A pricing model for the Guaranteed Lifelong Withdrawal Benefit Option

Gabriella Piscopo

Università degli studi di Napoli Federico II
Dipartimento di Matematica e Statistica
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Main References


• **Clements J., 2004.** For a conservative investment, variable annuities are too costly. Wall Street Journal C1, January 21


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What is a Variable Annuity?

• “As VAs are essentially a new product class in the U.K., an industry standard definition does not yet exist...We shall define a VA as any unit-linked or manage fund vehicle which offers optional guarantee benefits.” (Ledlie et al., 2008, Institute of Actuaries, London)
The embedded options

• **GMDB**: The Guaranteed Minimum Death Benefith option guarantees a minimum return of the principal invested upon the death of the policyholder.

• **GMAB**: The Guaranteed Minimum Accumulation Benefith option offers the policyholder a guaranteed minimum at maturity $T$ if he is still alive.

• **GMIB**: The Guaranteed Minimum Income Benefith option offers a minimum income stream from a specified future point in time.

• **GMWB**: The Guaranteed Minimum Withdrawal Benefith options gives the policyholder the possibility to withdraw a pre-specified amount annually until the maturity $T$ even if the fund value has fallen below this value.
What is a GLWB?

The GLWB option is a GMWB for Life option: it offers a lifelong withdrawal guarantee; therefore, there is no limit for the total amount that is withdrawn over the term of the policy, because if the account value becomes zero while the insured is still alive he can continue to withdraw the guaranteed amount annually until death.
The aims of the work

• Develop a pricing model and define a fair price for the GLWB option

• Verify if the current GLWB price on the USA market is fair or the market seems to be underpriced or overpriced.
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The financial assumption

The single premium $\omega_0$ paid by the policyholder is invested in a fund. Following the standard assumption in the literature (Boyle and Schwartz 1997), we model the evolution of the fund as:

$$dW_t = (\mu - \delta)W_t dt - \gamma_t dt + \sigma W_t dZ_t$$

Where $Z_t$ is a Brownian motion, $\delta$ is the insurance fee paid for the GLWB option and $\gamma_t$ is the withdrawal at time $t$. In a static approach

$$\gamma_t = G = g \omega_0 \forall t$$

where $g$ is the guaranteed rate.
The GLWB payoff

The policyholder receives the amount guaranteed until he is still alive; moreover, at the date of death the beneficiary will receive any remaining fund value. The discounted value at $t=0$ of the GLWB $V_0$ is the sum of the discounted values of living and death benefits:

$$V_0 = LB_0 + DB_0$$

$LB_0$ is the discounted value of a life annuity. Since the random time $\tau$ of death and the fund value at this time are independent

$$DB_0 = E_t \left\{ E \left\{ e^{-r\tau} \max[W\tau;0] \mid \tau = t \right\} \right\}$$
The Death Benefit (1)

If we fix the date $T$, the death benefit can be calculated by Ito’s Lemma

$$DB_T = e^{\left(\mu - \delta - \frac{\sigma^2}{2}\right)T + \sigma Z_T} \times \max \left[\left(\omega_0 - G \int_0^T e^{-\left(\mu - \delta - \frac{\sigma^2}{2}\right)t - \sigma Z_t} \ dt\right); 0\right]$$

and rewritten as a payoff of a Quanto Asian Put (QAP) Option:

$$DB_T = e^{\left(\mu - \delta - \frac{\sigma^2}{2}\right)T + \sigma Z_T} \times G T \times \max \left[\left(\frac{\omega_0}{GT} - \frac{1}{T} \int_0^T e^{-\left(\mu - \delta - \frac{\sigma^2}{2}\right)T - \sigma Z_T} \ dt\right); 0\right]$$
The Death Benefit (2)

Using a standard technique in literature, the No-arbitrage time-zero value of the death benefit at time \( T \) is:

\[
DB_0(\tau = T) = E^Q \left[ e^{\left( r - \delta - \frac{\sigma^2}{2} \right)T + \sigma Z_T} \times GT \times \max \left( \frac{\omega_0}{GT} - \frac{1}{T} \int_0^T e^{-\left( r - \delta - \frac{\sigma^2}{2} \right) t - \sigma Z_t} dt ; 0 \right) \right]
\]

\[
DB_0(\tau = T) = E^Q[QAP_T]
\]

where the expectation is under the risk neutral measure \( Q \). If we do not fix the date of death and consider both the expectations:

\[
DB_0 = \sum_{t=0}^{n-x} (x \ p_t \times q_{x+t} \times E_0[QAP_t])
\]

where \( x \) is the policyholder’s age and \( n \) is the final age.
The pricing formula

The zero-value of the GLWB option if the policyholder assumes a static strategy is:

\[ V_0 = \sum_{t=0}^{n-x} \left[ t p_x Ge^{-rt} + t p_x q_{x+t} E_0(QAP_t) \right] \]

Our main contribution lies in bifurcating the GLWB option into a life annuity plus a portfolio of QAP options with decreasing strikes and increasing maturities, where the weights of composition are the deferred probabilities of death.
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Application: The USA market

The guaranteed rates offered for GLWB options are:

<table>
<thead>
<tr>
<th>Guaranteed Rate</th>
<th>Policyholder’s age</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>60-69</td>
</tr>
<tr>
<td>0.06</td>
<td>70-79</td>
</tr>
<tr>
<td>0.07</td>
<td>80-85</td>
</tr>
</tbody>
</table>

*Table 1: The guaranteed rate in the USA Market*

- The average of the sub-account volatility for the universe of VA products is 0.18 (Morningstar Principia Pro)
- We use the latest USA mortality table (Human Mortality Database); we consider x=60 and n=110
- We set $\omega_0 = 100$ and $r = 0.05$
Let $P(\xi_t)$ the probability that $W_t$ hits zero at some point $t < \omega-x$. We compute $P(\xi_t)$ with Monte Carlo simulations if $g$ is 5% and the insurance fee is 60 b.p., which are hypothesis consistent with the current market.

<table>
<thead>
<tr>
<th>$P(\xi)$</th>
<th>$\mu = 4%$</th>
<th>$\mu = 6%$</th>
<th>$\mu = 8%$</th>
<th>$\mu = 10%$</th>
<th>$\mu = 12%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 10%$</td>
<td>52.1%</td>
<td>45.1%</td>
<td>39.2%</td>
<td>30.9%</td>
<td>24.6%</td>
</tr>
<tr>
<td>$\sigma = 15%$</td>
<td>58.5%</td>
<td>52.9%</td>
<td>48%</td>
<td>43.7%</td>
<td>39.5%</td>
</tr>
<tr>
<td>$\sigma = 18%$</td>
<td>61.9%</td>
<td>57%</td>
<td>52.1%</td>
<td>48.1%</td>
<td>44.5%</td>
</tr>
<tr>
<td>$\sigma = 20%$</td>
<td>64.3%</td>
<td>60%</td>
<td>54.9%</td>
<td>51%</td>
<td>48%</td>
</tr>
</tbody>
</table>

Table 1: The probability that the insurer has to pay the guaranteed amount
Monte Carlo simulation

• We have carried out many Monte Carlo Simulation under different scenarios generating for each of them 1000 paths of evolution of the fund

• We have calculated the fair insurance fee: once $r$, $g$ and $\sigma$ have been fixed we have searched the fair value of the fee making equal the initial premium to the zero value of the future cash flows $V_0$. 
Numerical Results

<table>
<thead>
<tr>
<th>Guaranteed Rate</th>
<th>$\sigma = 0.18$</th>
<th>$\sigma = 0.20$</th>
<th>$\sigma = 0.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>43 b.p.</td>
<td>54.5 b.p.</td>
<td>83 b.p.</td>
</tr>
<tr>
<td>0.05</td>
<td>79 b.p.</td>
<td>96.5 b.p.</td>
<td>138 b.p.</td>
</tr>
<tr>
<td>0.06</td>
<td>143 b.p.</td>
<td>167 b.p.</td>
<td>226 b.p.</td>
</tr>
</tbody>
</table>

Table 3: The fair fees for a policyholder aged 60
We pay attention to the fair insurance fee under the hypothesis $g=5\%$ and $\sigma=18\%$, which are consistent with the market. In this case the fair insurance is equal to 79 b.p., whereas the current market fee ranges between 60 b.p. and 70 b.p.

Although there is a common belief that the guarantees embedded in variable annuity policies are overpriced (see Clements (2004)), our analysis shows that the USA market of GLWB is underpriced, in line with the results obtained by Milevsky and Salisbury (2005) for the GMWB market.
Further Steps

Dynamic Strategy

The policyholder can withdraw more or less than G. We analyze two cases:

1) To withdraw less than G or an amount equal to G
2) To withdraw more than G
In contrast to a GMWB, for a GLWB withdrawing nothing or less than G can never be optimal. In fact, for a GMWB this strategy extends the life of guarantee; instead, in a GLWB there is a lifelong guarantee and no adjustments are made for future guaranteed withdrawals. Hence, when the policyholder withdraws less than G, the future guarantees are the same, but their values $F_s$ are lower because $W_t$ is greater. In addition, we have to consider that withdraw less than G involves a smaller $LB_t$ and a greater $DB_t$. However, due to the martingale property of the fund process and the fee deducted from the account value, the expected value of the additional death benefit is never greater than the withdrawal amount. So, the rational policyholder withdraws at least G.
2) In this case, we have to balance two effect: on one hand, withdraw more than \( G \) involves a greater \( LB_t \) and a smaller \( D_t \). However, due to the martingale property of the fund process and the fee deducted from the account value, the expected value of the decrease of the death benefit is never greater than the increase of the withdrawal amount. So, if it is optimal for the policyholder to withdraw more than \( G \), than it has to be optimal to withdraw the most it is possible, i.e. completely surrender the contract.

- Thus, we have to define a decision rule in order to establish the surrender time: for each possible scenario the rational policyholder would withdraw exactly the annual guaranteed amount until the value of the fund less the penalty exceeds the value of future benefits; then, he would surrender the contract.