A new approach to LIBOR modeling

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Outline of the talk

1 Interest rate markets
2 LIBOR model: Axioms
3 LIBOR and Forward price model
4 Affine processes
5 Affine martingales
6 Affine LIBOR model
7 Example: CIR martingales
8 Summary and Outlook
## Derivatives markets

According to the BIS Quarterly Review, the notional amounts outstanding for OTC derivatives in billions of US$, were:

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>Foreign exchange</td>
<td>40,239</td>
<td>48,620</td>
<td>56,238</td>
<td>62,983</td>
</tr>
<tr>
<td>Interest rate</td>
<td>291,115</td>
<td>346,937</td>
<td>393,138</td>
<td>458,304</td>
</tr>
<tr>
<td>Equity-linked</td>
<td>7,488</td>
<td>9,202</td>
<td>8,509</td>
<td>10,177</td>
</tr>
<tr>
<td>Commodity</td>
<td>7,115</td>
<td>7,567</td>
<td>9,000</td>
<td>13,229</td>
</tr>
<tr>
<td>Credit default swaps</td>
<td>28,650</td>
<td>42,580</td>
<td>57,894</td>
<td>57,325</td>
</tr>
<tr>
<td>Unallocated</td>
<td>39,682</td>
<td>61,501</td>
<td>71,225</td>
<td>81,708</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>414,290</td>
<td>516,407</td>
<td>596,004</td>
<td>683,725</td>
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</tbody>
</table>

- Interest rate derivatives are an important part of global financial markets.
Interest rates – Notation

- \( B(t, T) \): time-\( t \) price of a zero coupon bond for \( T \); \( B(T, T) = 1 \);
- \( L(t, T) \): time-\( t \) forward LIBOR for \([T, T + \delta]\);

\[
L(t, T) = \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right)
\]

- \( F(t, T, U) \): time-\( t \) forward price for \( T \) and \( U \); \( F(t, T, U) = \frac{B(t, T)}{B(t, U)} \)

“Master” relationship

\[
F(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)} = 1 + \delta L(t, T)
\]  \( (1) \)
Interest rates evolution

- Evolution of interest rate term structure, 2003 – 2004 (picture: Th. Steiner)
Calibration problems

1. Implied volatilities are constant neither across strike nor across maturity
2. Variance scales non-linearly over time (see e.g. D. Skovmand)
Economic thought dictates that LIBOR rates should satisfy:

**Axiom 1**

The LIBOR rate should be non-negative, i.e. \( L(t, T) \geq 0 \) for all \( t \).

**Axiom 2**

The LIBOR rate process should be a martingale under the corresponding forward measure, i.e. \( L(\cdot, T) \in \mathcal{M}(P_{T+\delta}) \).

Practical applications require:

- Models should be analytically tractable (\( \sim \) fast calibration).
- Models should have rich structural properties (\( \sim \) good calibration).

What axioms do the existing models satisfy?
LIBOR models I (Sandmann et al, Brace et al, . . . , Eberlein & Özkan)

Ansatz: model the LIBOR rate as the exponential of a semimartingale $H$:

$$L(t, T_k) = L(0, T_k) \exp \left( \int_0^t b(s, T_k) ds + \int_0^t \lambda(s, T_k) dH_{T_k}^{T_{k+1}} \right), \quad (2)$$

where $b(s, T_k)$ ensures that $L(\cdot, T_k) \in \mathcal{M}(P_{T_{k+1}})$.

$H$ has the $P_{T_{k+1}}$-canonical decomposition

$$H_t^{T_{k+1}} = \int_0^t \sqrt{c_s} dW_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu^{T_{k+1}})(ds, dx), \quad (3)$$

where the $P_{T_{k+1}}$-Brownian motion is

$$W_t^{T_{k+1}} = W_t^{T_*} - \int_0^t \left( \sum_{l=k+1}^N \frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \lambda(t, T_l) \right) \sqrt{c_s} ds, \quad (4)$$
LIBOR models II

and the $P_{T_{k+1}}$-compensator of $\mu^H$ is

$$\nu^{T_{k+1}}(ds, dx) = \left( \prod_{l=k+1}^{N} \frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \left( e^{\lambda(t, T_l)x} - 1 \right) + 1 \right) \nu^{T^*}(ds, dx).$$
LIBOR models II

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Consequences for continuous semimartingales:

1. caplets can be priced in closed form;
2. swaptions and multi-LIBOR products cannot be priced in closed form;
3. Monte-Carlo pricing is very time consuming $\Rightarrow$ coupled high dimensional SDEs!
LIBOR models II

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$$

Consequences for continuous semimartingales:

1. caplets can be priced in closed form;
2. swaptions and multi-LIBOR products cannot be priced in closed form;
3. Monte-Carlo pricing is very time consuming $\leadsto$ coupled high dimensional SDEs!

Consequences for general semimartingales:

1. even caplets cannot be priced in closed form!
2. ditto for Monte-Carlo pricing.
The equation for the dynamics yield the following matrix for the “dependence” structure.

\[
\begin{array}{ccccccc}
\vdots & L(t, T_{i-1}) & \vdots & \vdots & \vdots & \vdots & \vdots \\
L(t, T_{N-2}) & \ldots & \ldots & L(t, T_{N-2}) \\
L(t, T_{N-1}) & \ldots & \ldots & L(t, T_{N-1}) & L(t, T_{N-1}) \\
\vdots & L(t, T_N) & \ldots & \ldots & L(t, T_N) & L(t, T_N) \\
\vdots & L(t, T_i) & \ldots & \ldots & L(t, T_{N-3}) & L(t, T_{N-2}) & L(t, T_{N-1}) & L(t, T_N)
\end{array}
\]

Bottom line: LIBOR rates we wish to simulate.
LIBOR models IV: Remedies

1. **“Frozen drift” approximation**
   - Brace et al, Schlögl, Glassermann et al . . .
   - replace the random terms by their deterministic initial values:
     \[
     \frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \approx \frac{\delta_l L(0, T_l)}{1 + \delta_l L(0, T_l)} \tag{5}
     \]
   - (+) deterministic characteristics $\Rightarrow$ closed form pricing
   - (-) “ad hoc” approximation, no error estimates, compounded error . . .

2. **Strong Taylor approximation**
   - approximate the LIBOR rates in the drift by
     \[
     L(t, T_l) \approx L(0, T_l) + Y(t, T_l)_+ \tag{6}
     \]
   where $Y$ is the (scaled) exponential transform of $H$ ($Y = \log e^H$)
   - theoretical foundation, error estimates, simpler equations for MC
   - Siopacha and Teichmann; Hubalek, Papapantoleon & Siopacha
Forward price model I (Eberlein & Özkan, Kluge)

**Ansatz:** model the forward price as the exponential of a semimartingale $H$:

$$F(t, T_k) = F(0, T_k) \exp \left( \int_0^t b(s, T_k) ds + \int_0^t \lambda(s, T_k) dH_s^{T_{k+1}} \right), \quad (7)$$

where $b(s, T_k)$ ensures that $F(\cdot, T_k) = 1 + \delta L(\cdot, T_k) \in M(P_{T_{k+1}})$. $H$ has the $P_{T_{k+1}}$-canonical decomposition

$$H_t^{T_{k+1}} = \int_0^t \sqrt{c_s} dW_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu^{T_{k+1}})(ds, dx), \quad (8)$$

where the $P_{T_{k+1}}$-Brownian motion is

$$W_t^{T_{k+1}} = W_t^{T_*} - \int_0^t \left( \sum_{l=k+1}^N \lambda(t, T_l) \right) \sqrt{c_s} ds, \quad (9)$$
and the $P_{T_{k+1}}$-compensator of $\mu^H$ is

$$\nu^{T_{k+1}}(ds, dx) = \exp \left( x \sum_{l=k+1}^{N} \lambda(t, T_l) \right) \nu^{T_\ast}(ds, dx).$$

**Consequences:**

1. the model structure is preserved;
2. caps, swaptions and multi-LIBOR products priced in closed form.

**So, what is wrong?**
Forward price model II

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Negative LIBOR rates can occur!
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**Consequences:**

1. the model structure is preserved;
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So, what is wrong?

Negative LIBOR rates can occur!

**Aim:** design a model where the model structure is preserved and LIBOR rates are positive.

**Tool:** Affine processes on $\mathbb{R}^d_{\geq 0}$. 
Affine processes I

Let $X = (X_t)_{0 \leq t \leq T}$ be a conservative, time-homogeneous, stochastically continuous Markov process taking values in $D = \mathbb{R}^d_{\geq 0}$; and $(P_x)_{x \in D}$ a family of probability measures on $(\Omega, \mathcal{F})$, such that $X_0 = x$, $P_x$-a.s. for every $x \in D$. Setting

$$I_T := \left\{ u \in \mathbb{R}^d : E_x \left[ e^{\langle u, X_T \rangle} \right] < \infty, \text{ for all } x \in D \right\},$$

we assume that

(i) $0 \in I_T^0$;

(ii) the conditional moment generating function of $X_t$ under $P_x$ has exponentially-affine dependence on $x$; i.e. there exist functions $\phi_t(u) : [0, T] \times I_T \to \mathbb{R}$ and $\psi_t(u) : [0, T] \times I_T \to \mathbb{R}^d$ such that

$$E_x \left[ \exp \langle u, X_t \rangle \right] = \exp \left( \phi_t(u) + \langle \psi_t(u), x \rangle \right),$$

for all $(t, u, x) \in [0, T] \times I_T \times D$. 
Affine processes II


$$F(u) := \frac{\partial}{\partial t} \bigg|_{t=0^+} \phi_t(u) \quad \text{and} \quad R(u) := \frac{\partial}{\partial t} \bigg|_{t=0^+} \psi_t(u)$$

exist for all $u \in \mathcal{I}_T$ and are continuous in $u$. Moreover, $F$ and $R$ satisfy Lévy–Khintchine-type equations:

$$F(u) = \langle b, u \rangle + \int_D \left( e^{\langle \xi, u \rangle} - 1 \right) m(d\xi) \quad (13)$$

and

$$R_i(u) = \langle \beta_i, u \rangle + \left\langle \frac{\alpha_i}{2}, u, u \right\rangle + \int_D \left( e^{\langle \xi, u \rangle} - 1 - \langle u, h^i(\xi) \rangle \right) \mu_i(d\xi), \quad (14)$$

where $(b, m, \alpha_i, \beta_i, \mu_i)_{1 \leq i \leq d}$ are admissible parameters.
Affine processes III

The time-homogeneous Markov property of $X$ implies:

$$E_x[\exp\langle u, X_{t+s} \rangle | \mathcal{F}_s] = \exp (\phi_t(u) + \langle \psi_t(u), X_s \rangle),$$

(15)

for all $0 \leq t + s \leq T$ and $u \in \mathcal{I}_T$.

Lemma (Flow property)

The functions $\phi$ and $\psi$ satisfy the semi-flow equations:

$$\phi_{t+s}(u) = \phi_t(u) + \phi_s(\psi_t(u))$$

$$\psi_{t+s}(u) = \psi_s(\psi_t(u))$$

(16)

with initial condition

$$\phi_0(u) = 0 \quad \text{and} \quad \psi_0(u) = u,$$

(17)

for all suitable $0 \leq t + s \leq T$ and $u \in \mathcal{I}_T$. 
Affine processes IV

1. **Affine processes on** $\mathbb{R}$: the admissibility conditions yield

   \[ F(u) = bu + \frac{a}{2}u^2 + \int_{\mathbb{R}} (e^{zu} - 1 - uh(z)) m(dz) \]

   \[ R(u) = \beta u, \]

   for $a \in \mathbb{R}_{\geq 0}$ and $b, \beta \in \mathbb{R}$.

   - Every affine process on $\mathbb{R}$ is an Ornstein–Uhlenbeck (OU) process.

2. **Affine processes on** $\mathbb{R}_{\geq 0}$: the admissibility conditions yield

   \[ F(u) = bu + \int_{D} (e^{zu} - 1) m(dz) \]

   \[ R(u) = \beta u + \frac{\alpha}{2}u^2 + \int_{D} (e^{zu} - 1 - uh(z)) \mu(dz), \]

   for $b, \alpha \in \mathbb{R}_{\geq 0}$ and $\beta \in \mathbb{R}$.

   - There exist affine process on $\mathbb{R}_{\geq 0}$ which are not OU process, e.g. CIR.
Affine LIBOR model: martingales $\geq 1$

Idea:

1. insert an affine process in its moment generating function with inverted time; the resulting process is a martingale;
2. if the affine process is positive, the martingale is greater than one.

Theorem

The process $M^u = (M^u_t)_{0 \leq t \leq T}$ defined by

$$M^u_t = \exp \left( \phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle \right), \quad (18)$$

is a martingale. Moreover, if $u \in \mathcal{I}_T \cap \mathbb{R}_d^{\geq 0}$ then $M_t \geq 1$ a.s. for all $t \in [0, T]$, for any $X_0 \in \mathbb{R}_d^{\geq 0}$. 
Proof.

Using (17) and (15), we have that:

\[
E_x \left[ M^u_T | F_t \right] = E_x \left[ \exp \langle u, X_T \rangle | F_t \right]
\]
\[
= \exp \left( \phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle \right) = M^u_t.
\]

Regarding \( M^u_t \geq 1 \) for all \( t \in [0, T] \): note that if \( u \in \mathcal{I}_T \cap \mathbb{R}_d \), then

\[
M^u_t = E_x \left[ \exp \langle u, X_T \rangle | F_t \right] \geq 1. \tag{19}
\]
Example (Lévy process)

Consider a Lévy subordinator, then

\[
M_t^u = \exp \left( \phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle \right) \\
= \exp \left( (T - t)\kappa(u) + u \cdot X_t \right) \geq 1 \\
= \exp(T \kappa(u)) \exp(u \cdot X_t - t\kappa(u)) \in \mathcal{M},
\]

which is a martingale \( \geq 1 \) for \( u \in \mathbb{R}^d_{\geq 0} \).
Affine LIBOR model: Ansatz

Consider a discrete tenor structure $0 = T_0 < T_1 < T_2 < \cdots < T_N$; discounted bond prices must satisfy:

$$\frac{B(\cdot, T_k)}{B(\cdot, T_N)} \in \mathcal{M}(P_{T_N}), \quad \text{for all } k \in \{1, \ldots, N - 1\}. \quad (21)$$

**Ansatz**

We model quotients of bond prices using the martingales $M$:

$$\frac{B(t, T_1)}{B(t, T_N)} = M_{t}^{u_1}$$

$$\vdots$$

$$\frac{B(t, T_{N-1})}{B(t, T_N)} = M_{t}^{u_{N-1}}, \quad (22)$$

with initial conditions:

$$\frac{B(0, T_k)}{B(0, T_N)} = M_{0}^{u_k}, \quad \text{for all } k \in \{1, \ldots, N - 1\}. \quad (23)$$
Let $L(0, T_1), \ldots, L(0, T_N)$ be a tenor structure of non-negative initial LIBOR rates; let $X$ be an affine process starting at the canonical value 1.

1. If $\gamma_X := \sup_{u \in I_T \cap \mathbb{R}^d \geq 0} E_1 \left[ e^{\langle u, X_T \rangle} \right] > \frac{B(0, T_1)}{B(0, T_N)}$, then there exists a decreasing sequence $u_1 \geq u_2 \geq \cdots \geq u_N = 0$ in $I_T \cap \mathbb{R}^d_{\geq 0}$, such that

$$M_{0}^{u_k} = \frac{B(0, T_k)}{B(0, T_N)}, \quad \text{for all } k \in \{1, \ldots, N\}. \quad (24)$$

In particular, if $\gamma_X = \infty$, then the affine LIBOR model can fit any term structure of non-negative initial LIBOR rates.

2. If $X$ is one-dimensional, the sequence $(u_k)_{k \in \{1, \ldots, N\}}$ is unique.

3. If all initial LIBOR rates are positive, the sequence $(u_k)_{k \in \{1, \ldots, N\}}$ is strictly decreasing.
Affine LIBOR model: forward prices

Forward prices have the following form

\[
\frac{B(t, T_k)}{B(t, T_{k+1})} = \frac{B(t, T_k)}{B(t, T_N)} \frac{B(t, T_N)}{B(t, T_{k+1})} = \frac{M_t^{u_k}}{M_t^{u_{k+1}}}
\]

\[
= \exp \left( \phi_{T_N-t}(u_k) - \phi_{T_N-t}(u_{k+1}) \right)
\]

\[
+ \left\langle \psi_{T_N-t}(u_k) - \psi_{T_N-t}(u_{k+1}), X_t \right\rangle. \tag{25}
\]

Now, \( \phi_t(\cdot) \) and \( \psi_t(\cdot) \) are order-preserving, i.e.

\[
u \leq v \Rightarrow \phi_t(u) \leq \phi_t(v) \text{ and } \psi_t(u) \leq \psi_t(v).
\]

Consequently: positive initial LIBOR rate yields positive LIBOR rates for all times.
Affine LIBOR model: forward measures

Forward measures are related via:

\[
\frac{dP_{T_k}}{dP_{T_{k+1}}}_{\mathcal{F}_t} = \frac{F(t, T_k, T_{k+1})}{F(0, T_k, T_{k+1})} = \frac{B(0, T_{k+1})}{B(0, T_k)} \times \frac{M_{t}^{u_{k}}}{M_{t}^{u_{k+1}}} \tag{26}
\]

or equivalently:

\[
\frac{dP_{T_{k+1}}}{dP_{T_N}}_{\mathcal{F}_t} = \frac{B(0, T_N)}{B(0, T_{k+1})} \times \frac{B(t, T_{k+1})}{B(t, T_N)} = \frac{B(0, T_N)}{B(0, T_{k+1})} \times M_{t}^{u_{k+1}}. \tag{27}
\]

Hence, we can easily see that

\[
\frac{B(\cdot, T_k)}{B(\cdot, T_{k+1})} = \frac{M_{t}^{u_{k}}}{M_{t}^{u_{k+1}}} \in \mathcal{M}(P_{T_{k+1}}), \quad \text{for all } k \in \{1, \ldots, N - 1\}. \tag{28}
\]
Affine LIBOR model: dynamics under forward measures

The moment generating function of $X_t$ under any forward measure is

$$E_{P_{T_{k+1}}} [e^{vX_t}] = M_{0}^{u_{k+1}} E_{P_{T_N}} [M_{t}^{u_{k+1}} e^{vX_t}]$$

$$= \exp \left( \phi_t \left( \psi_{T_N-t}(u_{k+1}) + v \right) - \phi_t \left( \psi_{T_N-t}(u_{k+1}) \right) 
+ \langle \psi_t \left( \psi_{T_N-t}(u_{k+1}) + v \right) - \psi_t \left( \psi_{T_N-t}(u_{k+1}) \right), x \rangle \right).$$

Denote by \( \frac{M_{t}^{u_{k}}}{M_{t}^{u_{k+1}}} = e^{A_k+B_k \cdot X_t} \); the moment generating function is

$$E_{P_{T_{k+1}}} \left[ e^{v(A_k+B_k \cdot X_t)} \right] = \frac{B(0, T_N)}{B(0, T_{k+1})} \times \exp \left( v \phi_{T_N-t}(u_k) + (1 - v) \phi_{T_N-t}(u_{k+1}) 
+ \phi_t \left( v \psi_{T_N-t}(u_k) + (1 - v) \psi_{T_N-t}(u_{k+1}) \right) 
+ \langle \psi_t \left( v \psi_{T_N-t}(u_k) + (1 - v) \psi_{T_N-t}(u_{k+1}) \right), x \rangle \right).$$
Affine LIBOR model

Affine LIBOR model: caplet pricing

We can re-write the payoff of a caplet as follows (here $\mathcal{K} := 1 + \delta K$):

$$
\delta (L(T_k, T_k) - K)^+ = (1 + \delta L(T_k, T_k) - 1 + \delta K)^+
= \left( \frac{M_{T_k}^{u_k}}{M_{T_k}^{u_{k+1}}} - \mathcal{K} \right)^+
= \left( e^{A_k + B_k \cdot X_{T_k}} - \mathcal{K} \right)^+. \quad (31)
$$

Then we can price caplets by Fourier-transform methods:

$$
\mathbb{C}(T_k, K) = B(0, T_{k+1}) E_{P_{T_{k+1}}} \left[ \delta (L(T_k, T_k) - K)^+ \right]
= \frac{\mathcal{K} B(0, T_{k+1})}{2\pi} \int_{\mathbb{R}} \mathcal{K}^{iv-R} \frac{\Lambda_{A_k+B_k \cdot X_{T_k}} (R - iv)}{(R - iv)(R - 1 - iv)} dv \quad (32)
$$

where $\Lambda_{A_k+B_k \cdot X_{T_k}}$ is given by (30).
CIR martingales

The Cox-Ingersoll-Ross (CIR) process is given by

\[ dX_t = -\lambda (X_t - \theta) \, dt + 2\eta \sqrt{X_t} \, dW_t, \quad X_0 = x \in \mathbb{R}_{\geq 0}, \quad (33) \]

where \( \lambda, \theta, \eta \in \mathbb{R}_{\geq 0} \). This is an affine process on \( \mathbb{R}_{\geq 0} \), with

\[ E_x \left[ e^{uX_t} \right] = \exp \left( \phi_t(u) + x \cdot \psi_t(u) \right), \quad (34) \]

where

\[ \phi_t(u) = -\frac{\lambda \theta}{2\eta} \log \left( 1 - 2\eta b(t)u \right) \quad \text{and} \quad \psi_t(u) = \frac{a(t)u}{1 - 2\eta b(t)u}, \quad (35) \]

with

\[ b(t) = \begin{cases} t, & \text{if } \lambda = 0 \\ \frac{1 - e^{-\lambda t}}{\lambda}, & \text{if } \lambda \neq 0 \end{cases}, \quad \text{and} \quad a(t) = e^{-\lambda t}. \]
CIR martingales: closed-form formula I

Definition

A random variable $Y$ has location-scale extended non-central chi-square distribution, $Y \sim \text{LSNC-} \chi^2(\mu, \sigma, \nu, \alpha)$, if $\frac{Y - \mu}{\sigma} \sim \text{NC-} \chi^2(\nu, \alpha)$.

Then we have that

$$X_t \overset{P_{T_N}}{\sim} \text{LSNC-} \chi^2 \left(0, \eta b(t), \frac{\lambda \theta}{\eta}, \frac{xa(t)}{\eta b(t)}\right),$$

and

$$X_t \overset{P_{T_{k+1}}}{\sim} \text{LSNC-} \chi^2 \left(0, \frac{\eta b(t)}{\zeta(t, T_N)}, \frac{\lambda \theta}{\eta}, \frac{xa(t)}{\eta b(t) \zeta(t, T_N)}\right),$$

hence

$$\log \left(\frac{B(t, T_k)}{B(t, T_{k+1})}\right) \overset{P_{T_{k+1}}}{\sim} \text{LSNC-} \chi^2 \left(A_k, \frac{B_k \eta b(t)}{\zeta(t, T_N)}, \frac{\lambda \theta}{\eta}, \frac{xa(t)}{\eta b(t) \zeta(t, T_N)}\right).$$
CIR martingales: closed-form formula II

Then, denoting by \( M = \log \left( \frac{B(T_k, T_k)}{B(T_k, T_{k+1})} \right) \) the log-forward rate, we arrive at:

\[
C(T_k, K) = B(0, T_{k+1}) \mathbb{E}_{P_{T_{k+1}}} \left[ \left( e^M - K \right)^+ \right] 
\]

\[
= B(0, T_{k+1}) \left\{ \mathbb{E}_{P_{T_{k+1}}} \left[ e^M 1\{M \geq \log K\} \right] - K \mathbb{P}_{T_{k+1}} [M \geq \log K] \right\} 
\]

\[
= B(0, T_k) \cdot \chi^2_{\nu, \alpha_1} \left( \frac{\log K - A_k}{\sigma_1} \right) - K^* \cdot \chi^2_{\nu, \alpha_2} \left( \frac{\log K - A_k}{\sigma_2} \right),
\]

(36)

where \( K^* = K \cdot B(0, T_{k+1}) \) and \( \chi^2_{\nu, \alpha}(x) = 1 - \chi^2_{\nu, \alpha}(x) \), with \( \chi^2_{\nu, \alpha}(x) \) the non-central chi-square distribution function,

\[
\nu = \frac{\lambda \theta}{\eta}, \quad \sigma_{1,2} = \frac{B_k \eta b(T_k)}{\zeta_{1,2}}, \quad \alpha_{1,2} = \frac{x a(T_k)}{\eta b(T_k) \zeta_{1,2}},
\]

and

\[
\zeta_1 = 1 - 2 \eta b(T_k) \psi_{T_N-T_k}(u_k), \quad \zeta_2 = 1 - 2 \eta b(T_k) \psi_{T_N-T_k}(u_{k+1}).
\]
Examples of implied volatility surfaces for the CIR martingales.
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Thank you for your attention!