Optimal Portfolio Selection when Stock Price Returns follow Jump-Diffusions

Daniel Michelbrink

School of Mathematical Sciences
The University of Nottingham, UK
pmxdm@nottingham.ac.uk
Introduction and Model Setup

Stochastic Control vs Martingale Approach

Admissible Strategies and Budget Constraints

The Optimisation Problem

Example: Power Utility
Outline

1. Introduction and Model Setup
2. Stochastic Control vs Martingale Approach
3. Admissible Strategies and Budget Constraints
4. The Optimisation Problem
5. Example: Power Utility
An investor wants to invest money and needs some decision criteria.
During the investment he/she also might want to withdraw some money for consumption (e.g. buying a new car)
The investor’s wealth is the collection of all the investor’s assets.
Key question: How to choose an appropriate portfolio?

**Expected Utility Maximisation**

\[
\max \mathbb{E} \left( \int_0^T U_1(t, c_t) dt + U_2(V_T) \right)
\]
Definition

A utility function $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a $C^1$ function such that

1. $U(\cdot)$ is strictly increasing and strictly concave,
2. $U'(\infty) = 0$ and $U'(0+) = \infty$.

- Measures the amount of ‘utility’ an individual gains from some good.
- Utility of money: Money becomes less important the more we have.
Some Example Utility Functions

Logarithmic Utility

\[ U(x) = \log(x) \]

Used to solve the St. Petersburg paradox and important for maximising long term growth rate.

Exponential Utility

\[ U(x) = -e^{-\alpha x} \]

Power Utility

\[ U(x) = \frac{x^p}{p} \]

Used in Merton’s portfolio problem. Markowitz mean-variance problem is equivalent to a quadratic utility problem (p=2)!
The Market Model

**Stock Price Model**

\[
\frac{dS_t}{S_t} = \alpha_t dt + \sigma_t dB_t + \int_E \gamma(t, z) N(dt, dz)
\]

where \( N(dt, dz) \) is a Poisson random measure with intensity \( \nu(dz) \). \( E \) is the set of possible jump sizes and \( \gamma(t, z) \geq -1 \).

**Bond Price Model**

\[
\frac{dS^0_t}{S^0_t} = r_t dt
\]

**Wealth Process**

\[
dV_t = r_t V_t^- dt - c_t dt + \pi_t V_t^- \left( (\alpha_t - r_t) dt + \sigma_t dB_t + \int_E \gamma(t, z) N(dt, dz) \right)
\]

\[
V_0 = x \quad \text{(initial endowment)}
\]
Outline

1. Introduction and Model Setup
2. Stochastic Control vs Martingale Approach
3. Admissible Strategies and Budget Constraints
4. The Optimisation Problem
5. Example: Power Utility
How to solve the problem?

1. **Stochastic Control**
   - Merton (1971): Geometric Brownian motion model
   - Karatzas, Lehoczky, Sethi, Shreve (1990): Geometric Brownian motion model
   - Framstad, Øksendal, Sulem (1999): Jump-Diffusion model

2. **Martingale Approach**
   - Bardhan, Chao (1995): Jump-Diffusion model in complete market
   - Barbachan (2003): Lévy process model
How does stochastic control work?

Hamilton-Jacobi-Bellman Equation

Solve

\[ A^{\pi, c} \phi(t, v) + U_1(t, c) \leq 0 \quad t \leq T, \]
\[ A^{\pi^*, c^*} \phi(t, v) + U_1(t, c^*) = 0 \quad t \leq T, \]
\[ \phi(T, v) = U_2(v). \]

where \( A^{\pi, c} \) is the generator of the wealth process, e.g. in case of a Itô-Lévy model

\[ A^{\pi, c} \phi(s, x) = \partial_t \phi + (\pi(\alpha - r)x - c) \partial_x \phi \]
\[ + \frac{1}{2} \pi^2 x^2 \sigma^2 \partial_{xx} \phi + \int_E (\phi(s, x + \pi x z) - \phi(s, x)) \nu(dz) \]

Recall:

\[ Af(s, x) := \lim_{t \downarrow 0} \frac{\mathbb{E}^{s,x}(f(t, X_t)) - f(s, x)}{t} \]
How does the martingale approach work?

Using tools from stochastic calculus to find a solution.

**Itô Formula for Jump-Diffusions**

Let \( f \in C^{1,2}([0, \infty) \times \mathbb{R}) \) and \( X(t) \) as above. Then

\[
    df(t, X) = \frac{\partial}{\partial t} f(t, X) dt + \alpha(t, X) \frac{\partial}{\partial x} f(t, X) dB + \frac{1}{2} \sigma(t, X)^2 \frac{\partial^2}{\partial x^2} f(t, X) dt \\
    + f(t, X + \Delta X) - f(t, X)
\]

**Martingale Representation Theorem**

Any \((P, \mathcal{F}_t)\)-martingale \( M(t) \) has the representation

\[
    M(t) = M(0) + \int_0^t a_D(s) dB(s) + \int_0^t \int_E a_J(s, y) \tilde{N}(ds, dy)
\]

where \( a_D \) is predictable and square integrable and \( a_J \) is a \( \mathcal{F}_t \)-predictable, marked process, that is integrable with respect to \( \nu(ds, dy) \).
Change of Probability Measure (Girsanov’s Theorem)

We can change the probability measure from $\mathbb{P}$ to $Q$ by using a martingale $Z$ as Radon-Nikodym density:

$$\frac{dQ}{d\mathbb{P}} = Z_T.$$

In our case $Z$ will be

$$dZ_t = Z_t \theta^D_t dB - Z_t \int_E (1 - \theta^J(t, y)) \tilde{N}(dt, dy); \quad Z_0 = 1.$$

New Brownian motion and Poisson Measure

Under the new measure $Q$ we have

- $dB^Q = dB - \theta^D dt$ is a $Q$-Brownian motion
- $\tilde{N}^Q(dt, dz) = N(dt, dz) - \theta^J(z) \nu(dz) dt$ is a $Q$-Poisson random measure.
Measure Change and the Wealth Process

Wealth Process under $P$

\[
dV_t = r_t V_t dt - c_t dt + \pi_t V_t \left( (\alpha_t - r_t) dt + \sigma_t dB_t + \int_E \gamma(t, y) N(dt, dy) \right)
\]

Discounted Wealth Process under $Q$

\[
d\frac{V_t}{S_0^t} = -\frac{c_t}{S_0^t} dt + \pi_t \frac{V_t}{S_0^t} \left( \sigma_t dB_t^Q + \int_E \gamma(t, y) \tilde{N}^Q(dt, dy) \right)
\]

Conditions on $\theta^D$ and $\theta^J$

\[
\alpha_t - r_t + \sigma_t \theta_t^D + \int_E \gamma(t, y) \theta^J(t, y) \nu(dy) = 0
\]
Outline

1. Introduction and Model Setup
2. Stochastic Control vs Martingale Approach
3. Admissible Strategies and Budget Constraints
4. The Optimisation Problem
5. Example: Power Utility
Admissible Investment-Consumption Strategies

A pair \((\pi, c)\) is called admissible, write \((\pi, c) \in \mathcal{A}\), if

\[ V_t \geq 0, \quad t \in [0, T]. \]

Define the state price density as \(H(t) := Z(t)/S^0(t)\), where \(Z(t)\) is the process from the measure change \(\frac{dQ}{dP}|_{\mathcal{F}_t} = Z_t\).

Budget Constraint

If \((\pi, c) \in \mathcal{A}\). Then

\[
\mathbb{E}\left( \int_0^T H_t c_t dt + H_T V_T \right) \leq x
\]
Reversing the Budget Constraint

For the geometric Brownian motion model or in the case that the stock price also has a Poisson jump with a single jump size it is possible to reverse the budget constraint in the following way:

**Lemma**

Let \( c \) be a consumption process and \( Y \) a \( \mathcal{F}_T \) measurable random variable such that

\[
\mathbb{E}\left( \int_0^T H_t c_t dt + H_T Y \right) = x
\]

Then there exists a portfolio process \( \pi \) such that \( (\pi, c) \in \mathcal{A} \) and \( Y = V(T) \).

However, in more sophisticated models like for jump-diffusions it is more difficult to prove the Lemma.
Let $\phi(t)$ denote the amount of money invested into the stock. Then we need that

$$
\phi(t)\sigma(t) = \frac{1}{H(t)} \left( (J(t) - M(t))\theta_D(t) + a_D(t) \right),
$$

$$
\phi(t)\gamma(t, y) = \frac{1}{H(t)} \left( (M(t) - J(t)) \frac{1 - \theta_J(t, y)}{\theta_J(t, y)} + \frac{a_J(t, y)}{\theta_J(t, y)} \right), \quad \forall y \in E,
$$

where

$$
M(t) := \mathbb{E} \left( \int_0^T H(s)c(s)ds + H(T)Y \mid \mathcal{F}_t \right)
$$

$$
J(t) := \int_0^t H(s)c(s)ds
$$

and $a_D$ and $a_J$ are the martingale representation coefficient of $M(t)$. 
Conditions on the trading strategy (2)

We have 2 equations for 3 unknown: $\phi$, $\theta_D$, and $\theta_J$

\[
\phi(t)\sigma(t) = \frac{1}{H(t)}\left((J(t) - M(t))\theta_D(t) + a_D(t)\right),
\]

\[
\phi(t)\gamma(t, y) = \frac{1}{H(t)}\left((M(t) - J(t))\frac{1 - \theta_J(t, y)}{\theta_J(t, y)} + \frac{a_J(t, y)}{\theta_J(t, y)}\right), \quad \forall y \in E,
\]

However, we also have the following condition on $\theta_D$ and $\theta_J$:

\[
\alpha(t) - r(t) + \sigma(t)\theta_D(t) + \int_E \gamma(t, y)\theta_J(t, y)v(dy) = 0
\]
Outline

1. Introduction and Model Setup
2. Stochastic Control vs Martingale Approach
3. Admissible Strategies and Budget Constraints
4. The Optimisation Problem
5. Example: Power Utility
The Problem

Optimisation Problem and admissible Strategies

Maximise

\[ J(x; \pi, c) = \mathbb{E}\left( \int_0^T U_1(s, c_s) ds + U_2(V_T) \right) \]

with respect to \( \pi \) and \( c \) that satisfy

\[ \tilde{A}(x) := \left\{ (\pi, c) \in A(x) : \mathbb{E}\left( \int_0^T U_1(s, c_s) - ds + U_2(V_T) \right) > -\infty \right\} \]

Optimal Value Function

For the above problem, we define the optimal value function as

\[ \Phi(x) := \sup_{(\pi, c) \in \tilde{A}(x)} J(x; \pi, c) = J(x, \pi^*, c^*) \]
Solving the Optimisation Problem

Transfer to Constraint Optimisation

\[ \max_{(\pi, c) \in \tilde{A}} \mathbb{E}\left( \int_0^T U_1(s, c_s)ds + U_2(V_T) \right) \quad \text{s.t.} \]
\[ \mathbb{E}\left( \int_0^T H(u)c_u du + H(T)V(T) \right) = x \]

Use constraint optimisation techniques: Lagrange multiplier

Lagrange Multiplier

\[ \mathcal{L}(c, Y, \lambda) = \mathbb{E}\left( \int_0^T U_1(s, c_s)ds + U_2(V_T) \right) \]
\[ -\lambda \left[ \mathbb{E}\left( \int_0^T H(t)c_t dt + H(T)V(T) \right) - x \right] \]
The Solution

Optimal Values

Optimal consumption: \[ c^*(t) = \tilde{l}_1(t, H(t)X^{-1}(x)) \]

Optimal terminal wealth: \[ V^*(T) = \tilde{l}_2(H(T)X^{-1}(x)) \]

Optimal value: \[ \Phi(x) = \mathcal{Y}(X^{-1}(x)) \]

where \( \mathcal{X} \) and \( \mathcal{Y} \) are some known functions. \( \tilde{l}_1 \) and \( \tilde{l}_2 \) are the inverse of \( U'_1 \) and \( U'_2 \) respectively.
1. Introduction and Model Setup

2. Stochastic Control vs Martingale Approach

3. Admissible Strategies and Budget Constraints

4. The Optimisation Problem

5. Example: Power Utility
Example: Maximising with discounted Power Utility

Portfolio Wealth Process:

\[
dV(t) = \left( r_t V(t) - c_t \right) dt \\
+ \pi(t) V(t) \left( \alpha_t - r_t \right) dt + \sigma_t dB_t + \int_E \gamma(t, y) N(dt, dy)
\]

The maximisation Problem

\[
\max \quad \frac{1}{\beta} \mathbb{E} \left( \int_0^T e^{-\delta_1 t} c_t^\beta dt + e^{-\delta_2 T} V_T^\beta \right).
\]

with \( \delta_i \geq 0, i = 1, 2 \), and \( \beta < 1 \). Note \( \beta = 0 \) leads to log utility.
The trading strategy and the market prices of risk

Conditions on $\phi, \theta_D, \theta_J$

$$
\phi(t)\sigma(t) = \frac{1}{H(t)}\left(\left(J(t) - M(t)\right)\theta_D(t) + a_D(t)\right),
$$
$$
\phi(t)\gamma(t, y) = \frac{1}{H(t)}\left(\left(M(t) - J(t)\right)\frac{1 - \theta_J(t, y)}{\theta_J(t, y)} + \frac{a_J(t, y)}{\theta_J(t, y)}\right), \quad \forall y \in E,
$$
$$
\alpha(t) - r(t) + \sigma(t)\theta_D(t) + \int_E \gamma(t, y)\theta_J(t, y)\nu(dy) = 0
$$

Conditions on $\pi, \theta_D, \theta_J$ for Power Utility Problem

$$
\pi(t)\xi(t) = \frac{1}{\beta - 1}\theta_D(t), \quad (1)
$$
$$
\pi(t)\gamma(t, y) = \theta_J(t, y)^{1/\beta - 1} - 1, \quad y \in E, \quad (2)
$$
$$
\alpha(t) - r(t) + \sigma(t)\theta_D(t) + \int_E \gamma(t, y)\theta_J(t, y)\nu(dy) = 0 \quad (3)
$$
Optimal trading strategy

The optimal strategy $\pi(t)$ satisfies

$$\alpha(t) - r(t) + \sigma(t)^2(\beta - 1)\pi(t) + \int_E \gamma(t, y)(1 + \pi(t)\gamma(t, y))^{1-\beta} \nu(dy) = 0.$$ 

Note that a similar result can be found in Framstad, Øksendal, Sulem (1999).

