to 50 percent with increasing board size .

[8]To evaluate the heuristic approximately, it is possible to apply much faster iterative solvers.

1. E9 [(a)]	2. c9	29. H12	30. a15 [(d)]	57. C5	58. a5	85. P8	86. q9
3. C11 [(b)]	4. a11	31. B14	32. j11	59. B6	60. o5	87. P10	88. q11
5. A13 [(c)]	6. b12	33. H10	34. j9	61. N6	62. o7	89. Q1	90. s1
7. C13	8. e13	35. F16	36. m17	63. N8	64. o9	91. R2	92. s3
9. D12	10. e11	37. K16	38. m15	65. N10	66. o11	93. R4	94. s5
11. D10	12. g9	39. K14	40. m13	67. N12	68. o13	95. P12	96. q13
13. F10	14. g11	41. K12	42. m11	69. N14	70. o15	97. R6	98. s7
15. F12	16. g13	43. K10	44. m9	71. O3	72. n4	99. R8	100. s9
17. G15	18. f14	45. M7	46. k8	73. M3	74. k4	101. R10	102. s11
19. E15	20. d14	47. H8	48. j7	75. J3	76. h4	103. R12	104. s13
21. C15	22. j15	49. J5	50. h6	77. G3	78. f4	105. S15	106. r14
23. H14	24. j13	51. G5	52. f6	79. E3	80. q3	107. Q15	108. p14
25. J17	26. h16	53. E5	54. k6	81. P4	82. q5	109. O17	110. n16
27. G17	28. g7	55. M5	56. d6	83. P6	84. q7		

[(a)] Symmetric sequence: 1. M9 2. o9.
[(b)] Symmetric sequence: 3. C7 4. a7.
[(c)] This generates a chain (A15, B14).
[(d)] Removal of the chain (equivalent sequence: 30. b14 31. A15)

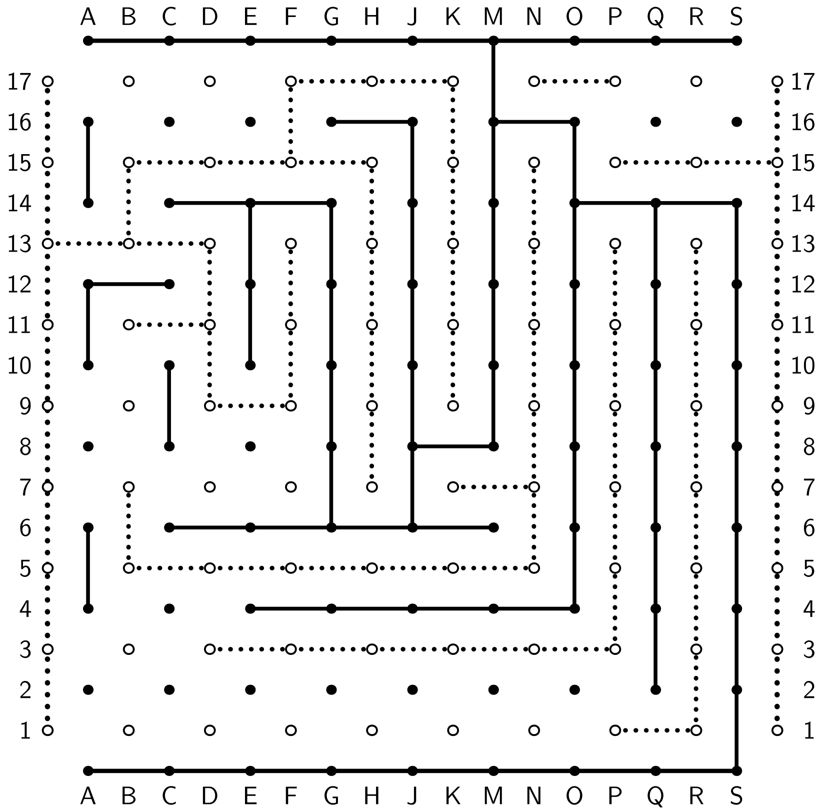


Table 2: Example match for a lost game with board size $M = 9$ (Shannon vs. Shannon).

Nevertheless, using simple heuristics can help to analyse complex situations. Anshelevich (2002) used a more advanced resistors-model (with iterative adding of virtual connections) for the similar connection game Hex. His program Hexy is based on the evaluation of the total resistance of circuits in combination with automatic reduction; and with its alpha-beta search it is one of the strongest Hex-playing programs (Anshelevich (2000)).

In the analysis of the quite similar game Connections[9], Sameith (2002b) used a slightly different heuristic, namely counting the number of missing edges for a win. In the resistors model this means calculating shortest paths. With a k-best alpha-beta search this heuristic gives a strong playing program (Sameith (2002a)).

[9]Connections has an additional local winning condition: A player also wins by generating a cycle in its graph. It was shown, that this does not change the arguments by Lehman, i. e. the first player has a winning strategy too (Sameith (2002b)).

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