Generalized hyperbolic Napier cycles and their hyperbolic kernels

Johannes Böhm in Jena

Part III

Types of Napier cycles

In part II the types of orthoschemes are considered and for each dimension the number of these types are calculated. For this purpose geometric permutations are usefull. Furthermore, an orthoscheme type generates a type of Napier cycles and therefore also a type of hyperbolic kernels. In this part the types of Napier cycles are considered and their numbers are recursively calculated for each dimension. For this purpose periodic permutations are used. Therefore, the first section explains the possibility to recognize the periodicity of a permutation with the help of permutation differences. Then one can prove, that the types of Napier cycles of a fixed dimension generate classes of Napier cycles with special period length. Thus the number of Napier cycle types is calculable. The proofs of the Lemmata are given in the Appendix of this part.

In Part I and Part II the following notions and their properties are explained
(cf. Part I, p. 6, 7; Part II, p. 2, 3, 5, 6):

Π_d:	Set of all permutations $\pi = \begin{pmatrix} 01\dots d+2 \\ i_o i_1 \dots i_{d+2} \end{pmatrix} =: i_o i_1 \dots i_{d+1} i_{d+2}$
$\hat{\Pi}_d \subset \Pi_d$:	Set of all normed permutations
$P_d \subset \hat{\Pi}_d$:	Set of all geometric normed permutations
$S_d \subset P_d$:	Set of all symmetric geometric normed permutations

1. Periodic permutations

Permutation differences are an important tool for the following investigations. With their help the periodicity of a permutation can be made visible.

Consider the permutation $\pi = i_o i_1 \dots i_{d+1} i_{d+2} \in \Pi_d$.

Definition 1. The permutation difference $D(\pi)$ belonging to π is

$$D(\pi) = D_+(\pi) = i'_o i'_1 \dots i'_{d+1} i'_{d+2}$$

$$0 \le i'_j \le d+2, \ \ i'_j \ \operatorname{mod}(d+3) \ \ (j = 0, 1, \dots d+2)$$

$$i'_j = i_{j+1} - i_j \qquad (j = 0, 1, \dots d+1)$$

$$i'_{d+2} = i_o - i_{d+2}$$

Definition 2. The permutation difference of second kind is

$$D_-(\pi) = i''_o i''_1 \dots i''_{d+1} i''_{d+2}$$

$$0 \le i''_j \le d+2,\ i''_j \ \mathrm{mod}(d+3)\ (j = 0, 1, \dots d+2)$$

$$i''_j = i_j - i_{j+1}\ (= -i'_j) \qquad (j = 0, 1, \dots d+1)$$

$$i''_{d+2} = i_{d+2} - i_o\ (= -i'_{d+2})$$

Lemma 1. There hold the following properties:

(i) $i_b = i_a + s \implies \sum_{j=a}^{b-1} i'_j = s \pmod{(d+3)}$

(ii) $\sum_{j=a}^{b-1} i''_j = -s \pmod{(d+3)}$

(iii) $\pi \mathrel{\hat{=}} \pi' \implies D_+(\pi) = D_+(\pi')$ or $D_+(\pi) = D_-(\pi')$

Especially there is $\sum_{j=o}^{d+2} i'_j = \sum_{j=o}^{d+2} i''_j = 0 \pmod{(d+3)}$ $(*)$

Definition 3. $-)$ $D(\pi)$ is called periodic with the least period

$$D_l(\pi) = i'_o i'_1 i'_2 \dots i'_{l-1} \qquad (=: \boxed{i'_o i'_1 i'_2 \dots i'_{l-1}})$$

if holds

$$D(\pi) = \underbrace{i'_o i'_1 i'_2 \dots i'_{l-1}}_{l \text{ places}}\ \underbrace{i'_o i'_1 i'_2 \dots i'_{l-1}}_{l \text{ places}} \ \dots\ \underbrace{i'_o i'_1 i'_2 \dots i'_{l-1}}_{l \text{ places}} \qquad (\tfrac{d+3}{l} \text{ periods})$$

and

$$D_l(\pi) = i'_o i'_1 i'_2 \dots i'_{l-1} \quad \text{is not periodic.}$$

$-)$ l is called the period length of $D(\pi)$.

$-)$ π is called periodic permutation with period length l.

Examples 1:

1. $\pi_{oo}^{(2q-1)} = 0\ q+1\ q\ q-1\ \dots\ 5\ 4\ 3\ 2\ 2q+1\ 2q\ 2q-1\ 2q-2\ \dots\ q+4\ q+3\ q+2\ 1\ \in S_d$,
is not periodic.

2. $\pi = 01478|10|11|14|23569|12|13| \in P_{12}$ is periodic with period length $l = 5$:

There is $D(\pi) = 133121331213312$ and $D_5(\pi) = \boxed{13312}$.

With the help of difference permutations it is possible to see whether a permutation is geometric and which properties (0),(Ia), or (Ib)) yield (cf. Part II, chapter 1). This is explained in Lemma 2.

Consider the permutation $\pi = i_o i_1 \ldots i_{d+1} i_{d+2} \in \hat{\Pi}_d,\ \ d > 1$ and its difference permutation

$$D(\pi) = i'_o i'_1 \ldots i'_{d+1} i'_{d+2}$$

with

$$i'_j = \begin{cases} i_{j+1} - i_j \ \text{ for } i_{j+1} > i_j \\ i_{j+1} - i_j + (d+3) \ \text{ for } i_{j+1} < i_j \end{cases} (j = 0, 1, \ldots, d+2\,,\ \ (\bmod (d+3))\ )$$

and

$$S := \Sigma_{j=0}^{d+2}\ i'_j.$$

Since $S = (i_1 - i_o + \epsilon_o(d+3)) + (i_2 - i_1 + \epsilon_1(d+3)) + \ldots + (i_{d+2} - i_{d+1} + \epsilon_{d+1}(d+3)) +$
$+ (i_o - i_{d+2} + \epsilon_{d+2}(d+3)) = \Sigma_{j=0}^{d+2}\ \epsilon_j(d+3)$
with $\epsilon_j \in \{0, 1\}, \epsilon_o = 0, \epsilon_{d+2} = 1,$

there holds

$$S = k(d+3), \quad 1 \leq k \leq d+2.$$

Lemma 2. Consider the permutation $\pi = i_o i_1 \ldots i_{d+1} i_{d+2} \in \hat{\Pi}_d,\ d > 1$ with the difference permutation $D(\pi)$ and $S = k(d+3)$.

There holds:

$k = 1 \iff \pi$ is geometric and has property (0)

$k = 2 \iff \pi$ is geometric and has property (Ia)

$k = d+1 \iff \pi$ is geometric and has property (Ib)

$3 \leq k \leq d \iff \pi$ is not geometric

$k = d+2$ is only possible for
$\pi = 0\, d+2\, d+1 \ldots 2\, 1\, (\hat{=}\ \pi_o^{(d)})$, but π is not normed.

Remark 1. Lemma 2 is also essentially valid for $\pi \in \Pi_d$.

The following Lemma is usefull for the further considerations.

Lemma 3. $\pi \in P_d \setminus \{\pi_{oo}^{(d)}\}$ has property (Ib)
$\Longrightarrow$ π_r has property (Ia)
(cf. Part II, p. 10 (Lemma 11), Appendix Part II, p. 19-21).

The next Lemma describes several properties of periodic permutations.

Lemma 4. Assume $\pi \in P_d$, π has property (Ia)
$\wedge$ π periodic with period length $l < d+3$

There yields:

(i) $l/(d+3)$

(ii) $2l \not| \ (d+3)$

(iii) For $d+3 = 2^{\mu_o} p_1^{\mu_1} \dots p_h^{\mu_h} (> 4)$ with

$2 < p_1 < \ldots < p_h$, p_k prime numbers
$\mu_o \geq 0, \mu_k > 0$ for $k = 1, 2, \ldots h$

$L(d)$: Set of all possible period length l for P_d

$\bar{L}(d) := L(d) \cup \{d+3\}$

there is

$l \in \bar{L}(d) \iff l = 2^{\nu_o} p_1^{\nu_1} \dots p_h^{\nu_h}$
$\nu_o = \mu_o, \ \nu_k \leq \mu_k, \ k = 1, 2, \ldots h$

2. Periodic Napier cycles

Definition 4. NC is called a <u>Napier cycle</u> generated by the permutation $\pi \in P_d$ $:\iff$

$$NC = NC_\pi = (\pi_o, \pi_1, \pi_2, \ldots, \pi_{d+1}, \pi_{d+2}) \qquad \pi_j \in P_d$$

$$\pi = \pi_o \mathrel{\hat{=}} i_o i_1 \ldots i_{d+2}, \ \pi_j \mathrel{\hat{=}} i_j i_{j+1} \ldots i_{d+2} i_o i_1 \ldots i_{j-2} i_{j-1}$$
$$(j = 0, 1, \ldots d+2 \ , \quad \text{mod } (d+3))$$

(cf. Part I, p. 4, 13 (Lemma 7), Appendix Part I, p. 10).

Definition 5. NC_{π_o} is called a periodic Napier cycle if the generating permutation π_o is a periodic permutation.

Lemma 5. Let be NC_{π_o} a periodic Napier cycle, π_o a periodic permutation with period length $l < d+3$.

Then there hold the following properties:

(i) All permutations π_j $(j = 0, 1, \dots d+2)$ of NC_{π_o} are periodic with period length l.

(ii) NC_{π_o} has period length l:

$$NC_{\pi_o} = (\underbrace{\pi_o, \pi_1, \pi_2 \dots \pi_{l-1}}_{l \text{ permutations}}\,\underbrace{\pi_o, \pi_1, \pi_2 \dots \pi_{l-1}}_{l \text{ permutations}} \;\dots\; \underbrace{\pi_o, \pi_1, \pi_2 \dots \pi_{l-1}}_{l \text{ permutations}})$$

$(\frac{d+3}{l}$ periods)

(iii) Both Napier cycles NC_{π_o} and $NC_{\pi_{or}}$ have the same period length l.

A geometric permutation $\pi \in P_d$ together with its reflected permutation π_r describes a type of the Napier cycles. Thus the Napier cycle NC_{π_r} consisting of all the reflected permutations of NC_π describes the same type of Napier cycles as NC_π.

Lemma 6. Only if NC_π contains no symmetric permutations the two Napier cycles NC_π and NC_{π_r} (being of the same type) are different.

The intersection of the corresponding orthoschemes of NC_π is called the hyperbolic kernel. It generates a type of hyperbolic kernels. The hyperbolic kernel of NC_{π_r} belongs to the same type. The number of types of hyperbolic kernels is the same as the number of the types of Napier cycles (cf. Part II, p. 12-17).

3. The structure of the periods

The following Lemmata describe structure and properties of the periods. Especially symmetric permutations play a special roll. They are important for the calculation of the types of Napier cyles. Therefore, the permutation differences for symmetric permutations are considered here.

Lemma 7. Let be $\pi \in P_d$ with period $\boxed{a_1 a_2 \dots a_l}$, $l < d+3$, $l/d+3$.

There yield the properties:

(i) $\Sigma_{i=1}^{l} a_i = 2l$ (π property (Ia))

(ii) For $\Sigma_{i=1}^{l'} a_{i+k} = l$, $k \in \{0, 1, \dots l-1\}$ exists no $l' < l$.

(iii) $\boxed{l+1\,1 \dots 1}$ is allowed for $l < d+3$, but for $l = d+3$ forbidden

Remark 2:
If $l = d+3$ then $\boxed{l+1\,1 \dots 1}$ is identical with $\boxed{1 \dots 1} = \boxed{1}$.

Now symmetric permutations with property (Ia) are considered.

Lemma 8. Let be $\pi \in S_d \setminus \pi_{oo}^{(d)}$

Then there is

(i) For $d = 2q-1$: $D(\pi) = i'_o i'_1 \dots i'_{q-1} i'_q i'_{q-1} \dots i'_1 i'_o i'_{2q+1}$

(ii) For $d = 2q$: $D(\pi) = i'_o i'_1 \dots i'_q i'_q \dots i'_1 i'_o i'_{2q+2}$

Consequence: If π is periodic with period length $l < d+3$ there yields for the period $D_l(\pi)$:

(i) $\boxed{a_o a_1 \dots a_{\frac{l-4}{2}} a_{\frac{l-2}{2}} a_{\frac{l-4}{2}} \dots a_1 a_o a_{l-1}}$ for l even

(ii) $\boxed{a_o a_1 \dots a_{\frac{l-3}{2}} a_{\frac{l-3}{2}} \dots a_1 a_o a_{l-1}}$ for l odd

Examples 2:

1. For <u>$l = 6$</u> all possible periods D_l for arbitrary dimension $d > 6$ are given:

711111	sym: 11111 7 and sym: 11711 1	(only for $d > 3$)
<u>6</u> -		
<u>5</u>21112	sym: 21112 5 and sym: 12521 1	
<u>4</u>31121	refl.: 412113 (there exist no symmetric permutations)	
<u>3</u>22122	sym: 22122 3 and sym: 22322 1	

2. For $\underline{d=15}$ two possible special periods with length $l=6$ which belong to symmetric permutations but do not exist for $d=3$ are given (cf. Remark):

$\boxed{111117}$: 0 1 2 3 4 5 12 13 14 15 16 17 6 7 8 9 10 11

$\boxed{117111}$: 0 1 2 9 10 11 12 13 14 3 4 5 6 7 8 15 16 17

4. The structure of Napier cycles

From the last chapter the various cases of the structure of Napier cycles NC_π follow.

-) π not periodic ($l=d+3$)

(i) There exists a symmetric permutation in NC_π, say π_o.

Then the structure of NC_{π_o} is the following:

$d=2q-1(\geq 1)$		$d=2q\ (>1)$	
symmetric	π_o	symmetric	π_o
reflected pair	(π_1, π_{d+2})	reflected pair	(π_1, π_{d+2})
$\vdots$		$\vdots$	
reflected pair	$(\pi_{\frac{d+1}{2}}, \pi_{\frac{d+5}{2}})$	reflected pair	$(\pi_{\frac{d}{2}}, \pi_{\frac{d+6}{2}})$
symmetric	$\pi_{\frac{d+3}{2}}$	reflected pair	$(\pi_{\frac{d+2}{2}}, \pi_{\frac{d+4}{2}})$

These are exactly

$d=2q-1$	$d=2q$
2 symmetric and $d+1$ not symmetric	1 symmetric and $d+2$ not symmetric

permutations

These are

$d=2q-1$	$d=2q$
$\frac{d+5}{2}$	$\frac{d+4}{2}$

types

(ii) There exist no symmetric permutations

Then for arbitrary $d>1$ the structure of NC_{π_o} is the following:

$$\begin{array}{ll} \text{reflected pair} & (\pi_o, (\pi_0)_r) \\ \text{reflected pair} & (\pi_1, (\pi_1)_r) \\ \vdots & \\ \text{reflected pair} & (\pi_{d+2}, (\pi_{d+2})_r) \end{array}$$

These are exactly $d+3$ reflected pairs (types) of not symmetric permutations.

-) π periodic $(l < d+3)$

Special cases: $\underline{l=1}$:

$\underline{d = 2q-1 (\geq 1):}$ Period $\boxed{1}$ $\pi = \pi_o^{(d)}$ $(= 0\,1\,2\,3 \ldots d+1\; d+2)$

$\underline{d = 2q (> 1):}$ Period $\boxed{1}$ $\pi = \pi_o^{(d)}$

Period $\boxed{2}$ $\pi = 0\,2\,4 \ldots d+2\; 1\,3 \ldots d-1\; d+1$

In each case all the $d+3$ permutations of the NC are identical and symmetric.

$\underline{l=2}$ (only for $d = 2q-1\,(>1)$) :

Period $\boxed{13}$ (or $\boxed{31}$)

NC consists of $\frac{d+3}{2}$ blocks of two permutations π_o, π_1 with period $\boxed{13}$ and $\boxed{31}$.

$\underline{l=3}$ (only for $d = 2q = 6q' (\geq 6)$:

Period $\boxed{114}$

NC consists of $\frac{d+3}{3} = 2q'+1$ blocks of two types of permutations π_o (symmetric) and (π_1, π_2) (reflected pair).

(i) There exists a symmetric permutation in NC_π, say π_o, the period is l.

Then the structure of NC_{π_o} is the following:

l even (>1)		l odd (>1)	
symmetric	π_o	symmetric	π_o
reflected pair	(π_1, π_{l-1})	reflected pair	(π_1, π_{l-1})
$\vdots$		$\vdots$	
reflected pair	$(\pi_{\frac{l-2}{2}}, \pi_{\frac{l+2}{2}})$	reflected pair	$(\pi_{\frac{l-3}{2}}, \pi_{\frac{l+3}{2}})$
symmetric	$\pi_{\frac{l}{2}}$	reflected pair	$(\pi_{\frac{l-1}{2}}, \pi_{\frac{l+1}{2}})$

These are exactly

2 symmetric and	1 symmetric and
$l-2$ not symmetric	$l-1$ not symmetric

permutations

These are

$\frac{l+2}{2}$	$\frac{l+1}{2}$

types

Exactly $\frac{d+3}{l}$ such blocks belong to NC_{π_o}.

(ii) There exist no symmetric permutations

Then the strucure of NC_{π_o} is the following:

$$\begin{array}{ll} \text{reflected pair} & (\pi_o, (\pi_o)_r) \\ \text{reflected pair} & (\pi_1, (\pi_1)_r) \\ & \vdots \\ \text{reflected pair} & (\pi_l, (\pi_l)_r) \end{array}$$

These are exactly l reflected pairs of
not symmetric permutations.

Exactly $\frac{d+3}{l}$ such blocks belong to NC_{π_o}.

Figure 1 shows all the possible structures of Napier cycle types for arbitrary $d\,(\geq 1)$ and for periodic Napier cycle types with period length l, $l/d+3$.

The meaning of the notions used in Figure 1 is the following:

${}^{s}a_{d+3}^{(d+3)}$: Cardinality of all Napier cycle types
not periodic
containing symmetric permutations

${}^{r}a_{d+3}^{(d+3)}$: Cardinality of all Napier cycle types
not periodic
containing no symmetric permutations

${}^{s}a_{l}^{(l'+3)}$: Cardinality of all Napier cycle types
periodic with period length $l < l'+3,\ l/(l'+3)$
containing symmetric permutations

${}^{r}a_l^{(l'+3)}$: Cardinality of all Napier cycle types periodic with period length $l < l'+3,\ l/(l'+3)$ containing no symmetric permutations

The corresponding sets are called classes of Napier cycle types.

For the cardinality A_d of all Napier cycle types for d (≥ 1) follows

$$A_d = \Sigma_{l\in L}\ ({}^{s}a_l^{(d+3)}) + \Sigma_{l\in L}\ ({}^{r}a_l^{(d+3)}) + {}^{s}a_{d+3}^{(d+3)} + {}^{r}a_{d+3}^{(d+3)} \quad (\star\star)$$

Because of Remark 2 there is

$$\begin{aligned} {}^{s}a_l^{(l')} &= {}^{s}a_l^{(l)} + 1 \quad \text{for } l < l' \\ {}^{r}a_l^{(l')} &= {}^{r}a_l^{(l)} \qquad\quad \text{for } l < l' \end{aligned}$$

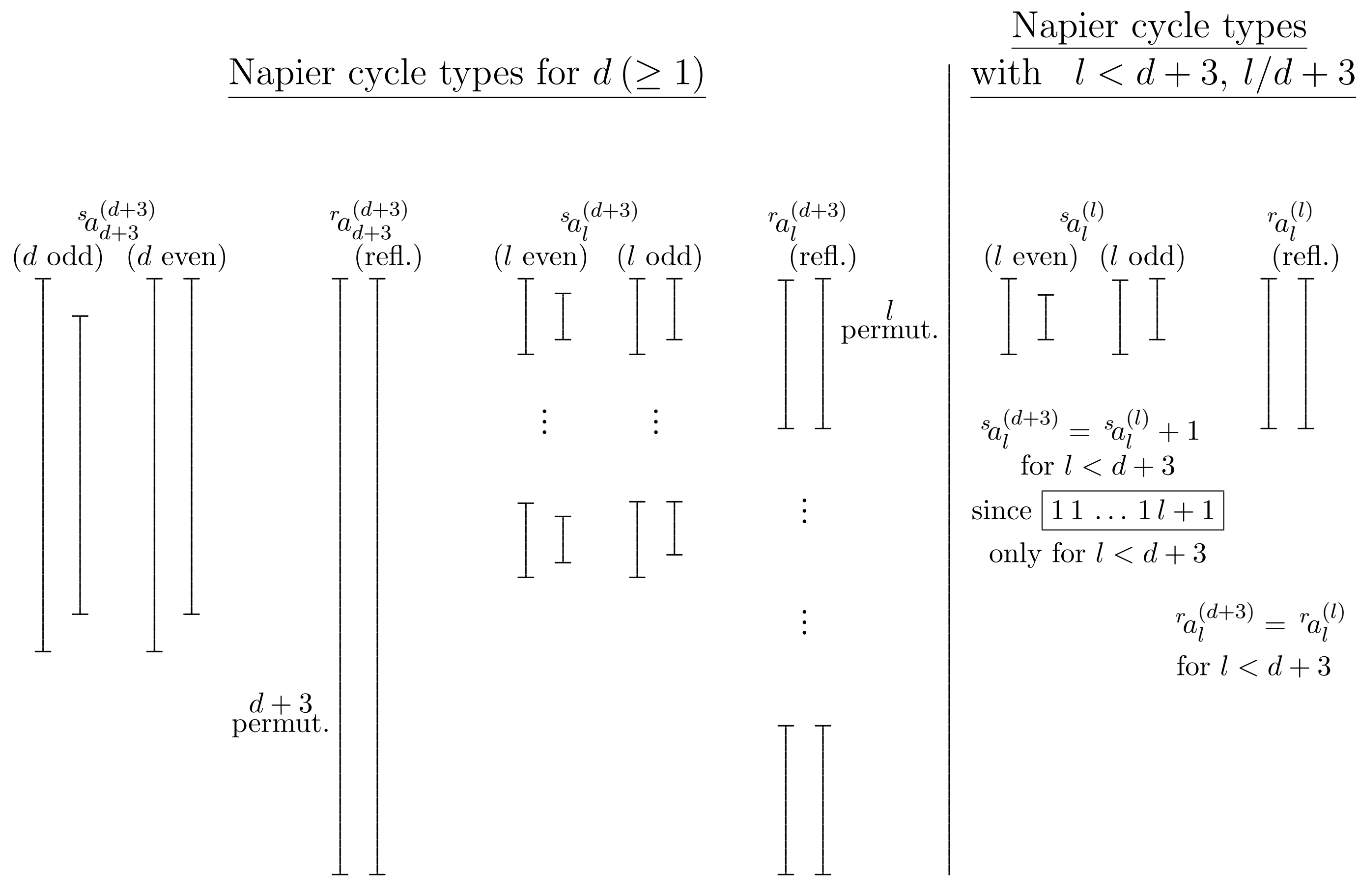

Figure 1
Kinds of Napier cycles

5. The numbers of Napier cycle types

At first two special cases are considered. With their help the numbers of special Napier cycle types can be calculated.

Theorem 1. Let be $d+3 = 2q+3 = p \geq 5$ $(q \geq 1)$, p odd prime number

There is $\bar{L} = \{1, p\}$ and it holds

for $l = 1$: $\boxed{1}$, $\boxed{2}$ (2 types)

for $l = p = 2q+3$ $({}^{s}a_p^{(p)} + {}^{r}a_p^{(p)}$ types)

$\Longrightarrow$ $A_d = A_{p-3} = \frac{2^{2q+1}+q+1}{2q+3}$ $(\in \mathbb{N})$

Theorem 2. Let be $d+3 = 2^z = 2q+2$ $(q > 1)$, $z \geq 3$ $(d > 1)$

There is $\bar{L} = \{1, 2^z\}$ and it holds

for $l = 1$: $\boxed{1}$ (1 type)

for $l = 2^z = 2q+2$ $({}^{s}a_{2q+2}^{(2q+2)} + {}^{r}a_{2q+2}^{(2q+2)}$ types)

$\Longrightarrow$ $A_d = \frac{1}{q+1} 2^{2q-1} + 2^{q-1}$ $(\in \mathbb{N})$

For low dimensions $d \leq 5$ the numbers of Napier cycle types are formerly given and here once more listed in Table 1.

Furthermore, from chapter 4 (π periodic, $l = 1$, $l = 2$ and $l = 3$) follows

$${}^{s}a_1^{(2q+2)} = 1, \quad {}^{s}a_1^{(2q+3)} = 2, \quad {}^{r}a_1^{(2q+2)} = {}^{r}a_1^{(2q+3)} = 0,$$
$${}^{s}a_2^{(2q+2)} = 1, \quad {}^{r}a_2^{(2q+2)} = 0,$$
$${}^{s}a_2^{(2q+3)} \text{ and } {}^{r}a_2^{(2q+3)} \text{ do not exist because of } 2 \not| \ (2q+3),$$
$${}^{s}a_3^{(2q+3)} = 1, \quad {}^{r}a_3^{(2q+3)} = 1$$

The numbers for ${}^{s}a_{d+3}^{(d+3)}$ and ${}^{r}a_{d+3}^{(d+3)}$ can be find

for $d = 1$ in Part II, Lemma 1,
for $d = 2$ in Part II, page 14,
for $d = 3$ in Part II, page 15,
for $d = 4$ and $d = 5$ in Part I, Theorem 2.

d	$d+3$	$\bar{L}$	${}^{s}a_1^{(d+3)}$	${}^{r}a_1^{(d+3)}$	${}^{s}a_2^{(d+3)}$	${}^{r}a_2^{(d+3)}$	${}^{s}a_{d+3}^{(d+3)}$	${}^{r}a_{d+3}^{(d+3)}$	A_d
1	4	$\{1,4\}$	1	0	1	0	0	0	2
2	5	$\{1,5\}$	2	0	-	-	2	0	4
3	6	$\{1,2,6\}$	1	0	1	0	2	1	5
4	7	$\{1,7\}$	2	0	-	-	6	1	9
5	8	$\{1,8\}$	1	0	-	0	7	4	12

Table 1

Now the general case $d \geq 1,\ d+3 \neq 2^z,\ d+3 \neq p$ is considered.

To find A_d corresonding formula $(\star\star)$ the numbers ${}^{s}a_l^{(d+3)}$ and ${}^{r}a_l^{(d+3)}$ with $l \in \bar{L}$ must be known.

There yields:

Theorem 3. Assume for all $j < d$ the numbers A_j and for all $j < d+3$ the numbers ${}^{s}a_l^{(d+3)}$ and ${}^{r}a_l^{(d+3)}$ are known. Then the numbers ${}^{s}a_{d+3}^{(d+3)}$, ${}^{r}a_{d+3}^{(d+3)}$ and A_d can be calculated.

d	$d+3$	A_d
1	4	2
2	5	4
3	6	5
4	7	9
5	8	12
6	9	23
7	10	34
8	11	63
9	12	102
10	13	190
11	14	325
12	15	612
13	16	1088
14	17	2056
15	18	3771
16	19	7155
17	20	13364
18	21	25482

d	$d+3$	A_d
19	22	48175
20	23	92205
21	24	175792
22	25	337594
23	26	647326
24	27	1246863
25	28	2400842
26	29	4636390
27	30	8956060
28	31	17334801
29	32	33570816
30	33	65108062
31	34	126355336
32	35	245492244
33	36	477284182
34	37	928772650
35	38	1808662528

Table 2

In Table 2 the numbers A_d of all Napier cycle types for $1 \le d \le 35$ are given.

Theorem 3 especially means that the numbers A_d $(d \ge 1)$ of all Napier cycle types are recursively calculable.

6. Proofs of the Theorems

As shown in the former Parts the numbers of all reflected pairs of geometric normed permutations (orthoscheme types) $(N_d := \mathrm{card}(O_d))$ and the numbers of all geometric normed symmetric permutations $(F_d : = \mathrm{card}(S_d))$ are known for each $d \ge 1$ (cf. Part II, Theorem 3 and Theorem 4):

$$N_d = \mathrm{card}(O_d) = 2^{[\frac{d+1}{2}]}(2^{[\frac{d+2}{2}]} - [\frac{d+3}{2}]) \ , \ d > 1, \ N_1 = 3.$$

$$F_d = \mathrm{card}(S_d) = 2^{[\frac{d+3}{2}]} - \frac{1}{2}(1-(-1)^d) \ , \quad d \ge 1$$

Therefore, the numbers of all reflected but not symmetric pairs of geometric normed permutations are $R_d = N_d - F_d$ and give

$$R_d = 2^{[\frac{d+1}{2}]}(2^{[\frac{d+2}{2}]} - 1) - [\frac{d+1}{2}] - \frac{1}{2}(1+(-1)^d), \quad d > 1, \ \ R_1 = 0.$$

Especially there yields

$$R_{2q} = 2^q(2^{q+1}-1) - (q+1) \ (q \ge 1), \quad R_{2q-1} = 2^q(2^q - 1) - q, \ (q > 1).$$

Consider d, $\bar{L} = \{1, l_1, l_2, \ldots, l_t, d+3\}$.

period length l of NC	structure of NC sym.	structure of NC refl. pairs	total numbers sym.	total numbers refl. pairs	numbers of NC with period l_i
1	1	0	${}^s a_1^{(d+3)}$	0	${}^s a_1^{(d+3)}$
l_1	$\frac{3+(-1)^{d+1}}{2}$	$[\frac{l_1-1}{2}]$	$(\frac{3+(-1)^{d+1}}{2}) {}^s a_{l_1}^{(d+3)}$	$[\frac{l_1-1}{2}] \ {}^s a_{l_1}^{(d+3)}$	${}^s a_{l_1}^{(d+3)}$
l_1	0	l_1	0	$l_1 \ {}^r a_{l_1}^{(d+3)}$	${}^r a_{l_1}^{(d+3)}$
$\vdots$					
d+3	$\frac{3+(-1)^{d+1}}{2}$	$[\frac{d+2}{2}]$	$(\frac{3+(-1)^{d+1}}{2}) {}^s a_{d+3}^{(d+3)}$	$([\frac{d+2}{2}]) {}^s a_{d+3}^{(d+3)}$	${}^s a_{d+3}^{(d+3)}$
d+3	0	d+3	0	$(d+3) {}^r a_{d+3}^{(d+3)}$	${}^r a_{d+3}^{(d+3)}$
sum:			F_d	R_d	A_d

Table 3

For the proofs of the theorems the connections of the cardinalities of the sets of the concerning various classes of Napier cycle types are to consider. These connections are demonstrated in Table 3. Then for the special values of d the concerning theorems can be proved. Observe the special values for

$$ {}^{s}a_1^{(2q+2)} \,,\; {}^{s}a_2^{(2q+2)} \,,\; {}^{s}a_1^{(2q+3)} \,,\; {}^{s}a_2^{(2q+3)},\; {}^{s}a_3^{(2q+3)} \,,\; {}^{r}a_3^{(2q+3)} \text{ (cf. page 11).} $$

Proof of Theorem 1. Let be $d+3 = 2q+3 = p \geq 5$ $(q \geq 1)$, p odd prime number. There is $\bar{L} = \{1, p\}$.

To show: $A_d = A_{p-3} = \frac{2^{2q+1}+q+1}{2q+3} + 2^q \quad (\in \mathbb{N})$.

There is ${}^{s}a_1^{(d+3)} = 2$, $F_d = 2^{q+1}$, $R_d = 2^q(2^{q+1}-1) - (q+1)$

From Table 3 follows for $d+3 = 2q+3 = p$, $d = 2q$ then the Table 3a:

1	1	0	2	0	2
$2q+3$	1	$q+1$	${}^{s}a_{d+3}^{(d+3)}$	$(q+1)\cdot{}^{s}a_{d+3}^{(d+3)}$	${}^{s}a_{d+3}^{(d+3)}$
$2q+3$	0	$2q+3$	0	$(2q+3)\cdot{}^{r}a_{d+3}^{(d+3)}$	${}^{r}a_{d+3}^{(d+3)}$
sum:			F_d	R_d	A_d

Table 3a

A balance gives

-) $F_d = 2 + {}^{s}a_{d+3}^{(d+3)} + 0 = 2^{q+1} \implies {}^{s}a_{d+3}^{(d+3)} = 2^{q+1} - 2$

-) $R_d = 0 + (q+1)\cdot{}^{s}a_{d+3}^{(d+3)} + (2q+3)\cdot{}^{r}a_{d+3}^{(d+3)} = (q+1)(2^{q+1}-2) + (2q+3)\cdot{}^{r}a_{d+3}^{(d+3)}$
$= 2^q(2^{q+1}-1) - (q+1)$

$$\implies (2q+3)\cdot{}^{r}a_{d+3}^{(d+3)} = 2^q(2^{q+1}-1) - (q+1) - 2(q+1)2^q + 2(q+1) = 2^q(2^{q+1} - (2q+3)) + q + 1$$

-) $A_d = 2 + {}^{s}a_{d+3}^{(d+3)} + {}^{r}a_{d+3}^{(d+3)} = \frac{2^{2q+1}+q+1}{2q+3} + 2^q.$

Thus Theorem 1 is proved.

Remark 3: Since A_d is a natural number there must be

$$\frac{2^{2q+1}+q+1}{2q+3} = \frac{2^{p-1}+(p-1)}{2p}$$

a natural number if p is a prime number $(p > 3)$. This can be seen in the following way announced by H. Menzer:

To show: $2p/(2^{p-1}+(p-1))$ for $p>2$, p prime.

Namely Fermat's Theorem

$$a^{p-1}\equiv 1 \bmod p,\ p \text{ odd prime number},\ a\in\mathbb{N}$$

yields for $a=2$:

$$2^{p-1}-1+p\equiv 0\,(p)$$
$$\Longrightarrow\ p/(2^{p-1}+p-1).$$

Because of $2/(2^{p-1}+p-1)$ there is

$$2p/(2^{p-1}+(p-1)) \text{ for } p \text{ odd prime number.}$$

This is the assertion.

Proof of Theorem 2. Let be $d+3=2^z=2q+2\ (q>1),\ z\geq 3\ (d>1)$

$$\text{There is } \bar{L}=\{1,2^z\}$$

$$\text{To show:}\quad A_d=A_{2^z-3}=\tfrac{1}{q+1}2^{2q-1}+2^{q-1}\quad(\in\mathbb{N})$$

There is ${}^{s}a_1^{(d+3)}=1$, $F_d=2^{q+1}-1$, $R_d=2^q(2^q-1)-q$

From Table 3 follows for $d+3=2^z=2q+2$, $d=2q-1$ then the Table 3b:

1	1	0	1	0	1
$2q+2$	2	q	$2\cdot{}^{s}a_{d+3}^{(d+3)}$	$q\cdot{}^{s}a_{d+3}^{(d+3)}$	${}^{s}a_{d+3}^{(d+3)}$
$2q+2$	0	$2q+2$	0	$(2q+2)\cdot{}^{r}a_{d+3}^{(d+3)}$	${}^{r}a_{d+3}^{(d+3)}$
sum:			F_d	R_d	A_d

Table 3b

A balance gives

-) $F_d=1+2\cdot{}^{s}a_{d+3}^{(d+3)}+0=2^{q+1}-1\ \Longrightarrow\ {}^{s}a_{d+3}^{(d+3)}=2^q-1$

-) $R_d=0+q\cdot{}^{s}a_{d+3}^{(d+3)}+(2q+2)\cdot{}^{r}a_{d+3}^{(d+3)}=(2q+2)\cdot{}^{r}a_{d+3}^{(d+3)}+q(2^q-1)$

$$=2^q(2^q-1)-q$$
$$\Longrightarrow\ (2q+2)\cdot{}^{r}a_{d+3}^{(d+3)}=2^q(2^q-(q+1))$$

-) $A_d = 1 +^s a_{d+3}^{(d+3)} +^r a_{d+3}^{(d+3)} = \frac{2^q(2^q+q+1)}{2q+2} = \frac{1}{q+1} 2^{2q-1} + 2^{q-1}$

Thus Theorem 2 is proved.

Remark 4:

Since A_d is a natural number there must be

$$\frac{2^{2q-1}}{q+1} = \frac{1}{4(2q+2)} 2^{2q+2} = \frac{2^{d+3}}{4(d+3)} = \frac{2^{2^z}}{4 \cdot 2^z} = 2^{2^z-(z+2)}$$

a natural number for $z \in \mathbb{N}$ and $z \geq 3$. This can be seen in the following way:

For $z \in \mathbb{N},\ z \geq 3$ follows

$$\begin{aligned} Z := 2^z - (z+2) &= (1+1)^z - (z+2) \\ &= 1+z+\tfrac{z(z-1)}{2}+\cdots+1-(z+2) = \tfrac{z(z-1)}{2}+\ldots > 0 \text{ and } Z \in \mathbb{N}. \end{aligned}$$

Thus $2^Z \in \mathbb{N}$ and therefore $(q+1)/2^{2q-1}$.

This is the assertion.

Proof of Theorem 3. Assume there are known

$$\begin{aligned} &{}^s a_j^{(d+3)} \text{ and } {}^r a_j^{(d+3)} \text{ for all } j < d+3\ , \\ &A_j \text{ for all } j < d. \end{aligned}$$

To show: ${}^s a_{d+3}^{(d+3)}$, ${}^r a_{d+3}^{(d+3)}$, A_d can be calculated.

Consider Table 3. A balance gives

-) $F_d =^s a_1^{(d+3)} + \frac{3+(-1)^{d+1}}{2} (\sum_{l_j \in \bar{L}}\ {}^s a_{l_j}^{(d+3)}) = 2^{\frac{[d+3]}{2}} - \frac{1}{2}(1-(-1)^d)$

$$\Longrightarrow\ {}^s a_{d+3}^{(d+3)} = \mathrm{f}({}^s a_{l_j}^{(d+3)})_{l_j \in L}$$

-) $R_d = \sum_{l_j \in \bar{L}} [\frac{l_j-1}{2}]\, {}^s a_{l_j}^{(d+3)} + \sum_{l_j \in L} l_j\, {}^r a_{l_j}^{(d+3)} + (d+3)\, {}^r a_{d+3}^{(d+3)}$

$$= 2^{[\frac{d+1}{2}]}(2^{[\frac{d+2}{2}]} - 1) - [\tfrac{d+1}{2}] - \tfrac{1}{2}(1+(-1)^d)$$

$$\Longrightarrow\ (d+3)\, {}^r a_{d+3}^{(d+3)} = \mathrm{f}({}^s a_{l_j}^{(d+3)}, {}^r a_{l_j}^{(d+3)}, {}^s a_{d+3}^{(d+3)})_{l_j \in L}$$

-) $A_d = \sum_{l_j \in \bar{L}} ({}^s a_{l_j}^{(d+3)} +^r a_{l_j}^{(d+3)})$

Thus Theorem 3 is proved.

Special cases were considered in Theorem 1 and Theorem 2. They have given special general values.

Example 3:

$\underline{d=27}$: There is $d+3=30$, $L=\{1,2,6,10\}$. Calculating ${}^s a_{30}^{(30)}$, ${}^r a_{30}^{(30)}$, A_{27} .

The calculation for $d=6$ and $d=10$ has given

$${}^s a_6^{(6)} = 2,\ {}^r a_6^{(6)} = 1\ ,\ A_3 = 5\ .$$

(cf. also Example 2,1. and Part II, Lemma 1)

$${}^s a_{10}^{(10)} = 14,\ {}^r a_{10}^{(10)} = 18\ ,\ A_7 = 34\ .$$

There follows : ${}^s a_6^{(30)} = {}^s a_6^{(6)} + 1 = 3\ ,\ {}^r a_6^{(30)} = {}^r a_6^{(6)} = 1$

${}^s a_{10}^{(30)} = {}^s a_{10}^{(10)} + 1 = 15\ ,\ {}^r a_{10}^{(30)} = {}^r a_{10}^{(10)} = 18$

Furthermore, there is ${}^s a_1^{(30)} = 1,\ {}^s a_2^{(30)} = 1\ ,\ {}^r a_1^{(30)} = 0,\ {}^r a_2^{(30)} = 0$

Therefore, there is the corresponding table

1	1	0	1	0	1
2	2	0	2	0	1
6	2	2	6	6	3
6	0	6	0	6	1
10	2	4	30	60	15
10	0	10	0	180	18
30	2	14	$2 \cdot {}^s a_{30}^{(30)}$	$14 \cdot {}^s a_{30}^{(30)}$	${}^s a_{30}^{(30)}$
30	0	30	0	$30 \cdot {}^r a_{30}^{(30)}$	${}^r a_{30}^{(30)}$
sum:			$F_{27} = 32767$	$R_{27} = 268419058$	A_{27}

Table 3c

A balance gives

-) $F_{27} = 1+2+6+30+2 \cdot {}^s a_{30}^{(30)} = 32767 \implies 2 \cdot {}^s a_{30}^{(30)} = 32728$

$$\implies {}^s a_{30}^{(30)} = 16364$$

-) $R_{27} = 6+6+60+180+14 \cdot {}^s a_{30}^{(30)} + 30 \cdot {}^r a_{30}^{(30)} = 268419058$

$$\implies 30 \cdot {}^r a_{30}^{(30)} = 268419058 - 229348 = 268189710$$

$$\implies {}^r a_{30}^{(30)} = 8939657$$

-) $A_{27} = 1+1+3+1+15+18+{}^s a_{30}^{(30)} + {}^r a_{30}^{(30)} = 39+16364+8939657 = 8956060$

7. Asymptotical behaviour

Since the cardinality A_d of the Napier cycle types for each d can be recursively calculated there it is of interest to destinate at least the asymptotical behaviour of the sequence $\{A_d\}$. Thus at first two special cases are considered using the knowledge of the two Theorems 1 and 2.

(i) Let be $d+3 = 2q+3 = p \geq 5$, p odd prime number. Then Theorem 1 can also be written in the form

$$A_d = \frac{2^{2q+1}+q+1}{2q+3} + 2^q = \frac{2^{d+1}}{d+3} + 2^{\frac{d}{2}} + \frac{d+2}{2(d+3)} = \frac{2^{d+1}}{d+3} + 2^{[\frac{d}{2}]} + O(1)$$

(ii) Let be $d+3 = 2^z = 2q+2$, $z \geq 3$. Then Theorem 2 can also be written in the form

$$A_d = \frac{1}{q+1}2^{2q-1} + 2^{q-1} = \frac{2^{d+1}}{d+3} + 2^{\frac{d-1}{2}} = \frac{2^{d+1}}{d+3} + 2^{[\frac{d}{2}]}$$

Now concerning the asymptotical behaviour of the sequence $\{A_d\}$ the general case is described by the following theorem.

Theorem 4. For the cardinality of the Napier cycle types yields for $d \geq 1$

$$A_d = \frac{1}{d+3}2^{d+1} + 2^{[\frac{d}{2}]} + O(2^c), \quad c \leq \frac{d}{3} \quad (\diamond)$$

For the proof of Theorem 4 now Lemma 9 is useful which describes the behaviour of the cardinalities ${}^s a_l^{(d+3)}$ and ${}^r a_l^{(d+3)}$.

Lemma 9. For the cardinality of the Napier cycle types ${}^s a_l^{(d+3)}$ and ${}^r a_l^{(d+3)}$ there is to distinguish between odd $d = 2q-1$ and even $d = 2q$.

$\underline{d = 2q-1}$: $q = \frac{d+1}{2}$, $q \geq 2$:

$$\textstyle\sum_{l \in \bar{L}} {}^s a_l^{(d+3)} = 2^q \quad \Longrightarrow \quad {}^s a_{d+3}^{(d+3)} = 2^q - \cdots$$

$$(d+3) \cdot {}^r a_{d+3}^{(d+3)} = 2^{2q} - 2^q - \tfrac{d+1}{2} \cdot {}^s a_{d+3}^{(d+3)} - \cdots \quad \Longrightarrow \quad {}^r a_{d+3}^{(d+3)} = \tfrac{1}{d+3}2^{2q} - 2^{q-1} - \ldots$$

$$l < d+3: \quad {}^s a_l^{(d+3)} = O(2^c),\ c \leq \tfrac{d}{6}, \qquad {}^r a_l^{(d+3)} = O(2^c),\ c \leq \tfrac{d}{3}$$

$\underline{d = 2q}$: $\quad q = \frac{d}{2}$, $q \geq 2$:

$$\textstyle\sum_{l\in\bar{L}} {}^s a_l^{(d+3)} = 2^{q+1} \quad \Longrightarrow \quad {}^s a_{d+3}^{(d+3)} = 2^{q+1} - \cdots$$

$$(d+3)\cdot {}^r a_{d+3}^{(d+3)} = 2^{2q+1} - 2^q - \tfrac{d+2}{2} \cdot {}^s a_{d+3}^{(d+3)} - \cdots \quad \Longrightarrow \quad {}^r a_{d+3}^{(d+3)} = \tfrac{1}{d+3} 2^{2q+1} - 2^q - \ldots$$

$$l < d+3: \quad {}^s a_l^{(d+3)} = O(2^c),\ c \leq \tfrac{d}{6}, \qquad {}^r a_l^{(d+3)} = O(2^c),\ c \leq \tfrac{d}{3}$$

Proof of Theorem 4.

Formerly there was shown (cf. also Table 2)

$$A_1 = 2 \text{ and } A_2 = 4.$$

The formula ($\diamond$) in Theorem 4 gives

$$A_1 = \tfrac{1}{4} 2^2 + 2^0 + O(2^c) = 2 + O(2^c), \quad c \leq \tfrac{1}{3}$$

$$A_2 = \tfrac{1}{5} 2^3 + 2^1 + O(2^c) = 3,6 + O(2^c), \quad c \leq \tfrac{2}{3}$$

This is compatible with the true values above.

Now consider $d \geq 3$. Since

$$A_d = {}^s a_{d+3}^{(d+3)} + {}^r a_{d+3}^{(d+3)} + \sum_{l\in L} {}^s a_l^{(d+3)} + \sum_{l \in L} {}^r a_l^{(d+3)}$$

follows for $\underline{d = 2q-1}$

$$A_d = \frac{1}{d+3} 2^{2q} - 2^{q-1} + 2^q + O(2^c),\ c \leq \frac{d}{3}$$

$$= \frac{1}{d+3} 2^{d+1} + 2^{\frac{d-1}{2}} + O(2^c)$$

and for $\underline{d = 2q}$

$$A_d = \frac{1}{d+3} 2^{2q+1} - 2^q + 2^{q+1} + O(2^c),\ c \leq \frac{d}{3}$$

$$= \frac{1}{d+3} 2^{d+1} + 2^{\frac{d}{2}} + O(2^c)$$

and therefore for arbitrary $d \geq 3$ yields ($\diamond$)

Thus Theorem 4 is proved.

Finaly the growth of the sequence $\{A_d\}$ is of interest. This is described in the next Theorem.

Theorem 5. For the cardinality of the Napier cycle types A_d yields

$$1 \leq d < d' \implies A_d < A_{d'}.$$

Remark 5:

Theorem 5 means: $\{A_d\}$ is a monotonly strict increasing sequence.

From Table 3 follows

Lemma 10. For $d \geq 9$ yields

$$\frac{1}{d+3} R_d < A_d < \frac{1}{d} R_d$$

Proof of Theorem 5.

For $d \leq 35$ Theorem 5 follows from Table 2. For $d \geq 9$ Lemma 10 can be used. There follows

$$A_d - A_{d-1} > \frac{1}{d+3} R_d - \frac{1}{d-1} R_{d-1} := R$$

To show: $R > 0$.

$\underline{d = 2q}$ $(q \geq 5)$

$$R = 2^{2q} \frac{2q-5}{(2q+3)(2q-1)} + 2^q \frac{4}{(2q+3)(2q-1)} + \frac{2q+1}{(2q+3)(2q-1)} > 0$$

$\underline{d = 2q-1}$ $(q \geq 5)$

$$R = 2^{q-1}(2^q - 1) \frac{2q-6}{(2q+2)(2q-2)} + \frac{4q}{(2q+2)(2q-2)} > 0$$

Thus Theorem 5 is proved.

Remark 6:

From Theorem 4 follows for $d \geq 2$

$$\frac{A_d}{A_{d-1}} \approx 2 : 1$$

8. Open questions

1. In this note there is explained how for general d the cardinality A_d of the Napier cycle types can be recursively calculated. The question is whether with a theorie of elementary number theory each cardinality A_d can be directly given in the form of a general formula.

2. In their papers H.Ch. Im Hof $[IH1], [IH2]$ and R. Kellerhals $[K1]$ have considered Coxeter polytopes, these are polytopes with dihedral angles having all values of sice $\frac{\pi}{n}$, $n \in \mathbb{N}$. R. Kellerhals had given many such examples for the polyhedra which are hyperbolic kernels and $d \leq 4$. The question is to give all possibilities of Coxeter polyhedra being hyperbolic kernels.

9. Literature

[B1] J. Böhm, Generalized hyperbolic Napier cycles and their hyperbolic kernels, Part I, Part II. Friedrich Schiller-Universität Jena. Jenaer Schriften zur Mathematik und Informatik. Math/Inf/01/08, 1-17, 1-17 (2008).

[B2] J. Böhm, Generalized hyperbolic Napier cycles and their hyperbolic kernels, Part I, Part II, Appendix. http://www.minet.uni-jena.de/preprints/boehm_08/Napiercycles.pdf. Report of the Department of Mathematics and Computer Science, Friedrich-Schiller-University Jena, report no. 08-01.

[IH1] HCh. Im Hof, A class of hyperbolic Coxeter groups, Expo. Math 3, 179-186 (1985).

[IH2] HCh. Im Hof, Napier cycles and hyperbolic Coxeter groups. Bull Soc. Math. Belgique 42, 523-545 (1990).

[K1] R. Kellerhals, Ueber den Inhalt hyperbolischer Polyeder in den Dimensionen drei und vier. Diss. Basel 1988.