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Local characterization of generalized 2-microlocal spaces *

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Abstract

This paper deals with a generalization of 2-microlocal spaces in the sense of weighted Besov spaces. We define 2-microlocal Besov spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ and describe first properties.
The key theorem is the local means characterization for these spaces. Using this characterization we prove some conclusions as a pointwise multiplier assertion and the invariance of the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ under the action of diffeomorphisms.

Keywords: Besov spaces, 2-microlocal spaces, local means
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1 Introduction

The concept of 2-microlocal analysis or 2-microlocal function spaces is due to J.M. Bony (see [Bo84]). It is an appropriate instrument to describe the local regularity and the oscillatory behavior of functions near to singularities.
The approach is Fourier-analytical using Littlewood-Paley-analysis of distributions. The theory has been elaborated and widely used in fractal analysis and signal processing by several authors. We refer to [Ja91], [JaMey96], [LVSeu04], [Mey97], [MeyXu97], [MoYa04] and [Xu96].
The main achievements are closely related to the use of wavelet methods and, as a consequence, wavelet characterizations of 2-microlocal spaces. Here, we intend to give a unified Fourier-analytical approach to generalize 2-microlocal Besov spaces and we are interested in local characterizations of the spaces under consideration.
Therefore, let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a smooth resolution of unity (see Subsection 2.2 for the precise definition) and let $\{w_j\}_{j\in\mathbb{N}_0}$ be a sequence of weight functions satisfying

$$0 < w_j(x) \leq \mathrm{C} w_j(y)(1+2^j|x-y|)^\alpha \tag{1.1}$$

$$2^{-\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x) \ , \tag{1.2}$$

for $x, y \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $\alpha, \alpha_1, \alpha_2 \geq 0$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and its inverse in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions, respectively. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then we introduce $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| w_j \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \right| L_p(\mathbb{R}^n)\big\|^q \right)^{1/q} < \infty \tag{1.3}$$

for $0 < q < \infty$ and

$$\left\| f \right| B^{s,mloc}_{p\infty}(\mathbb{R}^n, \boldsymbol{w})\big\| = \sup_{j\in\mathbb{N}_0} 2^{js} \left\| w_j \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \right| L_p(\mathbb{R}^n)\big\| < \infty \ , \tag{1.4}$$

for $q = \infty$. As a special case, let $w_j(x) = (1+2^j|x-x_0|)^{s'}$ for $s' \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$. If $p = q = 2$ we obtain the spaces $H^{s,s'}_{x_0}$ considered by Bony in [Bo84]. The case $p = q = \infty$ yields the 2-microlocal spaces $C^{s,s'}_{x_0}$ introduced by Jaffard in [Ja91] and extensively treated by Meyer, Jaffard and Lévy-Vehel ([JaMey96],[LVSeu04]).
The more general case $1 \leq p, q \leq \infty$, and characterizations of chirp-like signals as well as relations to gravitational wave signals, has been studied by Xu, Meyer and Moritoh,Yamada ([Xu96],[MeyXu97],[MoYa04]).
We can rewrite

$$[\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)](x) = [(\mathcal{F}^{-1}\varphi_j) * f](x) \ . \tag{1.5}$$

The functions $\mathcal{F}^{-1}\varphi_j$ do not have compact support. In particular, to compute the building blocks $(\mathcal{F}^{-1}\varphi_j\mathcal{F}f)$ in $x \in \mathbb{R}^n$ we need f globally. Roughly speaking, we shall show that the functions $\mathcal{F}^{-1}\varphi_j$ in (1.5) and (1.3),(1.4), respectively, can be replaced by smooth functions with compact support in a ball of radius $c2^{-j}$ (c is a constant). This leads to local characterizations of our spaces. Characterizations of such a type are well known for weighted and unweighted Besov spaces (see for instance [Tri92] and [Tri06]) and turned out to be very useful to solve some key problems as the behavior by pointwise multiplication and invariance under diffeomorphisms. Moreover, it paves the way to atomic and wavelet representations as well as to discretizations (see [Tri06] for classical Besov spaces) and isomorphisms to corresponding sequence spaces.

The paper is organized as follows. Section 2 contains all definitions and some basic properties such as the independence of $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ of the choice of the resolution of unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$ and the lift property. Here we rely on Fourier multiplier theorems for weighted spaces of entire analytic functions which can be found in [SchmTri87].

The main part is Section 3, where we give the characterization by local means. We use maximal function and inequalities and follow ideas in [Tri92], [Ry99] and [Vyb06] in a different context.

The final Section 4 deals with embedding theorems for different metrics based on weighted Nikols'kij inequalities ([SchmTri87]). Moreover, we apply the results of Section 3 (local means) to prove a theorem on pointwise multiplication. Finally, we use the local means characterization to prove that the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ are invariant under a special class of diffeomorphisms.

2 The 2-microlocal Besov spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, w)$

2.1 Preliminaries

As usual $\mathbb{R}^n$ symbolizes the $n-$dimensional Euclidean space, $\mathbb{N}$ is the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$ and $\mathbb{C}$ stand for the sets of integers and complex numbers, respectively.
The points of the Euclidian space $\mathbb{R}^n$ are denoted by $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), \ldots$.
If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ is a multi-index, then its length is denoted by $|\beta| = \sum_{j=1}^n \beta_j$.
The derivatives $D^\beta = \partial^{|\beta|}/\partial^{\beta_1} \cdots \partial^{\beta_n}$ have to be understood in the distributional sense.
We put $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of all complex valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. Its topology is generated by the norms

$$\|\varphi\|_{k,l} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^k \sum_{|\beta| \le l} |D^\beta \varphi(x)| \quad ,k, l \in \mathbb{N}_0 \ . \tag{2.1}$$

A linear mapping $f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is called a tempered distribution, if there is a constant $c > 0$ and $k, l \in \mathbb{N}_0$ such that

$$|f(\varphi)| \le c\|\varphi\|_{k,l}$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The collection of all such mappings is denoted by $\mathcal{S}'(\mathbb{R}^n)$. The Fourier transform is defined on both spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ and is given by

$$(\mathcal{F}f)(\varphi) := f(\mathcal{F}\varphi) \ , \qquad \varphi \in \mathcal{S}(\mathbb{R}^n) \ ,f \in \mathcal{S}'(\mathbb{R}^n)$$

where

$$\mathcal{F}\varphi(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} \, \varphi(x) dx \ .$$

Here $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ stands for the inner product. The inverse Fourier transform is denoted by $\mathcal{F}^{-1}\varphi$ or $\varphi^\vee$ and we often write $\hat{\varphi}$ instead of $\mathcal{F}\varphi$.

Vector-valued sequence spaces

As usual $L_p(\mathbb{R}^n)$ for $0 < p \le \infty$ stands for the Lebesgue spaces on $\mathbb{R}^n$ normed by (quasi-normed for $p < 1$)

$$\|\, f |\, L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} \qquad \text{for } 0 < p < \infty \text{ and}$$

$$\|\, f |\, L_\infty(\mathbb{R}^n)\| = \operatorname*{ess-sup}_{x \in \mathbb{R}^n} |f(x)| \ .$$

If w is a non-negative measurable function on $\mathbb{R}^n$, we denote the weighted Lebesgue spaces by $L_p(\mathbb{R}^n, w)$ and they are defined for $0 < p \leq \infty$ by

$$\| f| \, L_p(\mathbb{R}^n, w)\| = \| wf| \, L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x) dx \right)^{1/p} ,$$

with the usual modification if $p = \infty$. For a complex valued sequence $a = \{a_j\}_{j=0}^\infty$ the sequence spaces l_q for $0 < q \leq \infty$ are normed by (quasi-normed for $q < 1$)

$$\| a| \, l_q\| = \left(\sum_{j=0}^\infty |a_j|^q \right)^{1/q} \quad \text{for } 0 < q < \infty \text{ and}$$
$$\| a| \, l_\infty\| = \sup_{j \in \mathbb{N}_0} |a_j| \, .$$

Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of complex valued measurable functions, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then we put

$$\| f_k(x)| \, l_q(L_p)\| = \| \{f_k\}_{k\in\mathbb{N}_0}| \, l_q(L_p)\| = \left(\sum_{k=0}^\infty \left(\int_{\mathbb{R}^n} |f_k(x)|^p dx \right)^{q/p} \right)^{1/q} = \left(\sum_{k=0}^\infty \| f_k| \, L_p\|^q \right)^{1/q} ,$$

also with the above modifications for $p = \infty$ or $q = \infty$.
The constant c adds up all unimportant constants. So the value of the constant c may change from one occurrence to another. By $a_k \sim b_k$ we mean that there are two constants $c_1, c_2 > 0$ such that $c_1 a_k \leq b_k \leq c_2 a_k$ for all admissible k.

2.2 Definitions and basic properties

In this section we present the Fourier analytical definition of generalized 2-microlocal Besov spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ and we prove the basic properties in analogy to the classical Besov spaces. To this end we need smooth resolutions of unity and we introduce our admissible weight sequences $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0}$.

Definition 2.1 (Admissible weight sequence)**:** *Let $\alpha, \alpha_1, \alpha_2 \geq 0$. We say that a sequence of non-negative measurable functions $\boldsymbol{w} = \{w_j\}_{j=0}^\infty$ belongs to the class $\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ if and only if*

(i) There exists a constant $\mathrm{C} > 0$ *such that*

$$0 < w_j(x) \leq \mathrm{C} w_j(y) \left(1 + 2^j|x - y|\right)^\alpha \quad \text{for all } j \in \mathbb{N}_0 \text{ and all } x, y \in \mathbb{R}^n. \qquad (2.2)$$

(ii) For all $j \in \mathbb{N}_0$ we have

$$2^{-\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x) \quad \text{for all } x \in \mathbb{R}^n. \qquad (2.3)$$

Such a system $\{w_j\}_{j=0}^{\infty} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *is called* admissible weight sequence.

Remark 2.2: A non-negative measurable function ϱ is called an *admissible weight function* if there exist constants $\alpha_\varrho \geq 0$ and $\mathrm{C}_\varrho > 0$, such that

$$0 < \varrho(x) \leq \mathrm{C}_\varrho \varrho(y)(1+|x-y|)^{\alpha_\varrho} \quad \text{holds for every } x, y \in \mathbb{R}^n. \tag{2.4}$$

If $\boldsymbol{w} = \{w_j\}_{j=0}^{\infty}$ is an admissible weight sequence, each w_j is an admissible weight function, but in general the constant C_{w_j} depends on $j \in \mathbb{N}_0$.

Remark 2.3: If we use $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ without any restrictions, then $\alpha, \alpha_1, \alpha_2 \geq 0$ are arbitrary but fixed numbers.

Remark 2.4: If $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $\widetilde{\boldsymbol{w}} \in \mathcal{W}^{\beta}_{\beta_1,\beta_2}$ and $\lambda > 0$, it is easy to check:

(a) The sequence $\boldsymbol{w}^{-1} = \{w_j^{-1}\}_{j=0}^{\infty}$ belongs to the class $\mathcal{W}^{\alpha}_{\alpha_2,\alpha_1}$.

(b) The sequence $\lambda \boldsymbol{w}$ belongs to the class $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$.

(c) The sequence $\boldsymbol{w}^{\boldsymbol{\lambda}} = \{w_j^{\lambda}\}_{j=0}^{\infty}$ belongs to the class $\mathcal{W}^{\lambda\alpha}_{\lambda\alpha_1,\lambda\alpha_2}$.

(d) The sequence $\boldsymbol{w} + \widetilde{\boldsymbol{w}}$ belongs to the class $\mathcal{W}^{\max(\alpha,\beta)}_{\max(\alpha_1,\beta_1),\max(\alpha_2,\beta_2)}$.

(e) The sequence $\boldsymbol{w} \cdot \widetilde{\boldsymbol{w}}$ belongs to the class $\mathcal{W}^{\alpha+\beta}_{\alpha_1+\beta_1,\alpha_2+\beta_2}$.

Example 2.5: Let $U \neq \emptyset$ be a subset of $\mathbb{R}^n$. We denote by $\operatorname{dist}(x,U) = \inf_{z\in U}|x-z|$ the distance of $x \in \mathbb{R}^n$ from U. A typical admissible weight sequence is for fixed $U \subset \mathbb{R}^n$ and $s' \in \mathbb{R}$ given by

$$w_j(x) := \big(1 + 2^j \operatorname{dist}(x,U)\big)^{s'} \quad \text{for } j \in \mathbb{N}_0. \tag{2.5}$$

We have for $s' \geq 0$

$$w_j(x) \leq w_{j+1}(x) \leq 2^{s'} w_j(x) \quad \text{and for } s' < 0 \quad 2^{s'} w_j(x) \leq w_{j+1}(x) \leq w_j(x)\ .$$

Hence, for all $j \in \mathbb{N}_0$ and all fixed $s' \in \mathbb{R}$

$$2^{-\max(0,-s')} w_j(x) \leq w_{j+1}(x) \leq 2^{\max(0,s')} w_j(x) \qquad \text{for every } x \in \mathbb{R}^n. \tag{2.6}$$

From the inequality $\operatorname{dist}(x,U) \leq |x-y| + \operatorname{dist}(y,U)$ we derive for $s' \geq 0$

$$\begin{aligned} w_j(x) &= \big(1 + 2^j \operatorname{dist}(x,U)\big)^{s'} \\ &\leq \big(1 + 2^j|x-y| + 2^j \operatorname{dist}(y,U)\big)^{s'}\ . \end{aligned}$$

Since $a + b \leq 2ab$ for $a, b \geq 1$, we get

$$\begin{aligned} w_j(x) &\leq \left[2\left(1 + 2^j \operatorname{dist}(y,U)\right)(1 + 2^j|x-y|)\right]^{s'} \\ &= 2^{s'} w_j(y)(1 + 2^j|x-y|)^{s'}\ , \end{aligned}$$

for all $x, y \in \mathbb{R}^n$ and all $j \in \mathbb{N}_0$. If $s' < 0$ we can do the same calculation for the inverse weight sequence $\boldsymbol{w^{-1}}$ and according to Remark 2.4(a) we find

$$w_j(x) \le 2^{-s'} w_j(y)(1 + 2^j|x - y|)^{-s'} \ ,$$

for all $x, y \in \mathbb{R}^n$ and all $j \in \mathbb{N}_0$. Finally, we have for fixed $s' \in \mathbb{R}$ and all $j \in \mathbb{N}_0$

$$0 < w_j(x) \le 2^{|s'|} w_j(y)(1 + 2^j|x - y|)^{|s'|} \ , \tag{2.7}$$

for all $x, y \in \mathbb{R}^n$. Together with (2.6) we get $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ if $|s'| \le \alpha$, $\max(0, -s') \le \alpha_1$ and $\max(0, s') \le \alpha_2$.
A special case is $U = \{x_0\}$ for $x_0 \in \mathbb{R}^n$. Then $\operatorname{dist}(U, x) = |x - x_0|$ and we get the well known two-microlocal weights [JaMey96]:

$$w_j(x) = (1 + 2^j|x - x_0|)^{s'} \qquad \text{for } j \in \mathbb{N}_0. \tag{2.8}$$

If U is an open subset of $\mathbb{R}^n$, we get the weight sequence Moritoh and Yamada used in [MoYa04].

Example 2.6: Let $w : \mathbb{R}^n \to [0, \infty)$ be a measurable function with the properties: There are constants $\mathcal{C}_1$, $\mathcal{C}_2 \ge 1$ and $\beta \ge 1$ such that for all $x, y \in \mathbb{R}^n$

$$0 \le w(x) \le \mathcal{C}_1 w(y) + \mathcal{C}_2 |x - y|^{\beta} \ . \tag{2.9}$$

For fixed $s' \in \mathbb{R}$ and all $j \in \mathbb{N}_0$ we define

$$w_j(x) = \big(1 + 2^j w(x)\big)^{s'/\beta} \quad \text{for all } x \in \mathbb{R}^n. \tag{2.10}$$

By analogy to Example 2.5 above we get

$$0 < w_j(x) \le (2\,\mathcal{C}_1\mathcal{C}_2)^{|s'|}\, w_j(y)\, \big(1 + 2^j|x - y|\big)^{|s'|} \quad \text{and} \tag{2.11}$$

$$2^{-\max(0,-s')} w_j(x) \le w_{j+1}(x) \le 2^{\max(0,s')} w_j(x) \quad \text{holds for all } x, y \in \mathbb{R}^n \text{ and } j \in \mathbb{N}_0. \tag{2.12}$$

Hence, we have $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ for all $\alpha \ge |s'|$ and $\alpha_1 \ge \max(0, -s')$, $\alpha_2 \ge \max(0, s')$.
As a special case we choose $w : \mathbb{R}^n \to [0, \infty)$ subadditiv, that is

$$0 \le w(x + y) \le \tilde{c}_1 \left(w(x) + w(y)\right) \quad \text{and in addition we need}$$
$$w(x) \le \tilde{c}_2|x|^{\beta} \quad \text{for all } x \in \mathbb{R}^n \text{ and fixed } \tilde{c}_1,\ \tilde{c}_2,\ \beta \ge 1.$$

Thus we have (2.9) with $\mathcal{C}_1 = \tilde{c}_1$ und $\mathcal{C}_2 = \tilde{c}_1\tilde{c}_2$ and we can define the admissible weight sequence as in (2.10).

Next we define the resolution of unity.

Definition 2.7 (Resolution of unity)**:** *A system* $\boldsymbol{\varphi} = \{\varphi_j\}_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ *belongs to the class* $\Phi(\mathbb{R}^n)$ *if and only if* *(i)* $\operatorname{supp}\varphi_0 \subseteq \{x \in \mathbb{R}^n : |x| \leq 2\}$ *and* $\operatorname{supp}\varphi_j \subseteq \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| \leq 2^{j+1}\}$

(ii) *For each* $\beta \in \mathbb{N}_0^n$ *there exist constants* $c_\beta > 0$ *such that*

$$2^{j|\beta|} \sup_{x \in \mathbb{R}^n} |D^\beta \varphi_j(x)| \leq c_\beta \quad \textit{holds for all } j \in \mathbb{N}_0.$$

(iii) *For all* $x \in \mathbb{R}^n$ *we have*

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \ .$$

Remark 2.8: Such a resolution of unity can easily be constructed. Consider the following example. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(x) = 1$ for $|x| \leq 1$ and $\operatorname{supp}\varphi_0 \subseteq \{x \in \mathbb{R}^n : |x| \leq 2\}$. For $j \geq 1$ we define

$$\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x) \ .$$

Now it is obvious that $\boldsymbol{\varphi} = \{\varphi_j\}_{j \in \mathbb{N}_0} \in \Phi(\mathbb{R}^n)$.

Definition 2.9: *Let* $(\varphi_j)_{j \in \mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ *be a resolution of unity and let* $\boldsymbol{w} = (w_j)_{j \in \mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$. *Further, let* $0 < p \leq \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *Then we define*

$$B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) = \left\{ f \in S' : \left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\|_{\varphi} < \infty \right\} \quad , \textit{ where} \tag{2.13}$$

$$\left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\|_{\varphi} = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| w_j (\varphi_j \hat{f})^{\vee} \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \quad , \tag{2.14}$$

with the usual modifications if p *or* q *are equal to infinity.*

Remark 2.10: One recognizes immediately that for $w_j \equiv 1$ one obtains the usual Besov spaces, see [Tri83]. If one defines the admissible weight sequence as $w_j(x) = \varrho(x)$ for each $j \in \mathbb{N}_0$ and ϱ being an admissible weight, we obtain the usual weighted Besov spaces, see [EdTri96, Chapter 4].

Firstly, we have to prove that Definition 2.9 is independent of the chosen system $(\varphi_j)_{j \in \mathbb{N}_0} \in \Phi(\mathbb{R}^n)$. We need a Fourier multiplier theorem for weighted Lebesgue spaces of entire analytic functions as in [SchmTri87]. We define the Sobolev spaces $W_2^k(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$. A function $f \in L_2(\mathbb{R}^n)$ belongs to $W_2^k(\mathbb{R}^n)$ if

$$\left\| f | \, W_2^k(\mathbb{R}^n) \right\| := \left(\sum_{|\gamma| \leq k} \| D^\gamma f | \, L_2(\mathbb{R}^n) \|^2 \right)^{1/2} < \infty \ . \tag{2.15}$$

Theorem 2.11 ([SchmTri87])**:** *Let $\varrho : \mathbb{R}^n \to \mathbb{R}$ be an admissible weight which satisfies (2.4) for some $\alpha_\varrho \geq 0$. Furthermore, let $B_b = \{y \in \mathbb{R}^n : |y| \leq b\}$ for $b > 0$ and $0 < p \leq \infty$. Then for every $\kappa \in \mathbb{N}$ with*

$$\kappa > n\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right) + \alpha_\varrho \tag{2.16}$$

there exists a constant $c > 0$ (depending on b) such that

$$\left\| \varrho\mathcal{F}^{-1}M\mathcal{F}f \right| L_p(\mathbb{R}^n)\big\| \leq c\,\|M|\,W_2^\kappa(\mathbb{R}^n)\|\,\|\varrho f|\,L_p(\mathbb{R}^n)\| \tag{2.17}$$

holds for all $f \in L_p(\mathbb{R}^n, \varrho) \cap \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp}\mathcal{F}f \subseteq B_b$ and all $M \in \mathcal{S}(\mathbb{R}^n)$.

Remark 2.12: Additionally, we need a corollary of the Theorem 2.11. Let $0 < p \leq \infty$ and let

$$B_b = \{y \in \mathbb{R}^n : |y| \leq b\} \quad \text{for } b > 0.$$

We assume that the weight satisfies

$$0 < \varrho(x) \leq \mathrm{C}_\varrho \varrho(y)\,(1 + ab|x-y|)^{\alpha_\varrho} \quad \text{for fixed } a > 0 \text{ and all } x, y \in \mathbb{R}^n.$$

If $f \in L_p(\mathbb{R}^n, \varrho)$ with $\operatorname{supp}\mathcal{F}f \subset B_b$, then $\operatorname{supp}\mathcal{F}[f(b^{-1}x)] \subset B_1$ and by the properties of the Fourier transform

$$\left(\varrho\mathcal{F}^{-1}M\mathcal{F}f\right)(x) = \left\{\varrho(b^{-1}\cdot)\mathcal{F}^{-1}\left[M(b\cdot)\left(\mathcal{F}f(b^{-1}\cdot)\right)(\cdot)\right]\right\}(bx)\,. \tag{2.18}$$

Therefore, we obtain

$$\begin{aligned}\left\| \left(\varrho\mathcal{F}^{-1}M\mathcal{F}f\right)(x) \right| L_p(\mathbb{R}^n)\big\| &= \left\| \left\{\varrho(b^{-1}\cdot)\mathcal{F}^{-1}\left[M(b\cdot)\left(\mathcal{F}f(b^{-1}\cdot)\right)(\cdot)\right]\right\}(bx) \right| L_p(\mathbb{R}^n)\big\| \\ &= b^{-\frac{n}{p}} \left\| \left\{\varrho(b^{-1}\cdot)\mathcal{F}^{-1}\left[M(b\cdot)\left(\mathcal{F}f(b^{-1}\cdot)\right)(\cdot)\right]\right\}(x) \right| L_p(\mathbb{R}^n)\big\|\,.\end{aligned}$$

For the weight function $r(x) = \varrho(b^{-1}x)$ we have $\alpha_r = \alpha_\varrho$ and

$$0 < r(x) \leq \max(1,a)\mathrm{C}_\varrho\, r(y)\,(1 + |x-y|)^{\alpha_\varrho} = \mathrm{C}'_\varrho\, r(y)\,(1+|x-y|)^{\alpha_\varrho}\,.$$

We can apply Theorem 2.11 and obtain

$$\begin{aligned}\left\| \left(\varrho\mathcal{F}^{-1}M\mathcal{F}f\right)(x) \right| L_p(\mathbb{R}^n)\big\| &\leq c\cdot \mathrm{C}'_\varrho b^{-\frac{n}{p}}\,\|M(b\cdot)|\,W_2^\kappa(\mathbb{R}^n)\|\,\left\|\varrho(b^{-1}\cdot)f(b^{-1}\cdot)\right| L_p(\mathbb{R}^n)\big\| \\ &= c\mathrm{C}_\varrho\,\|M(b\cdot)|\,W_2^\kappa(\mathbb{R}^n)\|\,\|\varrho f|\,L_p(\mathbb{R}^n)\| \end{aligned} \tag{2.19}$$

for $\kappa > n\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right) + \alpha_\varrho$.

Now, we are ready to show that Definition 2.9 of the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ is independent of the chosen resolution of unity $\boldsymbol{\varphi} \in \Phi(\mathbb{R}^n)$.

Theorem 2.13 (Independence of the resolution of unity)**:** *Let* $\boldsymbol{\varphi} = (\varphi_j)_{j\in\mathbb{N}_0}$, $\boldsymbol{\phi} = (\phi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ *be two resolutions of unity and let* $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *be an admissible weight sequence. If* $0 < p \leq \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$, *then we have*

$$\left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\|_{\varphi} \sim \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\|_{\phi} \quad \textit{for all } f \in \mathcal{S}'(\mathbb{R}^n).$$

Proof: It is sufficient to show that there is a $c > 0$ such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ we have $\left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\|_{\phi} \leq c \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\|_{\varphi}$. Interchanging φ and ϕ we derive the result we aim at.
Putting $\varphi_{-1} = 0$ we see

$$\phi_j(x) = \phi_j(x) \sum_{k=-1}^{1} \varphi_{j+k}(x) \quad \text{for all } j \in \mathbb{N}_0.$$

By the properties of the Fourier transform

$$w_j\{\mathcal{F}^{-1}\phi_j(\mathcal{F}f)\} = \sum_{k=-1}^{1} w_j \left\{\mathcal{F}^{-1}\phi_j \left(\mathcal{F}\left[\mathcal{F}^{-1}\varphi_{j+k}(\mathcal{F}f)\right]\right)\right\} .$$

Now, we apply (2.19) with $b = 2^{j+2}$, $M = \phi_j$ und $f = \mathcal{F}^{-1}\varphi_{j+k}(\mathcal{F}f)$ for $k \in \{-1, 0, 1\}$. We get for every $j \in \mathbb{N}_0$

$$\begin{aligned} &\left\| w_j \left\{\mathcal{F}^{-1}\phi_j \left(\mathcal{F}\left[\mathcal{F}^{-1}\varphi_{j+k}(\mathcal{F}f)\right]\right)\right\} \right| L_p(\mathbb{R}^n)\big\| \\ &\qquad \leq c \left\| \phi_j(2^{j+2}\cdot) \right| W^{\kappa}_2(\mathbb{R}^n)\big\| \left\| w_j \left\{\mathcal{F}^{-1}\varphi_{j+k}(\mathcal{F}f)\right\} \right| L_p(\mathbb{R}^n)\big\| , \qquad (2.20) \end{aligned}$$

with $\kappa > n\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right) + \alpha$. By (2.2) and formula (2.19) the constant c does not depend on $j \in \mathbb{N}$. Since $\operatorname{supp} \phi_j(2^{j+2}\cdot) \subseteq B_1$ and using the properties of the resolution of unity, we have

$$\sup_{l\in\mathbb{N}_0} \left\| \phi_l(2^{l+2}\cdot) \right| W^{\kappa}_2(\mathbb{R}^n)\big\| \leq c \sup_{l\in\mathbb{N}_0} \sup_{|\beta|\leq\kappa} \sup_{x\in\mathbb{R}^n} 2^{l|\beta|} |(D^{\beta}\phi_l)(x)| < c_{\kappa} .$$

We conclude that

$$\begin{aligned} \left\| w_j\{\mathcal{F}^{-1}\phi_j(\mathcal{F}f)\} \right| L_p(\mathbb{R}^n)\big\| &\leq c \sum_{k=-1}^{1} \left\| w_j \left\{\mathcal{F}^{-1}\phi_j \left(\mathcal{F}\left[\mathcal{F}^{-1}\varphi_{j+k}(\mathcal{F}f)\right]\right)\right\} \right| L_p(\mathbb{R}^n)\big\| \\ &\leq c' \sum_{k=-1}^{1} \left\| w_j \left\{\mathcal{F}^{-1}\varphi_{j+k}(\mathcal{F}f)\right\} \right| L_p(\mathbb{R}^n)\big\| . \end{aligned}$$

Finally, multiplying by 2^{js}, using the property (2.3) of the admissible weight sequence and taking the l_q quasi-norm with respect to j, we see that

$$\left(\sum_{j=0}^{\infty} 2^{jsq} \left\| w_j\{\mathcal{F}^{-1}\phi_j(\mathcal{F}f)\} \right| L_p(\mathbb{R}^n)\big\|^q\right)^{1/q} \leq c'(2^{s+\alpha_2} + 1 + 2^{-s+\alpha_1}) \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\|_{\varphi} .$$

This completes the proof. □

Remark 2.14: As in Theorem 2.3.3 in [Tri83] we can prove that $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ is a quasi-Banach space for all $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ and even a Banach space in the case $p, q \geq 1$.

2.3 Lift property and equivalent norms

We introduce the lift operator as in the classical case of Besov spaces, [Tri83]. If $\sigma \in \mathbb{R}$, the operator I_σ is defined by

$$I_\sigma : f \mapsto \left(\langle \xi \rangle^\sigma \hat{f} \right)^\vee \tag{2.21}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Theorem 2.15: *Let $s, \sigma \in \mathbb{R}$ and $\boldsymbol{w} = (w_j)_{j \in \mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$. Moreover, let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then I_σ maps $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ isomorphically onto $B^{s-\sigma,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ and $\left\| I_\sigma f \right| B^{s-\sigma,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \|$ is an equivalent quasi-norm on $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.*

Proof: To prove the theorem we show that

$$\left\| I_\sigma f \right| B^{s-\sigma,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \| = \left(\sum_{j=0}^{\infty} 2^{j(s-\sigma)q} \left\| w_j (\varphi_j \langle \xi \rangle^\sigma \hat{f})^\vee \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \sim \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \| \ .$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\phi(x) = 1 \qquad \text{if} \qquad \frac{1}{2} \leq |x| \leq 2 \qquad \text{and}$$
$$\operatorname{supp} \phi \subseteq \left\{ \xi \in \mathbb{R}^n : \frac{1}{4} \leq |\xi| \leq 4 \right\} \ .$$

Then we have for $j \geq 1$

$$\left(\varphi_j \langle \xi \rangle^\sigma \hat{f} \right)^\vee = \left(\langle \xi \rangle^\sigma \phi(2^{-j}\xi) \varphi_j \hat{f} \right)^\vee \quad ,$$

and we define

$$M_j(\xi) := 2^{-\sigma j} \langle \xi \rangle^\sigma \phi(2^{-j}\xi) \ , \quad \text{whereas} \ , \quad \operatorname{supp} \varphi_j \hat{f} \subseteq \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{j+1} \} \ .$$

Now, we can apply (2.19) with $b = 2^{j+2}$ and $\kappa \in \mathbb{N}$ with $\kappa > n \left(\frac{1}{\min(1,p)} - \frac{1}{2} \right) + \alpha$ and we obtain

$$\left\| w_j \left(2^{-\sigma j} \langle \xi \rangle^\sigma \phi(2^{-j}\xi) \varphi_j \hat{f} \right)^\vee \right| L_p(\mathbb{R}^n) \Big\| \leq c \sup_{l \in \mathbb{N}_0} \left\| M_l(2^{l+2} \cdot) \right| W_2^\kappa(\mathbb{R}^n) \| \left\| w_j (\varphi_j \hat{f})^\vee \right| L_p(\mathbb{R}^n) \Big\| \tag{2.22}$$

for all $j \in \mathbb{N}_0$ and $0 < p \leq \infty$. If $\beta \in \mathbb{N}_0^n$ is a multi-index with $|\beta| \leq \kappa$, we have

$$\begin{aligned} \left| D^\beta \left(M_l(2^{l+2} \cdot) \right)(x) \right| &= \left| D^\beta \left(2^{-\sigma l} \langle 2^{l+2} \cdot \rangle^\sigma \phi(4 \cdot) \right)(x) \right| \\ &\leq 2^{2\sigma} \sum_{\gamma \leq \beta} c_{\beta,\gamma} \left| D^\gamma \left(2^{-2(l+2)} + |x|^2 \right)^{\sigma/2} \right| \left| \left(D^{\beta-\gamma} \phi \right)(4x) \right| 4^{|\beta-\gamma|} \\ &\leq 2^{2(\sigma+\kappa)} \sup_{|\delta| \leq \kappa} \sup_{y \in \mathbb{R}^n} \left| \left(D^\delta \phi \right)(y) \right| \sum_{\gamma \leq \beta} c_{\beta,\gamma} \left| D^\gamma \left(2^{-2(l+2)} + |x|^2 \right)^{\sigma/2} \right| \ . \end{aligned} \tag{2.23}$$

Since $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{supp} \phi \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{4} \le |\xi| \le 4\}$ we obtain

$$\sup_{|\delta| \le \kappa} \sup_{y \in \mathbb{R}^n} \left| \left(D^\delta \phi \right)(y) \right| \le c \; . \tag{2.24}$$

Furthermore, we have

$$\left| D^\gamma \left(2^{-2(l+2)} + |x|^2 \right)^{\sigma/2} \right| \le c_{\sigma,\gamma} \left(2^{-2(l+2)} + |x|^2 \right)^{\sigma/2 - |\gamma|/2} \qquad \text{and} \tag{2.25}$$

$$\operatorname{supp} M_l(2^{l+2} \cdot) \subseteq \left\{ x \in \mathbb{R}^n : \frac{1}{16} \le |x| \le 1 \right\} \; . \tag{2.26}$$

Finally, we get from (2.23)-(2.27) for $0 < \sigma < \kappa$

$$\begin{aligned} \left| D^\beta \left(M_l(2^{l+2} \cdot) \right)(x) \right| &\le c \sum_{\gamma \le \beta} \binom{\beta}{\gamma} c_{\sigma,\gamma} \left(2^{-2(l+2)} + |x|^2 \right)^{\sigma/2 - |\gamma|/2} \\ &\le c \sum_{\gamma \le \sigma} \binom{\beta}{\gamma} c_{\sigma,\gamma} \left(2^{-2(l+2)} + 1 \right)^{\sigma/2 - |\gamma|/2} + \sum_{\sigma < \gamma \le \kappa} \binom{\beta}{\gamma} c_{\sigma,\gamma} \left(2^{-2(l+2)} + \frac{1}{16} \right)^{\sigma/2 - |\gamma|/2} \\ &\le c' \; . \end{aligned}$$

This implies for all $l \in \mathbb{N}_0$

$$\left\| M_l(2^{l+2} \cdot) \right| W_2^\kappa(\mathbb{R}^n) \right\| = \left(\sum_{|\beta| \le \kappa} \left\| D^\beta \left(M_l(2^{l+2} \cdot) \right) \right| L_2(\mathbb{R}^n) \right\|^2 \right)^{1/2} < \infty \; .$$

For $j = 0$ we have to define ϕ_0 as

$$\begin{aligned} \phi_0(x) = 1 \qquad \text{if} \qquad |x| \le 2 \qquad \text{and} \\ \operatorname{supp} \phi_0 \subseteq \{ \xi \in \mathbb{R}^n : |\xi| \le 4 \} \; . \end{aligned}$$

By a similar calculation as above ($j = 0$) we can show for all $j \in \mathbb{N}_0$

$$\left\| w_j \left(2^{-\sigma j} \langle \xi \rangle^\sigma \varphi_j \hat{f} \right)^\vee \right| L_p(\mathbb{R}^n) \right\| \le c \left\| w_j (\varphi_j \hat{f})^\vee \right| L_p(\mathbb{R}^n) \right\| \; , \tag{2.27}$$

where c is independent of $j \in \mathbb{N}_0$.
Now, taking the l_q quasi-norm in (2.22) leads to

$$\left\| I_\sigma f \right| B_{pq}^{s-\sigma,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \le c \left\| f \right| B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \; .$$

This proves the theorem. □

The next theorem is a characterization of $B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})$ by derivatives. We follow closely Theorem 2.3.8 in [Tri83] and use the weighted Fourier multiplier theorem 2.11.

Theorem 2.16: *Let $s \in \mathbb{R}$, $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < p,q \leq \infty$ and let $m \in \mathbb{N}_0$. Then*

$$\sum_{|\beta|\leq m} \left\| D^\beta f \,\middle|\, B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\| \quad \textit{and} \quad \left\| f \,\middle|\, B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\| + \sum_{i=1}^{n} \left\| \frac{\partial^m f}{\partial x_i^m} \,\middle|\, B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\|$$

are equivalent quasi-norms on $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$.

Proof: First Step: We define $\phi \in \mathcal{S}(\mathbb{R}^n)$ as

$$\phi(x) = 1 \quad \text{for} \quad 1/2 \leq |x| \leq 2 \quad \text{and} \quad \operatorname{supp}\phi \subseteq \{x \in \mathbb{R}^n : 1/4 \leq |x| \leq 4\} \ . \tag{2.28}$$

Then for all $j \in \mathbb{N}$

$$\begin{aligned}\mathcal{F}^{-1}\varphi_j \mathcal{F} D^\beta f &= c\mathcal{F}^{-1}\phi(2^{-j}\cdot)x^\beta \varphi_j \mathcal{F} f \\ &= c\mathcal{F}^{-1}\phi(2^{-j}\cdot)\frac{x^\beta}{(1+|x|^2)^{m/2}}\mathcal{F}\mathcal{F}^{-1}\varphi_j(1+|x|^2)^{m/2}\mathcal{F} f \ .\end{aligned}$$

Now, using (2.19) with $b = 2^{j+2}$ and $M = \phi(2^{-j}\cdot)\frac{x^\beta}{(1+|x|^2)^{m/2}}$ we get for $\kappa > n\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right) + \alpha$

$$\begin{aligned}\left\| w_j \mathcal{F}^{-1}\varphi_j \mathcal{F} D^\beta f \,\middle|\, L_p(\mathbb{R}^n)\right\| &= c\left\| w_j \mathcal{F}^{-1}\phi(2^{-j}\cdot)\frac{x^\beta}{(1+|x|^2)^{m/2}}\mathcal{F}\mathcal{F}^{-1}\varphi_j(1+|x|^2)^{m/2}\mathcal{F} f \,\middle|\, L_p(\mathbb{R}^n)\right\| \\ \leq \left\| \phi(4\cdot)\frac{(2^{j+2}x)^\beta}{(1+|2^{j+2}x|^2)^{m/2}} \,\middle|\, W_2^\kappa(\mathbb{R}^n)\right\| &\left\| w_j \mathcal{F}^{-1}\varphi_j(1+|x|^2)^{m/2}\mathcal{F} f \,\middle|\, L_p(\mathbb{R}^n)\right\| \ ,\end{aligned} \tag{2.29}$$

for all $0 < p \leq \infty$ and $j \in \mathbb{N}$. Since $\frac{(2^{j+2}x)^\beta}{(1+|2^{j+2}x|^2)^{m/2}} < c$ for $|\beta| \leq m$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, we get

$$\left\| \phi(4\cdot)\frac{(2^{j+2}x)^\beta}{(1+|2^{j+2}x|^2)^{m/2}} \,\middle|\, W_2^\kappa(\mathbb{R}^n)\right\| \leq c_{\kappa,m} \quad \text{independently of } j \in \mathbb{N}.$$

For $j = 0$ we obtain (2.29) by similar arguments and $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\phi_0(x) = 1$ for $|x| \leq 2$ and $\operatorname{supp}\phi_0 \subseteq \{x \in \mathbb{R}^n : |x| \leq 4\}$. Hence, we have for all $j \in \mathbb{N}_0$

$$\left\| w_j \mathcal{F}^{-1}\varphi_j \mathcal{F} D^\beta f \,\middle|\, L_p(\mathbb{R}^n)\right\| \leq c_{\kappa,m}\left\| w_j \mathcal{F}^{-1}\varphi_j(1+|x|^2)^{m/2}\mathcal{F} f \,\middle|\, L_p(\mathbb{R}^n)\right\| \ ,$$

where the constant $c_{\kappa,m}$ is independent of $j \in \mathbb{N}_0$ and $|\beta| \leq m$. Finally, multiplying by $2^{j(s-m)}$ and applying the l_q quasi-norm in respect to j, we get for all $|\beta| \leq m$

$$\left\| D^\beta f \,\middle|\, B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\| \leq c\left\| I_m f \,\middle|\, B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\| \leq c'\left\| f \,\middle|\, B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\| \ . \tag{2.30}$$

Second Step: Now, we assume that $f \in B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ and $\frac{\partial^m f}{\partial x_i^m} \in B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ for $i = 1,\ldots,n$. We want to show that f belongs to $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$. Theorem 2.15 shows

$$\begin{aligned}\left\| f \,\middle|\, B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\| &\leq c\left\| I_m f \,\middle|\, B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\| \\ &= c\left(\sum_{j=0}^{\infty} 2^{j(s-m)q}\left\| w_j \mathcal{F}^{-1}(1+|x|^2)^{m/2}\varphi_j \mathcal{F} f \,\middle|\, L_p(\mathbb{R}^n)\right\|^q\right) \ .\end{aligned} \tag{2.31}$$

From [Tri83] we adopt the construction of functions $\varrho_1, \dots, \varrho_n \in C^\infty(\mathbb{R}^n)$ from the third step of the proof of Theorem 2.3.8. If m is even, then $\varrho_i(x) = 1$ has the desired properties but for odd m the situation is a bit more complicated. These functions fulfill

$$1 + \sum_{i=1}^n \varrho_i(x) x_i^m \geq c(1+|x|^2)^{m/2} \qquad \text{for all } x \in \mathbb{R}^n.$$

Thus we have

$$M(x) := (1+|x|^2)^{m/2} \left[1 + \sum_{i=1}^n \varrho_i(x) x_i^m\right]^{-1} \leq c \quad \text{for all } x \in \mathbb{R}^n.$$

With the function $\phi \in \mathcal{S}(\mathbb{R}^n)$ as in (2.28) and (2.19) with $b = 2^{j+2}$ and $\kappa > 0$ large enough we get for all $j \in \mathbb{N}$

$$\begin{aligned} &\left\| w_j \mathcal{F}^{-1} (1+|x|^2)^{m/2} \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| \\ &\quad = \left\| w_j \mathcal{F}^{-1} M(x) \phi(2^{-j}\cdot) \mathcal{F}\mathcal{F}^{-1} \varphi_j \left[1 + \sum_{i=1}^n \varrho_i(x) x_i^m\right] \mathcal{F} f \right| L_p(\mathbb{R}^n) \right\| \\ &\quad \leq c \left\| M(2^{j+2}\cdot)\phi(4\cdot) \right| W_2^\kappa(\mathbb{R}^n) \big\| \left\| w_j \mathcal{F}^{-1} \varphi_j \left[1 + \sum_{i=1}^n \varrho_i(x) x_i^m\right] \mathcal{F} f \right| L_p(\mathbb{R}^n) \right\| . \end{aligned}$$

From the properties of M and $\phi \in \mathcal{S}(\mathbb{R}^n)$ we get that the Sobolev space norm is bounded, independent of $j \in \mathbb{N}$. For $j = 0$ we can do the same calculation with ϕ_0 instead of $\Phi(2^{-j}\cdot)$. By an analogous procedure as above we can use (2.19) again and obtain

$$\begin{aligned} &\left\| w_j \mathcal{F}^{-1} (1+|x|^2)^{m/2} \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| \leq c \left\| w_j \mathcal{F}^{-1} \varphi_j \left[1 + \sum_{i=1}^n \varrho_i(x) x_i^m\right] \mathcal{F} f \right| L_p(\mathbb{R}^n) \right\| \\ &\quad \leq c' \left\| w_j \mathcal{F}^{-1} \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| + c' \sum_{i=1}^n \left\| w_j \mathcal{F}^{-1} \varrho_i(x) \phi(2^{-j}\cdot) \mathcal{F}\mathcal{F}^{-1} x_i^m \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| \\ &\quad \leq c' \left\| w_j \mathcal{F}^{-1} \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| + c' \sum_{i=1}^n \left\| \varrho_i(2^{j+2}x) \phi(4\cdot) \right| W_2^\kappa(\mathbb{R}^n) \big\| \left\| w_j \mathcal{F}^{-1} x_i^m \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| , \end{aligned}$$

for $\kappa > n \left(\frac{1}{\min(1,p)} - \frac{1}{2} \right) + \alpha$. Since $\varrho_i \in C^\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ we get that the Sobolev space norm is bounded by a constant independently of $j \in \mathbb{N}$. If $j = 0$, then we use the usual replacement by ϕ_0. Finally, we have

$$\begin{aligned} &\left\| w_j \mathcal{F}^{-1} (1+|x|^2)^{m/2} \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| \\ &\qquad \leq c \left\| w_j \mathcal{F}^{-1} \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| + c \left\| w_j \mathcal{F}^{-1} x_i^m \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| \\ &\qquad = c \left\| w_j \mathcal{F}^{-1} \varphi_j \mathcal{F} f \right| L_p(\mathbb{R}^n) \big\| + c \left\| w_j \mathcal{F}^{-1} \varphi_j \mathcal{F} \frac{\partial^m f}{\partial x_i^m} \right| L_p(\mathbb{R}^n) \right\| \end{aligned}$$

for all $j \in \mathbb{N}_0$. Using (2.31) we get

$$\left\| f \right| B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \le c \left\| f \right| B_{pq}^{s-m,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| + c \sum_{i=1}^{n} \left\| \frac{\partial^m f}{\partial x_i^m} \right| B_{pq}^{s-m,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| .$$

Finally, this and (2.30) prove the theorem. □

Now, we present a characterization of the 2-microlocal spaces with the special weight sequence $w_j(x) = (1 + 2^j \operatorname{dist}(x, U))^{s'}$ for $U \subseteq \mathbb{R}^n$.

Definition 2.17: *Let $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be a resolution of unity. Let $U \subseteq \mathbb{R}^n$ and $s' \in \mathbb{R}$ be fixed. Further, let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then we define*

$$B_{pq}^{s,s'}(\mathbb{R}^n, U) = \left\{ f \in S' : \left\| f \right| B_{pq}^{s,s'}(\mathbb{R}^n, U) \right\| < \infty \right\} \quad , \textit{ where}$$

$$\left\| f \right| B_{pq}^{s,s'}(\mathbb{R}^n, U) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (1 + 2^j \operatorname{dist}(x, U))^{s'} (\varphi_j \hat{f})^{\vee} \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} ,$$

with the usual modifications if p or q are equal to infinity.

Remark 2.18: In slight abuse of notation we write $B_{pq}^{s,s'}(\mathbb{R}^n, x_0)$ if $U = \{x_0\} \subset \mathbb{R}^n$. If $U = \{x_0\} \subset \mathbb{R}^n$ then $B_{\infty\infty}^{s,s'}(\mathbb{R}^n, x_0) = C_{x_0}^{s,s'}$, see [JaMey96, Definition 1.1]. For $p = q = 2$ we get $B_{22}^{s,s'}(\mathbb{R}^n, x_0) = H_{x_0}^{s,s'}$. Both types are the 2-microlocal spaces introduced by Bony [Bo84] and Jaffard [Ja91].

Corollary 2.19: *Let $s, s' \in \mathbb{R}$ and let $U \subseteq \mathbb{R}^n$. Further, let $0 < p, q \le \infty$ and $m \in \mathbb{N}_0$, then the following statements are equivalent*

(i) $f \in B_{pq}^{s,s'}(\mathbb{R}^n, U)$

(ii) $D^\beta f \in B_{pq}^{s-m,s'}(\mathbb{R}^n, U)$ *for all* $0 \le |\beta| \le m$

(iii) $f \in B_{pq}^{s-m,s'}(\mathbb{R}^n, U)$ *and* $\frac{\partial^m f}{\partial x_i^m} \in B_{pq}^{s-m,s'}(\mathbb{R}^n, U)$ *for each* $i = 1, \dots, n$.

Remark 2.20: This corollary coincides essentially with Corollary 3.1 in [Mey97] for the special case $p = q = \infty$ and $U = \{x_0\} \subset \mathbb{R}^n$.

3 Local Means

3.1 Preliminaries

In this part we present the main technical tool. We characterize the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ by so called *local means*. We follow closely the method presented by Rychkov [Ry99] and by Vybiral [Vyb06].
Recall the specific system $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ which we fix now for the rest of our work:
Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with

$$\varphi_0(x) = \begin{cases} 1 & , \text{ if } |x| \leq 1 \\ 0 & , \text{ if } |x| \geq 2 \end{cases} .$$

We put $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and

$$\varphi_j(x) = \varphi(2^{-j}x) \qquad \text{for all } x \in \mathbb{R}^n \text{ and all } j \in \mathbb{N}.$$

3.1.1 The Peetre maximal operator

The Peetre maximal operator was introduced by Jaak Peetre in [Pe75]. The operator assigns to each system $\{\psi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$, to each distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ and to each number $a > 0$ the following quantities

$$\sup_{y\in\mathbb{R}^n} \frac{|(\psi_k \hat{f})^\vee(y)|}{1 + |2^k(y-x)|^a} , \quad x \in \mathbb{R}^n , k \in \mathbb{N}_0 . \tag{3.1}$$

Since $\psi_k \in \mathcal{S}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$ the operator is well-defined because $(\psi_k \hat{f})^\vee = c(\psi_k^\vee * f)$ is well-defined for every distribution $f \in \mathcal{S}'(\mathbb{R}^n)$.
Given a system $\{\psi_k\}_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$, we set $\Psi_k = \hat{\psi}_k \in \mathcal{S}(\mathbb{R}^n)$ and reformulate the Peetre maximal operator (3.1) for every $f \in \mathcal{S}'(\mathbb{R}^n)$ and $a > 0$ as

$$(\Psi_k^* f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|(\Psi_k * f)(y)|}{1 + |2^k(y-x)|^a} , \quad x \in \mathbb{R}^n \text{ and } k \in \mathbb{N}_0 . \tag{3.2}$$

3.1.2 Helpful lemmas

Before proving the local means characterization we present some technical lemmas without proof, which appeared in the papers of Rychkov [Ry99] and Vybiral [Vyb06]. The first lemma describes the use of the so called moment conditions.
Lemma 3.1: *Let $g, h \in \mathcal{S}(\mathbb{R}^n)$ and let $M \geq -1$ be an integer. Suppose that*

$$(D^\beta \hat{g})(0) = 0 \quad \text{for} \quad 0 \leq |\beta| \leq M . \tag{3.3}$$

Then for each $N \in \mathbb{N}_0$ there is a constant C_N such that

$$\sup_{z \in \mathbb{R}^n} |(g_t * h)(z)|(1 + |z|^N) \leq C_N t^{M+1} \ , \quad \textit{for} \quad 0 < t < 1 \ , \tag{3.4}$$

where $g_t(x) = t^{-n} g(x/t)$.

Remark 3.2: If $M = -1$, the condition (3.3) is empty.

The next lemma is a discrete convolution inequality which we will need later on.

Lemma 3.3: *Let $0 < p, q \leq \infty$ and $\delta > 0$. Let $\{g_k\}_{k \in \mathbb{N}_0}$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$ and let*

$$G_\nu(x) = \sum_{k=0}^{\infty} 2^{-|\nu - k|\delta} g_k(x) \ , \qquad x \in \mathbb{R}^n \ , \nu \in \mathbb{N}_0 \ . \tag{3.5}$$

Then there is some constant $c = c(p, q, \delta)$ such that

$$\| G_k | \, l_q(L_p) \| \leq c \, \| g_k | \, l_q(L_p) \| \ . \tag{3.6}$$

Lemma 3.4: *Let $0 < r \leq 1$ and let $\{\gamma_\nu\}_{\nu \in \mathbb{N}_0}$, $\{\beta_\nu\}_{\nu \in \mathbb{N}_0}$ be two sequences taking values in $(0, \infty)$. Assume that for some $N^0 \in \mathbb{N}_0$,*

$$\gamma_\nu = O(2^{\nu N^0}) \ , \quad \textit{for } \nu \to \infty \ .$$

Furthermore, we assume that for any $N \in \mathbb{N}$

$$\gamma_\nu \leq C_N \sum_{k=0}^{\infty} 2^{-kN} \beta_{k+\nu} \gamma_{k+\nu}^{1-r} \ , \quad \nu \in \mathbb{N}_0 \ , \quad C_N < \infty$$

holds, then for any $N \in \mathbb{N}$

$$\gamma_\nu^r \leq C_N \sum_{k=0}^{\infty} 2^{-kN} \beta_{k+\nu} \ , \quad \nu \in \mathbb{N}_0 \tag{3.7}$$

holds with the same constants C_N.

The proofs of the lemmas can be found in [Ry99] and [Vyb06].

3.1.3 Comparison of different Peetre maximal operators

In this subsection we present an inequality between different Peetre maximal operators. We start with two given functions $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\psi_j(x) = \psi_1(2^{-j+1} x) \ , \quad \text{for } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}. \tag{3.8}$$

Furthermore, for all $j \in \mathbb{N}_0$ we write $\Psi_j = \hat{\psi}_j$ and in an analogous manner we define Φ_j from two starting functions $\phi_0, \phi_1 \in \mathcal{S}(\mathbb{R}^n)$.

Using this notation we are ready to formulate the theorem.

Theorem 3.5: *Let* $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < p, q \le \infty$ *and* $s, a \in \mathbb{R}$ *with* $a > 0$. *Moreover, let* $R+1 \in \mathbb{N}_0$ *with* $R+1 > s + \alpha_2$,

$$D^{\beta}\psi_1(0) = 0 \ , \quad 0 \le |\beta| \le R \tag{3.9}$$

and for some $\varepsilon > 0$

$$|\phi_0(x)| > 0 \quad on \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \tag{3.10}$$
$$|\phi_1(x)| > 0 \quad on \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \tag{3.11}$$

then

$$\big\| 2^{ks}(\Psi_k^* f)_a w_k \big| \, l_q(L_p) \big\| \le c \, \big\| 2^{ks}(\Phi_k^* f)_a w_k \big| \, l_q(L_p) \big\| \tag{3.12}$$

holds for every $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof: We define the functions $\{\lambda_j\}_{j\in\mathbb{N}_0}$ by

$$\lambda_j(x) = \frac{\varphi_j\left(\frac{2x}{\varepsilon}\right)}{\phi_j(x)} \ .$$

It follows from the *Tauberian conditions* (3.10) and (3.11) that they satisfy

$$\sum_{j=0}^{\infty} \lambda_j(x)\phi_j(x) = 1 \ , \quad x \in \mathbb{R}^n \tag{3.13}$$
$$\lambda_j(x) = \lambda_1(2^{-j+1}x) \ , \quad x \in \mathbb{R}^n \ , \quad j \in \mathbb{N} \tag{3.14}$$
$$\operatorname{supp}\lambda_0 \subset \{x \in \mathbb{R}^n : |x| \le \varepsilon\} \quad \text{and} \quad \operatorname{supp}\lambda_1 \subset \{x \in \mathbb{R}^n : \varepsilon/2 \le |x| \le 2\varepsilon\} \ . \tag{3.15}$$

Furthermore, we denote $\Lambda_k = \hat{\lambda_k}$ for $k \in \mathbb{N}_0$ and obtain together with (3.13) the following identities (convergence in $\mathcal{S}'(\mathbb{R}^n)$)

$$f = \sum_{k=0}^{\infty} \Lambda_k * \Phi_k * f \ , \qquad \Psi_\nu * f = \sum_{k=0}^{\infty} \Psi_\nu * \Lambda_k * \Phi_k * f \ . \tag{3.16}$$

We have

$$\begin{aligned} |(\Psi_\nu * \Lambda_k * \Phi_k * f)(y)| &\le \int\limits_{\mathbb{R}^n} |(\Psi_\nu * \Lambda_k)(z)||(\Phi_k * f)(y-z)|dz \\ &\le (\Phi_k^* f)_a(y) \int\limits_{\mathbb{R}^n} |(\Psi_\nu * \Lambda_k)(z)|(1 + |2^k z|^a)dz \\ &=: (\Phi_k^* f)_a(y) I_{\nu,k} \ , \end{aligned} \tag{3.17}$$

where

$$I_{\nu,k} := \int\limits_{\mathbb{R}^n} |(\Psi_\nu * \Lambda_k)(z)|(1 + |2^k z|^a)dz \ .$$

According to Lemma 3.1 we get

$$I_{\nu,k} \le c \begin{cases} 2^{(k-\nu)(R+1)} & ,k \le \nu \\ 2^{(\nu-k)(a+|s|+1+\alpha_1)} & ,\nu \le k \ . \end{cases} \tag{3.18}$$

Namely, we have for $1 \le k < \nu$ with the change of variables $2^k z \mapsto z$

$$\begin{aligned} I_{\nu,k} &= 2^{-n} \int\limits_{\mathbb{R}^n} |(\Psi_{\nu-k} * \Lambda_1(\cdot/2))(z)|(1+|z|^a)dz \\ &\le c \sup_{z\in\mathbb{R}^n} |(\Psi_{\nu-k} * \Lambda_1(\cdot/2))(z)|(1+|z|)^{a+n+1} \le c2^{(k-\nu)(R+1)} \ . \end{aligned}$$

Similarly, we get for $1 \le \nu < k$ with the substitution $2^\nu z \mapsto z$

$$\begin{aligned} I_{\nu,k} &= 2^{-n} \int\limits_{\mathbb{R}^n} |(\Psi_1(\cdot/2) * \Lambda_{k-\nu})(z)|(1+|2^{k-\nu}z|^a)dz \\ &\le c2^{(\nu-k)(M+1-a)} \ . \end{aligned}$$

M can be taken arbitrarily large because Λ has infinite vanishing moments. Taking $M > 2a + |s| + \alpha_1$ we derive (3.18) for the cases $k, \nu \ge 1$ with $k \ne \nu$. The missing cases can be treated separately in an analogous manner. The needed moment conditions are always satisfied by (3.9) and (3.15). The case $k = \nu = 0$ is covered by the constant c in (3.18).
Furthermore, we have

$$\begin{aligned} (\Phi_k^* f)_a(y) &\le (\Phi_k^* f)_a(x)(1 + |2^k(x-y)|^a) \\ &\le (\Phi_k^* f)_a(x)(1 + |2^\nu(x-y)|^a)\max(1, 2^{(k-\nu)a}) \ . \end{aligned}$$

We put this into (3.17) and get

$$\sup_{y\in\mathbb{R}^n} \frac{|(\Psi_\nu * \Lambda_k * \Phi_k * f)(y)|}{1+|2^\nu(x-y)|^a} \le c(\Phi_k^* f)_a(x) \begin{cases} 2^{(k-\nu)(R+1)} & ,k \le \nu \\ 2^{(\nu-k)(|s|+1+\alpha_1)} & ,k \ge \nu \ . \end{cases}$$

Multiplying both sides with $w_\nu(x)$ and using

$$w_\nu(x) \le w_k(x) \begin{cases} 2^{(k-\nu)(-\alpha_2)} & ,k \le \nu \\ 2^{(\nu-k)(-\alpha_1)} & ,k \ge \nu \end{cases} , \tag{3.19}$$

leads us to

$$\sup_{y\in\mathbb{R}^n} \frac{|(\Psi_\nu * \Lambda_k * \Phi_k * f)(y)|}{1+|2^\nu(x-y)|^a} w_\nu(x) \le c(\Phi_k^* f)_a(x) w_k(x) \begin{cases} 2^{(k-\nu)(R+1-\alpha_2)} & ,k \le \nu \\ 2^{(\nu-k)(|s|+1)} & ,k \ge \nu \ . \end{cases}$$

This inequality together with (3.16) gives for $\delta := \min(1, R+1-\alpha_2 - s) > 0$

$$2^{\nu s}(\Psi_\nu^* f)_a(x) w_\nu(x) \le c \sum_{k=0}^\infty 2^{-|k-\nu|\delta} 2^{ks} (\Phi_k^* f)_a(x) w_k(x) \ , \quad x \in \mathbb{R}^n \ .$$

Then, Lemma 3.3 yields immediately the desired result.

□

Remark 3.6: The conditions (3.9) are usually called *moment conditions* while (3.10) and (3.11) are the so called *Tauberian conditions*.
If $R = -1$ in Theorem 3.5, then there are no moment conditions on ψ_1.

3.1.4 Boundedness of the Peetre maximal operator

We will present a theorem which describes the boundedness of the Peetre maximal operator. We use the same notation introduced in the beginning of the last subsection. Especially, we have the functions $\psi_k \in \mathcal{S}(\mathbb{R}^n)$ and $\Psi_k = \hat{\psi}_k \in \mathcal{S}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$.

Theorem 3.7: *Let* $\{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $a, s \in \mathbb{R}$ *and* $0 < p, q \leq \infty$. *For some* $\varepsilon > 0$ *we assume* $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$ *with*

$$|\psi_0| > 0 \quad on \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \tag{3.20}$$

$$|\psi_1| > 0 \quad on \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \ . \tag{3.21}$$

If $a > \frac{n}{p} + \alpha$, *then*

$$\left\| 2^{ks}(\Psi^*_k f)_a w_k \right| l_q(L_p)\| \leq c \left\| 2^{ks}(\Psi_k * f) w_k \right| l_q(L_p)\| \tag{3.22}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof: As in the last proof we find the functions $\{\lambda_j\}_{j\in\mathbb{N}_0}$ with the properties (3.14)-(3.15) and

$$\sum_{k=0}^{\infty} \lambda_k(2^{-\nu}x)\psi_k(2^{-\nu}x) = 1 \quad \text{for all } \nu \in \mathbb{N}_0 \ . \tag{3.23}$$

Instead of (3.16) we get the identity

$$\Psi_\nu * f = \sum_{k=0}^{\infty} \Lambda_{k,\nu} * \Psi_{k,\nu} * \Psi_\nu * f \ , \tag{3.24}$$

where

$$\Lambda_{k,\nu}(\xi) = [\lambda_k(2^{-\nu}\cdot)]^\wedge(\xi) = 2^{\nu n}\Lambda_k(2^\nu \xi) \qquad \text{for all } \nu, k \in \mathbb{N}_0 \ .$$

The $\Psi_{k,\nu}$ are defined similarly. For $k \geq 1$ and $\nu \in \mathbb{N}_0$ we have $\Psi_{k,\nu} = \Psi_{k+\nu}$ and with the notation

$$\sigma_{k,\nu}(x) = \begin{cases} \psi_0(2^{-\nu}x) & \text{, if } k = 0 \\ \psi_\nu(x) & \text{, otherwise} \end{cases}$$

we get $\psi_k(2^{-\nu}x)\psi_\nu(x) = \sigma_{k,\nu}(x)\psi_{k+\nu}(x)$. Hence, we can rewrite (3.24) as

$$\Psi_\nu * f = \sum_{k=0}^{\infty} \Lambda_{k,\nu} * \hat{\sigma}_{k,\nu} * \Psi_{k+\nu} * f \ . \tag{3.25}$$

For $k \geq 1$ we get from Lemma 3.1

$$|(\Lambda_{k,\nu} * \hat{\sigma}_{k,\nu})(z)| = 2^{\nu n}|(\Lambda_k * \Psi)(2^\nu z)| \leq C_M 2^{\nu n} \frac{2^{-kM}}{(1+|2^\nu z|^a)} \tag{3.26}$$

for all $k, \nu \in \mathbb{N}_0$ and arbitrarily large $M \in \mathbb{N}$. For $k = 0$ we get the estimate (3.26) by using Lemma 3.1 with $M = -1$. This together with (3.25) gives us

$$|(\Psi_\nu * f)(y)| \leq C_M 2^{\nu n} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \frac{2^{-kM}}{(1+|2^\nu(y-z)|^a)} |(\Psi_{k+\nu} * f)(z)| dz \ . \tag{3.27}$$

For fixed $r \in (0,1]$ we divide both sides of (3.27) by $(1+|2^\nu(x-y)|^a)$ and we take the supremum in respect to $y \in \mathbb{R}^n$. Using the inequalities

$$(1+|2^\nu(y-z)|^a)(1+|2^\nu(x-y)|^a) \geq c(1+|2^\nu(x-z)|^a) \ ,$$
$$|(\Psi_{k+\nu} * f)(z)| \leq |(\Psi_{k+\nu} * f)(z)|^r (\Psi^*_{k+\nu} f)_a(x)^{1-r}(1+|2^{k+\nu}(x-y)|^a)^{1-r}$$

and

$$\frac{(1+|2^{k+\nu}(x-z)|^a)^{1-r}}{(1+|2^\nu(x-y)|^a)} \leq \frac{2^{ka}}{(1+|2^{k+\nu}(x-y)|^a)^r} \ ,$$

we get

$$(\Psi^*_\nu f)_a(x) \leq C_M \sum_{k=0}^{\infty} 2^{-k(M+n-a)} (\Psi^*_{k+\nu} f)_a(x)^{1-r} \int_{\mathbb{R}^n} \frac{2^{(k+\nu)n}|(\Psi_{k+\nu} * f)(z)|^r}{(1+|2^{k+\nu}(x-y)|^a)^r} dz \ . \tag{3.28}$$

Now, we apply Lemma 3.4 with

$$\gamma_\nu = (\Psi^*_\nu f)_a(x) \ , \qquad \beta_\nu = \int_{\mathbb{R}^n} \frac{2^{\nu n}|(\Psi_\nu * f)(z)|^r}{(1+|2^\nu(x-y)|^a)^r} dz \ , \quad \nu \in \mathbb{N}_0$$

$N = M + n - a$, $C_N = C_M + n - a$ and N^0 giving the order of the distribution f.
By Lemma 3.4 we obtain for every $N \in \mathbb{N}$, $x \in \mathbb{R}^n$ and $\nu \in \mathbb{N}_0$

$$(\Psi^*_\nu f)_a(x)^r \leq C_N \sum_{k=0}^{\infty} 2^{-kNr} \int_{\mathbb{R}^n} \frac{2^{(k+\nu)n}|(\Psi_{k+\nu} * f)(z)|^r}{(1+|2^{k+\nu}(x-y)|^a)^r} dz \ . \tag{3.29}$$

We point out that (3.29) holds also for $r > 1$, where the proof is much simpler. We only have to take (3.27) with $a + n$ instead of a, divide both sides by $(1+|2^\nu(x-y)|^a)$ and apply Hölder's inequality with respect to k and then z.
Multiplying (3.29) by $w_\nu(x)^r$ we derive with the properties of our weight sequence

$$(\Psi^*_\nu f)_a(x)^r w_\nu(x)^r \leq C'_N \sum_{k=0}^{\infty} 2^{-k(N-\alpha_1)r} \int_{\mathbb{R}^n} \frac{2^{(k+\nu)n}|(\Psi_{k+\nu} * f)(z)|^r w_{k+\nu}(z)^r}{(1+|2^{k+\nu}(x-y)|^{a-\alpha})^r} dz \ , \tag{3.30}$$

for all $x \in \mathbb{R}^n$, $\nu \in \mathbb{N}_0$ and all $N \in \mathbb{N}$.
Now, choosing $r > 0$ with $\frac{n}{a-\alpha} < r < p$ the function

$$\frac{1}{(1+|z|)^{r(a-\alpha)}} \in L_1(\mathbb{R}^n)$$

and by the majorant property of the Hardy-Littlewood maximal operator (see [StWe71], Chapter 2) it follows

$$(\Psi_\nu^* f)_a(x)^r w_\nu(x)^r \le C_N' \sum_{k=0}^{\infty} 2^{-k(N-\alpha_1)r} M(|\Psi_{k+\nu} * f|^r w_{k+\nu}^r)(x) \ . \tag{3.31}$$

We choose $N > 0$ such that $N > -s + \alpha_1$ and denote

$$g_k(x) = 2^{krs} M(|\Psi_k * f|^r w_k^r)(x) \ .$$

From (3.31) we derive

$$G_\nu(x) = (\Psi_\nu^* f)_a(x)^r w_\nu(x)^r \le C \sum_{k \ge \nu}^{\infty} 2^{-k(N-\alpha_1)r} g_k(x) \ .$$

So, for $0 < \delta < N + s - \alpha_1$, we can apply Lemma 3.3 with the $l_{q/r}(L_{p/r})$ norm. This gives us

$$\left\| 2^{krs} (\Psi_k^* f)_a(x)^r w_k(x)^r \right| l_{q/r}(L_{p/r}) \Big\| \le c \left\| 2^{krs} M(|\Psi_k * f|^r w_k^r)(x) \right| l_{q/r}(L_{p/r}) \Big\| \tag{3.32}$$

Rewriting the left hand side of (3.32) and using the scalar Hardy-Littlewood Theorem [FeS71] (we recall $r < p$) on the right hand side, we finally get

$$\left\| 2^{ks} (\Psi_k^* f)_a w_k \right| l_q(L_p) \Big\| \le c \left\| 2^{ks} (\Psi_k * f) w_k \right| l_q(L_p) \Big\| \ ,$$

and the proof is complete. □

3.2 Local means characterization

In this section we only combine the two previous subsections to derive the usual local means characterization as in [Tri92] and [Ry99]. The Peetre maximal operator was defined in section 3.1.1 and the functions ψ_0, ψ_1 belong to $\mathcal{S}(\mathbb{R}^n)$.

Theorem 3.8: *Let $\boldsymbol{w} = \{w_k\}_{k \in \mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < p, q \le \infty$ and let $s, a \in \mathbb{R}$, $R + 1 \in \mathbb{N}_0$ with $a > \frac{n}{p} + \alpha$ and $R + 1 > s + \alpha_2$. If*

$$D^\beta \psi_1(0) = 0 \ , \quad \text{for } 0 \le |\beta| \le R \ , \tag{3.33}$$

and

$$|\psi_0(x)| > 0 \quad \text{on} \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \tag{3.34}$$

$$|\psi_1(x)| > 0 \quad \text{on} \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \tag{3.35}$$

for some $\varepsilon > 0$, then

$$\left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \Big\| \sim \left\| 2^{ks} (\Psi_k * f) w_k \right| l_q(L_p) \Big\| \sim \left\| 2^{ks} (\Psi_k^* f)_a w_k \right| l_q(L_p) \Big\|$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Remark 3.9:

(a) The proof of Theorem 3.8 is is just a reformulation of Theorem 3.5 and Theorem 3.7.

(b) If $R = -1$, then there are no moment conditions (3.33) on ψ_1.

(c) In [Vyb06] the proof for the local means characterization was made for the dominating mixed smoothness case. It is not hard to see that we can also generalize our weight functions in the following sense:
We can use tensor products of weights, i.e.

$$w_k(x) = \prod_{i=1}^{n} w_k^i(x_i)$$

where the one-dimensional measurable functions $w_k^i(t)$ have to satisfy the weight conditions

$$0 < w_k^i(t) \le C^i w_k^i(r)(1 + 2^k|t - r|)^{\alpha_i} \ ,$$
$$2^{-\alpha_1^i} w_k^i(t) \le w_{k+1}^i(t) \le 2^{\alpha_2^i} w_k^i(t) \ .$$

Finally, we get the weight class $\mathcal{W}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}}$ with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\alpha_1} = (\alpha_1^1, \ldots, \alpha_1^n)$, $\boldsymbol{\alpha_2} = (\alpha_2^1, \ldots, \alpha_2^n)$. The local means characterization with this weights can be seen directly if one compares the above Theorem with Theorem 1.23 in [Vyb06].

Next we reformulate the Theorem 3.8 in the sense of [Tri92].

Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball and $k \in \mathcal{S}(\mathbb{R}^n)$ a function with support in B. For a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ the corresponding local means are defined by (at least formally)

$$k(t,f)(x) = \int_{\mathbb{R}^n} k(y) f(x + ty) dy = t^{-n} \int_{\mathbb{R}^n} k\left(\frac{y - x}{t}\right) f(y) dy \ , \quad x \in \mathbb{R}^n, t > 0 \ . \quad (3.36)$$

Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ be two functions with

$$\operatorname{supp} k_0 \subseteq B \ , \qquad \operatorname{supp} k^0 \subseteq B \ , \quad (3.37)$$

and

$$\hat{k_0}(0) \neq 0 \ , \qquad \hat{k^0}(0) \neq 0 \ . \quad (3.38)$$

For $N \in \mathbb{N}_0$ we define the iterated Laplacian

$$k(y) := \Delta^N k^0(y) = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} \right)^N k^0(y) \ , \qquad y \in \mathbb{R}^n \ . \quad (3.39)$$

It follows easily that

$$\check{k}(x) = |x|^{2N} \check{k}^0(x) \qquad \text{and that implies} \tag{3.40}$$
$$D^\beta \check{k}(0) = 0 \qquad \text{for} \qquad 0 \le |\beta| < 2N \ . \tag{3.41}$$

Using this notation we come to the usual local means characterization.

Theorem 3.10: *Let* $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < p, q \le \infty$, $s \in \mathbb{R}$. *Furthermore, let* $N \in \mathbb{N}_0$ *with* $2N > s + \alpha_2$ *and let* $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ *and the function* k *be defined as above. Then*

$$\| k_0(1,f)w_0 | \, L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^\infty 2^{jsq} \left\| k(2^{-j},f)w_j \right| L_p(\mathbb{R}^n) \|^q \right)^{1/q} \sim \left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \tag{3.42}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof: We put

$$\psi_0 = k_0^\vee \ , \qquad \psi_1 = k^\vee(\cdot/2) \ .$$

Then the Tauberian conditions (3.34) and (3.35) are satisfied and due to (3.41) also the moment conditions (3.33) are fulfilled. If we define ψ_j for $j \in \mathbb{N}_0$ as in (3.8), then we get

$$(\psi_j \hat{f})^\vee(x) = c(\psi_j^\vee * f)(x) = c \int_{\mathbb{R}^n} (\mathcal{F}\psi_j)(y) f(x+y) dy \ . \tag{3.43}$$

For $j = 0$ we get $\mathcal{F}\psi_0 = k_0$ and for $j \ge 1$ we obtain

$$(\mathcal{F}\psi_j)(y) = (\mathcal{F}\psi_1(2^{-j+1}\cdot))(y) = 2^{(j-1)n}(\mathcal{F}\psi_1)(2^{j-1}y) = 2^{jn}k(2^j y) \ .$$

This and the equation (3.43) lead to

$$(\psi_j \hat{f})^\vee(x) = c 2^{jn} \int_{\mathbb{R}^n} k(2^j y) f(x+y) dy = ck(2^{-j}, f)(x) \ , \quad j \in \mathbb{N}_0 \ , \quad x \in \mathbb{R}^n \ .$$

Together with Theorem 3.8 the proof is complete. □

Remark 3.11: If we take $w_j \equiv 1$ for all $j \in \mathbb{N}_0$, we obtain the usual Besov spaces. If we now compare our result with section 2.5.3 in [Tri92], we get an improvement with respect to $N \in \mathbb{N}_0$. The condition in [Tri92] is $2N > \max(s, \sigma_p)$ where $\sigma_p = \max(0, n(1/p - 1))$. We derived $2N > s$ in Theorem 3.10 ($\alpha_2 = 0$ for $w_j \equiv 1$) wich seems to be more natural. Furthermore, we proved the equivalence of the (quasi-)norms for all $f \in \mathcal{S}'(\mathbb{R}^n)$ by this method where in [Tri92] the equivalence does only hold for $f \in B^s_{pq}(\mathbb{R}^n)$.

For the last modification of the local means representation we introduce some necessary notation. For $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ we denote by $Q_{\nu m}$ the cube centred at the point $2^\nu m = (2^\nu m_1, \ldots, 2^\nu m_n)$ with sides parallel to coordinate axes and of length $2^{-\nu}$. Hence

$$Q_{\nu m} = \{x \in \mathbb{R}^n : |x_i - 2^\nu m_i| \leq 2^{-\nu-1}, i = 1, \ldots, n\} , \qquad \nu \in \mathbb{N}_0 , m \in \mathbb{Z}^n . \tag{3.44}$$

If $\gamma > 0$, then $\gamma Q_{\nu m}$ denotes a cube concentric with $Q_{\nu m}$ with sides also parallel to coordinate axes and of length $\gamma 2^{-\nu}$.
Defining the Peetre maximal operator by (3.2), we get

$$(\Psi_\nu^* f)_a(x) \geq c \sup_{x-y \in \gamma Q_{\nu m}} |(\Psi_\nu * f)(y)| , \quad \nu \in \mathbb{N}_0 , x \in \mathbb{R}^n ,$$

where the constant c only depends on $a > 0$, $\gamma > 0$ and does not depend on $x \in \mathbb{R}^n$ and $\nu \in \mathbb{N}_0$.
With this simple observation we get immediately the following conclusion of Theorem 3.8.

Theorem 3.12: *Let $\boldsymbol{w} = \{w_k\}_{k \in \mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$. For $N \in \mathbb{N}_0$ with $2N > s + \alpha_2$ let k_0, k^0, k be as in Theorem 3.10. Then for every $\gamma > 0$*

$$\left\| f \,|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \sim \left\| \sup_{(x-y) \in \gamma Q_{0,0}} |k_0(1, f)(y)| \,\middle|\, L_p(\mathbb{R}^n, w_0) \right\| + \left(\sum_{j=1}^\infty 2^{jsq} \left\| \sup_{(x-y) \in \gamma Q_{j,0}} |k(2^{-j}, f)(y)| \,\middle|\, L_p(\mathbb{R}^n, w_j) \right\|^q \right)^{1/q} , \tag{3.45}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

4 Further properties

4.1 Embedding theorems

4.1.1 General embeddings

For the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ introduced above we want to show some general embedding theorems. We follow closely [Tri83], see Proposition 2.3.2/2 and Theorem 2.7.1. We say a Banach space A_1 is continuously embedded in another Banach space A_2, $A_1 \hookrightarrow A_2$, if $A_1 \subseteq A_2$ and there is a $c > 0$ such that $\|a|\, A_2\| \le c\, \|a|\, A_1\|$ for all $a \in A_1$.
First, we present an embedding theorem which connects the two-microlocal Besov spaces with the usual weighted Besov spaces [EdTri96]. We denote by $B^s_{p,q}(\mathbb{R}^n, \alpha)$ the weighted Besov spaces, with respect to the weight $\langle x\rangle^\alpha = (1+|x|^2)^{\alpha/2}$ for $\alpha \in \mathbb{R}$.

Theorem 4.1: *Let $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and $0 < p, q \le \infty$, then*

$$B^{s+\alpha_2}_{pq}(\mathbb{R}^n, \alpha) \hookrightarrow B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \hookrightarrow B^{s-\alpha_1}_{pq}(\mathbb{R}^n, -\alpha) \ .$$

Proof: Using the properties (2.2) and (2.3) we obtain

$$w_j(x) \le 2^{j\alpha_2} w_0(x) \le \mathrm{C} 2^{j\alpha_2} w_0(0)(1+|x|^2)^{\alpha/2}$$
$$w_j(x) \ge 2^{-j\alpha_1} w_0(x) \ge \frac{1}{\mathrm{C}} 2^{-j\alpha_1} w_0(0)(1+|x|^2)^{-\alpha/2}$$

for all $x \in \mathbb{R}^n$ and every $j \in \mathbb{N}_0$. It follows immediately

$$c_1 2^{-j\alpha_1} \left\| (1+|x|^2)^{-\alpha/2} (\varphi_j \hat{f})^\vee \right| L_p(\mathbb{R}^n) \Big\| \le \left\| w_j (\varphi_j \hat{f})^\vee \right| L_p(\mathbb{R}^n) \Big\|$$
$$\le c_2 2^{j\alpha_2} \left\| (1+|x|^2)^{\alpha/2} (\varphi_j \hat{f})^\vee \right| L_p(\mathbb{R}^n) \Big\|$$

and therefrom the theorem. □

The following is an easy consequence of the above theorem and $B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_{\max(1,p)}(\mathbb{R}^n)$ for $s > \sigma_p = n(1/p - 1)_+$.

Corollary 4.2: *Let and $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ and let $0 < p, q \le \infty$, then for $s > \sigma_p + \alpha_1$*

$$B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \hookrightarrow L_{\max(1,p)}(\mathbb{R}^n, \langle x\rangle^{-\alpha}) \ .$$

We need a special weighted version of Nikol'skij's inequality.

Proposition 4.3: *Let ϱ be an admissible weight satisfying $(a, b > 0)$*

$$0 < \varrho(x) \le \mathrm{C}_\varrho \varrho(y)(1 + ab|x-y|)^{\alpha_\varrho} \quad \text{for all } x, y \in \mathbb{R}^n \ .$$

Further let $0 < p \le q \le \infty$ and $B_b = \{x \in \mathbb{R}^n : |x| \le b\}$. If $\beta \in \mathbb{N}^n_0$ is a multi-index, then there exists a positive constant c such that

$$\left\| \varrho D^\beta \varphi \right| L_q(\mathbb{R}^n) \Big\| \le c b^{|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right)} \|\varrho\varphi|\, L_p(\mathbb{R}^n)\| \tag{4.1}$$

holds for all $\varphi \in L_p(\mathbb{R}^n, \varrho)$ with $\operatorname{supp}\hat{\varphi} \subseteq B_b$ where the c is independent of $b > 0$.

Proof: We substitute

$$\widetilde{\varrho}(x) := \varrho(b^{-1}x) \quad \text{and}$$
$$\widetilde{\varphi}(x) := \varphi(b^{-1}x) \ .$$

Now the weight $\widetilde{\varrho}$ satisfies

$$0 < \widetilde{\varrho}(x) \le \mathrm{C}'_{\varrho}\widetilde{\varrho}(y)(1+|x-y|)^{\alpha_\varrho} \quad \text{for all } x, y \in \mathbb{R}^n \ . \tag{4.2}$$

Further, the function $\widetilde{\varphi} \in L_p(\mathbb{R}^n, \widetilde{\varrho})$ with $\operatorname{supp} \widetilde{\varphi} \subset B_1$. Now, we can apply Proposition 1.4.3 in [SchmTri87]. After a resubstitution we derive the above statement (4.1). From Remark 2 in [SchmTri87, 1.4.2] we get that the constant c in (4.1) is independent of b and of the choice of the weight function (it depends only on C'_ϱ and α_ϱ). □

Theorem 4.4: *Let $s \in \mathbb{R}$ and $\boldsymbol{w}, \boldsymbol{\varrho} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ with $\frac{w_j(x)}{\varrho_j(x)} \le c$ for all $j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$.*

(i) For $0 < p \le \infty$ and $0 < q_1 \le q_2 \le \infty$ we have

$$B^{s,mloc}_{pq_1}(\mathbb{R}^n, \boldsymbol{\varrho}) \hookrightarrow B^{s,mloc}_{pq_2}(\mathbb{R}^n, \boldsymbol{w}) \ . \tag{4.3}$$

(ii) If $0 < p \le \infty$, $0 < q_1 \le \infty$, $0 < q_2 \le \infty$ and $\varepsilon > 0$, then

$$B^{s,mloc}_{pq_1}(\mathbb{R}^n, \boldsymbol{\varrho}) \hookrightarrow B^{s-\varepsilon,mloc}_{pq_2}(\mathbb{R}^n, \boldsymbol{w}) \ . \tag{4.4}$$

(iii) For $0 < p_1 \le p_2 \le \infty$, $0 < q \le \infty$ and $-\infty < s_2 \le s_1 < \infty$ with

$$s_1 - \frac{n}{p_1} \ge s_2 - \frac{n}{p_2} \quad \text{we have} \quad B^{s_1,mloc}_{p_1q}(\mathbb{R}^n, \boldsymbol{\varrho}) \hookrightarrow B^{s_2,mloc}_{p_2q}(\mathbb{R}^n, \boldsymbol{w}) \ . \tag{4.5}$$

Proof: The proof of 4.3 and 4.4 is the same as in Proposition 2.3.2/2 in [Tri83] one only has to plug in the weight sequence. To prove 4.5 we use Proposition 4.3 with $b = 2^{j+1}$, $\varrho = w_j$ and $\varphi = (\varphi_j \hat{f})^\vee$ for each $j \in \mathbb{N}_0$. Now, the substituted weight functions $\widetilde{w_j}$ satisfy a condition as in (4.2), where the constants C_{w_j} and α_{w_j} do not depend on $j \in \mathbb{N}_0$. Hence, Proposition 4.3 gives

$$\left\| w_j(\varphi_j \hat{f})^\vee \middle| L_{p_2}(\mathbb{R}^n) \right\| \le c 2^{jn\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \left\| w_j(\varphi_j \hat{f})^\vee \middle| L_{p_1}(\mathbb{R}^n) \right\| \ ,$$

for all $j \in \mathbb{N}_0$, where the constant c is independent of $j \in \mathbb{N}_0$. After multiplying the inequality by $2^{j(s_2-n/p_2)}$ and using the conditions on s_1, s_2, p_1, p_2 and the weight sequences, we get

$$2^{js_2} \left\| w_j(\varphi_j \hat{f})^\vee \middle| L_{p_2} \right\| \le c' 2^{js_1} \left\| \varrho_j(\varphi_j \hat{f})^\vee \middle| L_{p_1} \right\| \ .$$

Finally we apply the l_q quasi-norm to find the desired result. □

With minor modifications we have an analogous theorem to Theorem 2.3.3 in [Tri83].

Theorem 4.5: *Let* $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ *and* $0 < p, q \leq \infty$, *then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \quad \textit{holds.} \tag{4.6}$$

If $s \in \mathbb{R}$ *and* $0 < p, q < \infty$, *then* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.

Proof: The proof is essentially the same as in [Tri83, 2.3.3]. One only has to bring in the weight sequence and use its properties (2.2) and (2.3). Also the weighted Nikol'skij inequality (Proposition 4.3) and section 1.5 in [SchmTri87] has to be used as a replacement for the unweighted ones in the proof in [Tri83]. □

4.1.2 Embeddings for 2-microlocal Besov spaces

In this subsection we present some special embedding theorems for the weight sequence of 2-microlocal weights, $w_j(x) = (1 + 2^j \operatorname{dist}(x, U))^{s'}$ for fixed $U \subseteq \mathbb{R}^n$ and $s' \in \mathbb{R}$. The spaces $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ were defined in Definition 2.17. As shown in Example 2.5, the weight sequence belongs to $\mathcal{W}^{|s'|}_{\max(0,-s'),\max(0,s')}$. We recall the spaces $B^s_{pq}(\mathbb{R}^n, \alpha)$, with respect to the weight $\langle x \rangle^\alpha = (1 + |x|^2)^{\alpha/2}$ for $\alpha \in \mathbb{R}$. An easy consequence of Theorem 4.1 and Theorem 4.4 is the following.

Theorem 4.6: *Let* $s \in \mathbb{R}$ *and let* $0 < p, q \leq \infty$.

(i) For $s' \in \mathbb{R}$ *and* $U = \{x_0\} \in \mathbb{R}^n$ *we have*

$$B^{s,s'}_{pq}(\mathbb{R}^n, x_0) \hookrightarrow C^{s-\frac{n}{p},s'}_{x_0} .$$

(ii) For $s' \geq 0$ *and* $U \subseteq V \subseteq \mathbb{R}^n$ *we have*

$$B^{s+s'}_{pq}(\mathbb{R}^n, s') \hookrightarrow B^{s,s'}_{pq}(\mathbb{R}^n, U) \hookrightarrow B^{s,s'}_{pq}(\mathbb{R}^n, V) \hookrightarrow B^s_{pq}(\mathbb{R}^n, -s') .$$

(iii) For $s' \geq 0$ *and* $U \subseteq V \subseteq \mathbb{R}^n$ *we have*

$$B^{s,s'}_{pq}(\mathbb{R}^n, U) \hookrightarrow B^{s,s'}_{pq}(\mathbb{R}^n, V) \hookrightarrow B^s_{pq}(\mathbb{R}^n) \hookrightarrow B^{s,-s'}_{pq}(\mathbb{R}^n, V) \hookrightarrow B^{s,-s'}_{pq}(\mathbb{R}^n, U) .$$

(iv) For $s' \geq t'$ *and* $U \subseteq \mathbb{R}^n$ *we have*

$$B^{s,s'}_{pq}(\mathbb{R}^n, U) \hookrightarrow B^{s,t'}_{pq}(\mathbb{R}^n, U) .$$

Corollary 4.7: *Let* $s \geq s' \geq 0$ *and let* $0 < p, q \leq \infty$. *Further, if* $U \subseteq \mathbb{R}^n$, *then*

$$B^{s,s'}_{pq}(\mathbb{R}^n, U) \hookrightarrow B^s_{pq}(\mathbb{R}^n) \hookrightarrow B^{s,-s}_{pq}(\mathbb{R}^n, U) .$$

Remark 4.8: Corollary 4.7 coincides partially with Proposition 1.3 (1) and (2) in [JaMey96] for $p = q = \infty$ and $U = \{x_0\}$ and with Theorem 3.2 in [MoYa04] with $p = q \geq 1$ and U be an open subset or $U = \{x_0\} \subset \mathbb{R}^n$.
In the mentioned papers local versions of $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ have been used to treat further kinds of embeddings in the scale of $B^{s,s'}_{pq}(\mathbb{R}^n, U)$.

4.2 Pointwise multipliers

Let g be a bounded function on $\mathbb{R}^n$. We ask, under which conditions the mapping $f \mapsto gf$ makes sense and generates a bounded operator in a given space $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$. We follow closely [Tri92, 4.2.2] and adapt the proofs to our situation. First, we prove a lemma which is important for pointwise multipliers.

Lemma 4.9: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and let $0 < p, q \leq \infty$. Then for $s > \frac{n}{p} + \alpha + \alpha_1$ and all $\gamma > 0$ there is a constant $c_\gamma > 0$ such that*

$$\left\| w_0(\cdot) \sup_{|\cdot - y| \leq \gamma} |f(y)| \,\middle|\, L_p(\mathbb{R}^n) \right\| \leq c_\gamma \left\| f \,\middle|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \quad \textit{holds for all } f \in \mathcal{S}'(\mathbb{R}^n).$$

Proof: Let $\{\varphi_j\}_{j \in \mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be the chosen resolution of unity from the beginning of the chapter. Then we get for arbitrary $\varepsilon > 0$

$$\left\| w_0(\cdot) \sup_{|\cdot - y| \leq \gamma} |f(y)| \,\middle|\, L_p(\mathbb{R}^n) \right\| \leq c \sum_{j=0}^{\infty} 2^{j\varepsilon} \left\| w_0(\cdot) \sup_{|\cdot - y| \leq \gamma} |(\varphi_j \hat{f})^\vee(y)| \,\middle|\, L_p(\mathbb{R}^n) \right\| .$$

For all $a > 0$ we have

$$\sup_{|x-y| \leq \gamma} |(\varphi_j \hat{f})^\vee(y)| \leq c 2^{ja} \sup_{z \in \mathbb{R}^n} \frac{|(\varphi_j \hat{f})^\vee|(x-z)}{1 + |2^j z|^a}$$

where the constant only depends on $\gamma > 0$. Using the property (2.3) of the weight sequence and Theorem 3.8, we obtain for arbitrary $a > n/p + \alpha$ and $\varepsilon > 0$

$$\begin{aligned} \left\| w_0(\cdot) \sup_{|\cdot - y| \leq \gamma} |f(y)| \,\middle|\, L_p(\mathbb{R}^n) \right\| &\leq c \left\| (\varphi_j^* f)_a \,\middle|\, B^{a+\alpha_1+\varepsilon,mloc}_{p1}(\mathbb{R}^n, \boldsymbol{w}) \right\| \\ &\leq c' \left\| f \,\middle|\, B^{a+\alpha_1+\varepsilon,mloc}_{p1}(\mathbb{R}^n, \boldsymbol{w}) \right\| \\ &\leq c'' \left\| f \,\middle|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| , \end{aligned}$$

for $s > \frac{n}{p} + \alpha + \alpha_1$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. □

Let k_0, $k \in \mathcal{S}(\mathbb{R}^n)$ and $k(t, f)$ be the same functions as in (3.36)-(3.39). For $g \in C^m(\mathbb{R}^n)$ we study gf where $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$. First, we prove the theorem and after that we discuss, how gf has to be understood.

Theorem 4.10: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and let $0 < p, q \leq \infty$. If $m \in \mathbb{N}$ is sufficiently large, then there exists a positive number c_m such that*

$$\left\| gf \,\middle|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \leq c_m \sum_{|\beta| \leq m} \left\| D^\beta g \,\middle|\, L_\infty(\mathbb{R}^n) \right\| \left\| f \,\middle|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \qquad (4.7)$$

for all $g \in C^m(\mathbb{R}^n)$ and all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof: First Step: Firstly, we prove the theorem under the additional assumption $s > \frac{n}{p} + \alpha + \alpha_1$. We use the Taylor expansion of $g \in C^m(\mathbb{R}^n)$

$$g(x) = \sum_{|\beta| \le m-1} \frac{D^\beta g(y)}{\beta!}(x-y)^\beta + \sum_{|\beta|=m} \frac{D^\beta g(y+\theta(x-y))}{\beta!}(x-y)^\beta \ , \qquad (4.8)$$

for $\theta \in (0,1)$. By (3.36) we have

$$k(2^{-j}, f)(x) = \int_{\mathbb{R}^n} k(y) f(x+2^{-j}y) g(x+2^{-j}y) dy$$

$$= \sum_{|\beta| \le m-1} \frac{D^\beta g(x)}{\beta!} 2^{-j|\beta|} \int_{\mathbb{R}^n} y^\beta k(y) f(x+2^{-j}y) dy + 2^{-jm} \int_{\mathbb{R}^n} k(y) r_m(x, 2^{-j}, y) f(x+2^{-j}y) dy \ ,$$

where the remainder term in Taylor's expansion, $r_m(x, 2^{-j}, y)$, is in any case uniformly bounded. If we choose $N \in \mathbb{N}_0$ in (3.39) sufficiently large, for each $\beta \le m-1$ the function $k_\beta(y) = y^\beta k(y)$ is a new kernel for which Theorem 3.10 holds. Thus, choosing $m > s + \alpha_2$ and using Theorem 3.10 for every $|\beta| \le m-1$ we obtain

$$\left(\sum_{j=1}^{\infty} 2^{jsq} \left\| w_j k(2^{-j}, f) \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \le c \sum_{|\beta| \le m-1} \left\| D^\beta g \right| L_\infty(\mathbb{R}^n) \right\| \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\|$$

$$+ c \sum_{|\beta| \le m} \left\| D^\beta g \right| L_\infty(\mathbb{R}^n) \right\| \left\| w_0(\cdot) \sup_{|\cdot - y| \le 1} |f(y)| \right| L_p(\mathbb{R}^n) \right\|$$

Now, Lemma 4.9 with $\gamma = 1$ proves the theorem provided $s > \frac{n}{p} + \alpha + \alpha_1$.
Second Step: Let $-\infty < s \le \frac{n}{p} + \alpha + \alpha_1$ and let $l \in \mathbb{N}$ with $s + 2l > \frac{n}{p} + \alpha + \alpha_1$. From the lift property (see Section 2.3) we get, that any $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ can be represented as $f = (\mathrm{id} + (-\Delta)^l) h$, with

$$h \in B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \quad \text{and} \quad \left\| h \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \sim \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \ . \qquad (4.9)$$

We have

$$gf = (\mathrm{id} + (-\Delta)^l) gh + \sum_{|\beta| < 2l} D^\beta (g_\beta h) \ ,$$

where each g_β is a sum of terms of the type $D^\beta g$ with $|\beta| \le 2l$. Now, Theorem 2.16 shows

$$\left\| gf \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \le c \sum_{|\beta| \le 2l} \left\| g_\beta h \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \ .$$

If $l \in \mathbb{N}$ is sufficiently large, that is $m - 2l > s + 2l + \alpha_2$, we can apply the first step and and obtain

$$\left\| gf \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \le c \sum_{|\beta| \le m} \left\| D^\beta g \right| L_\infty(\mathbb{R}^n) \right\| \left\| h \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \ .$$

Finally, (4.9) proves the theorem. □

Remark 4.11: The interpretation of $g \cdot f$ is a bit sophisticated. We approximate f and g by smooth functions, f_j and g_j. The limit of $g_j \cdot f_j$ exists in $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$, see [Tri92, Remark 1/4.2.2], and we define $g \cdot f = \lim_{j\to\infty} g_j \cdot f_j$, where $g_j \cdot f_j$ has to be understood in the usual pointwise sense, as limit element. For a more detailed discussion of this procedure we refer also to [RuSi96, Chapter 4].

4.3 Invariance under Diffeomorphisms

In this section we show that the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ are invariant under diffeomorphisms. The result and the proof are closely related to Section 4.3 in[Tri92]. Let $m \in \mathbb{N}$, then we call an isomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ an m-diffeomorphism if the components $\psi_j(x)$ of $\psi(x) = (\psi_1(x), \ldots, \psi_n(x))$ have classical derivatives up to the order k and the functions $D^\beta \psi_j(x)$ are bounded for all $0 < |\beta| \le m$, $1 \le j \le n$ and all $x \in \mathbb{R}^n$. Furthermore, the Jacobian matrix ψ_* has to fulfill $|\det \psi_*(x)| \ge d > 0$ for all $x \in \mathbb{R}^n$. If $y = \psi(x)$ is a m-diffeomorphism for every $m \in \mathbb{N}$, then it is called diffeomorphism.
We want to prove that $f \to f \circ \psi$ is a linear and bounded operator in all spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$. If ψ is a diffeomorphism, then

$$f \circ \psi(x) = f(\psi(x)) \tag{4.10}$$

makes sense for all $f \in \mathcal{S}'(\mathbb{R}^n)$. If ψ is only an m-diffeomorphism, then (4.10) has to be understood as an approximation procedure with smooth functions (see also Remark 4.11). In the proof we use the local means characterization in the form of Theorem 3.10. First of all, we have to prove two lemmas which will be useful later on.
We need a modification of Theorem 3.10. Therefore, let k_0 and k^0 be kernels in the sense of (3.37)-(3.39) with $N \in \mathbb{N}_0$ large enough and $a(x)$ be an $n \times n$ matrix with real-valued continuous entries $a_{ij}(x)$, where $x \in \mathbb{R}^n$ and $i, j \in \{1, \ldots, n\}$. Further, there exist two numbers $d, d' > 0$ with

$$|a_{ij}(x)| \le d' \qquad \text{for all } x \in \mathbb{R}^n,\ i,j \in \{1,\ldots,n\} \text{ and} \tag{4.11}$$

$$|\det a(x)| \ge d > 0 \qquad \text{for all } x \in \mathbb{R}^n. \tag{4.12}$$

Since, $y \mapsto ya(x)$ is an isomorphic mapping for fixed $x \in \mathbb{R}^n$ we can generalize (3.36) by

$$k(a,t,f)(x) = \int_{\mathbb{R}^n} k(y) f(x + ta(x)y)dy\ . \tag{4.13}$$

Lemma 4.12: *Let $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and let$0 < p, q \le \infty$. Further, let $a(x)$ be the above matrix with (4.11), (4.12) and let k_0 and k be the functions from (3.37)-(3.39). Then there exists a constant c such that*

$$\| k_0(a,1,f)w_0 | L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^\infty 2^{jsq} \left\| k(a,2^{-j},f)w_j \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \le c \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \tag{4.14}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof: Let B be the collection of all matrices $b = \{b_{ij}\}_{i,j=1}^n$ satisfying (4.11) and (4.12). For fixed $b \in B$ we derive by this properties

$$k(b,t,f)(x) = k^b(t,f)(x) \qquad \text{whereas} \quad k^b(y) = ck(b^{-1}y) \tag{4.15}$$

is a modified kernel in the sense of (3.37)-(3.39). The same holds for k_0^b, so that we can apply now Theorem 3.10 with the new kernels, and get

$$\left\| k_0^b(1,f)w_0 \right| L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \left\| k^b(2^{-j},f)w_j \right| L_p(\mathbb{R}^n)\|^q \right)^{1/q} \sim c \left\| f \right| B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})\|$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$. Now, we obtain (4.14) from this formula in going over to the supremum over all $b \in B$ inside the L_p quasi-norms. □

The second Lemma is necessary for our weighted spaces. To get the invariance under diffeomorphisms of our spaces we also need a special restriction on the diffeomorphisms. From now on we consider only diffeomorphisms ψ which satisfy $\psi(x) = x$ for x near to infinity ($|x| > R$ for some $R > 0$).
With that restriction we are ready to formulate the next lemma.

Lemma 4.13: *Let w_0 be an admissible weight function. Let $R > 0$ and ψ be an m-diffeomorphism with $\psi(x) = x$ for $|x| > R$, then there exists a constant $c > 0$ such that*

$$(w_0 \circ \psi^{-1})(x) \le c w_0(x) \qquad \textit{holds for all } x \in \mathbb{R}^n.$$

Proof: If ψ is an m-diffeomorphism with the restriction above, then also ψ^{-1} is an m-diffeomorphism with $\psi^{-1}(x) = x$ for $|x| > R$. We define

$$a^* := \max_{1 \le i,j \le n} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial \psi_i^{-1}}{\partial x_j}(x) \right| . \tag{4.16}$$

Using the properties of the weight function w_0 and Taylor expansion of ψ^{-1} we obtain

$$\begin{aligned} w_0(\psi^{-1}(x)) &\le \mathrm{C} w_0(x)(1 + |x - \psi^{-1}(x)|)^\alpha \le \mathrm{C} w_0(x)(1 + |x - \psi^{-1}(0) - \psi_*^{-1}(..) \cdot x|)^\alpha \\ &\le \mathrm{C} w_0(x) 2^\alpha (1 + |\psi^{-1}(0)|)^\alpha (1 + |x - \psi_*^{-1}(..) \cdot x|)^\alpha \\ &\le \mathrm{C}' w_0(x)(1 + |x - \psi_*^{-1}(..) \cdot x|)^\alpha . \end{aligned}$$

Here $\psi_*^{-1}(..)$ is the Jacobian where in every line different arguments from the line segment between 0 and x are possible. In every case, the absolute values from all entries of $\psi_*^{-1}(..)$ are bounded by a^*. We can estimate from this property

$$|x - \psi_*^{-1}(..) \cdot x| \le |x|(1 + a^* n) \qquad \text{for all } x \in \mathbb{R}^n. \tag{4.17}$$

Finally, we get from $\psi^{-1}(x) = x$ for $|x| > R$ and the preceding calculation

$$w_0 \circ \psi^{-1}(x) = w_0(\psi^{-1}(x)) \le \begin{cases} C_{R,\alpha,\psi,n} w_0(x) & \text{for } |x| \le R \\ w_0(x) & \text{for } |x| > R \end{cases} ,$$

and this finishes the proof. □

Now, the main theorem can be stated.

Theorem 4.14: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < p, q \leq \infty$ and let $s \in \mathbb{R}^n$. Further, let ψ be a m-diffeomorphism for $m \in \mathbb{N}$ large enough and with $\psi(x) = x$ for large x. Then $f \mapsto f \circ \psi$ is an isomorphic mapping from $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ onto $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.*

Proof: First Step: It is enough to prove, that there exists a constant $c > 0$ such that

$$\left\| f \circ \psi \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \| \leq c \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \| \qquad \text{for all } f \in \mathcal{S}'(\mathbb{R}^n). \tag{4.18}$$

The reverse inequality follows immediately if we use ψ^{-1} in (4.18). Furthermore, we always assume that f is a smooth function.

Second Step: Let $s > \frac{n}{p} + \alpha + 2\alpha_1 + \alpha + 2$, then we can find a number $K \in \mathbb{N}$ with

$$\alpha_1 + \alpha_2 + 1 < K + \frac{n}{p} + \alpha + \alpha_1 < s \qquad \text{and} \qquad s + \alpha_2 < 2K \ . \tag{4.19}$$

We use the local means characterization, Theorem 3.10, with some kernels k_0, k and $N \in \mathbb{N}_0$ large enough. To simplify our notation we write $k(1,f) := k_0(1,f)$ and we put the first summand with k_0 and w_0 into the infinite summation with $j = 0$. So we get with this notation

$$\begin{aligned}\left\| f \circ \psi \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \| &\leq c \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| w_j k(2^{-j}, f \circ \psi) \right| L_p(\mathbb{R}^n) \|^q \right)^{1/q} \\ &\leq c \left(\sum_{j=0}^{\infty} 2^{jq(s+\alpha_2)} \left\| w_0(x) \int_{\mathbb{R}^n} k(y) f(\psi(x + 2^{-j}y)) dy \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} .\end{aligned} \tag{4.20}$$

We use Taylor expansion on ψ and obtain

$$\psi(x + 2^{-j}y) = \psi(x) + 2^{-j}\psi_*(x) \cdot y + \sum_{2 \leq |\beta| < 2K} 2^{-j|\beta|} \frac{D^\beta \psi(x)}{\beta!} y^\beta + 2^{-2Kj} R_{2K}(x, 2^{-j}, y) \ ,$$

where $D^\beta \psi$ and the remainder term R_{2K} stand for appropriate vectors. Again, we apply Taylor expansion now on f and derive

$$\begin{aligned} & f \left[\psi(x) + 2^{-j}\psi_*(x) \cdot y + \sum_{2 \leq |\beta| < 2K} + 2^{-2Kj} R_{2K} \right] \\ & \qquad = f \left[\psi(x) + 2^{-j}\psi_*(x) \cdot y + \sum_{2 \leq |\beta| < 2K} \right] + 2^{-2Kj} \widetilde{R_{2K}}(x, 2^{-j}, y) \cdot (\nabla f)(\xi) \ , \end{aligned} \tag{4.21}$$

where the last term is a scalar product with an immaterially modified remainder term. Now, putting the last summand of (4.21) into (4.20) and using $2K > s + \alpha_2$ we can estimate this by

$$c \left\| w_0(x) \sup_{|\psi(x) - z| < c'} |(\nabla f)(z)| \right| L_p(\mathbb{R}^n) \right\| .$$

An obvious substitution and Lemma 4.13, Lemma 4.9 and Theorem 2.16 show that this is bounded by $c\,\big\|\, f\,|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\|$. To handle the first term in (4.21) we use Taylor again and get

$$f\left[\psi(x) + 2^{-j}\psi_*(x)\cdot y + \sum_{2\le|\beta|<2K}\right]$$
$$= \sum_{0\le|\gamma|<K} \frac{D^\gamma f(\psi(x)+2^{-j}\psi_*(x)\cdot y)}{\gamma!}\left(\sum_{2\le|\beta|<2K}\right)^\gamma + \sum_{|\gamma|=K}\frac{D^\gamma f}{\gamma!}\left(\sum_{2\le|\beta|<2K}\right)^\gamma . \qquad (4.22)$$

From

$$\left|\left(\sum_{2\le|\beta|<2K}\right)^\gamma\right| \le c2^{-2Kj} \quad \text{for } |\gamma| = K ,$$

we can estimate the last term of (4.22) in (4.20) by

$$c\sum_{|\gamma|=K}\left\| w_0(x) \sup_{|\psi(x)-z|<c'} |D^\gamma f(z)| \,\middle|\, L_p(\mathbb{R}^n)\right\| .$$

The same substitution as above and Lemma 4.13, Lemma 4.9 and Theorem 2.16 show the boundedness by $c\,\big\|\, f\,|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\|$ when $s-K > \frac{n}{p}+\alpha+\alpha_1$. Finally, it remains to estimate the first term of (4.22) in (4.20). The resulting term is

$$c\sum_{0\le|\gamma|<K}\left(\sum_{j=0}^\infty 2^{jsq}\left\| w_j(x)2^{-jb}\int\limits_{\mathbb{R}^n} k(y)y^\delta D^\gamma f(\psi(x)+2^{-j}\psi_*(x)\cdot y)dy \,\middle|\, L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} ,$$

where $b\ge 2|\gamma|$ and $|\delta| \le (2K-1)|\gamma|$. For large $N\in\mathbb{N}_0$ we get that $\widetilde{k_\gamma}(y) := k(y)y^\delta$ are new kernels in the sense of Theorem 3.10 and we can estimate

$$\le c'\sum_{0\le|\gamma|<K}\left(\sum_{j=0}^\infty 2^{jq(s+\alpha_2-b)}\left\| w_0(x)\widetilde{k_\gamma}(\psi_*\circ\psi^{-1}, 2^{-j}, D^\gamma f)(\psi(x))\,\middle|\, L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} .$$

Substitution and usage of Lemma 4.13 makes us ready to use Lemma 4.12 and we derive

$$\le c'\sum_{0\le|\gamma|<K}\big\| D^\gamma f\,|\, B^{s+\alpha_1+\alpha_2-b,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\| .$$

This can be estimated by $c\,\big\|\, f\,|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\|$ if $K > \alpha_1+\alpha_2+1$ and therefore $s > \frac{n}{p}+\alpha+2\alpha_1+\alpha_2+2$.

Third Step: Let $s \le \frac{n}{p} + \alpha + 2\alpha_1 + \alpha_2 + 2$ then there is an $l \in \mathbb{N}$ such that $s + 2l > \frac{n}{p} + \alpha + 2\alpha_1 + \alpha_2 + 2$. As in the previous section we present $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ by

$$f = (\mathrm{id} + (-\Delta)^l)h \quad h \in B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \tag{4.23}$$

and

$$\left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \sim \left\| h \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \ . \tag{4.24}$$

We have

$$f(x) = \sum_{|\beta| \le 2l} c_\beta(x)(D^\beta h \circ \psi \circ \psi^{-1})(x) \ , \tag{4.25}$$

where c_β are some functions. We assume that they are smooth and that we can apply Theorem 4.10 and obtain

$$\left\| f \circ \psi \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \le c \sum_{|\beta| \le 2l} \left\| D^\beta h \circ \psi \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \le c' \left\| h \circ \psi \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \ .$$

Finally, the second step and (4.23) lead to the result we focused on. □

The restriction $\psi(x) = x$ for large x is not satisfactory. For the special case of the 2-microlocal Besov spaces $B^{s,s'}_{pq}(\mathbb{R}^n, x_0)$ with the weight sequence $w_j(x) = (1 + 2^j|x - x_0|)^{s'}$ a more moderate restriction on ψ can be used. Let us have a look on $w_0 \circ \psi^{-1}$ for $s' \ge 0$, we have

$$w_0 \circ \psi^{-1}(x) = w_0(\psi{-}1(x)) = \left(1 + |\psi^{-1}(x) - x_0|\right)^{s'} \ .$$

Now, using Taylor expansion on ψ^{-1} at the point x_0, we get

$$= \left(1 + |\psi^{-1}(x_0) + \psi_*^{-1}(..) \cdot (x - x_0) - x_0|\right)^{s'} \ .$$

Finally, demanding $\psi^{-1}(x_0) = x_0$ we obtain in the same manner as in (4.17)

$$= \left(1 + |\psi_*^{-1}(..) \cdot (x - x_0)|\right)^{s'} \le C_{\psi,n,s'}(1 + |(x - x_0)|)^{s'} = C_{\psi,n,s'} w_0(x) \ ,$$

which is the result we aimed at. We can use the above calculation instead of Lemma 4.13 in the proof of Theorem 4.14. So the following corollary holds.

Corollary 4.15: *Let $x_0 \in \mathbb{R}^n$, $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $s' \ge 0$, Further let ψ be an m-diffeomorphism with $m \in \mathbb{N}$ large enough and $\psi(x_0) = x_0$, then $f \mapsto f \circ \psi$ is an isomorphic mapping from $B^{s,s'}_{pq}(\mathbb{R}^n, x_0)$ onto $B^{s,s'}_{pq}(\mathbb{R}^n, x_0)$.*

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