Equilibrium in Incomplete Markets under Translation Invariant Preferences

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March 2009
We consider a dynamic partial equilibrium model in discrete time.

\( \mathcal{A} \) denotes a finite set of economic agents.

The agents can lend and borrow from the money market account at the same rate and invest in a financial market that comprises a finite number of stocks and securities. Stock prices follow an exogenous dynamics while securities prices will be determined endogenously by an equilibrium condition.

Uncertainty is represented by a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and the flow of information is captured by a filtration \( (\mathcal{F}_t) \).
Setup

- The **stocks** are liquidly traded assets whose price dynamics is unaffected by the agent’s demand and follows exogenous stochastic processes \((S_t^i)_{t=0}^{T} \) with \( S_t \in L^\infty(\mathcal{F}_t)^J \).

- The **securities** may be viewed as structured financial products that are only traded by the agents that belong to the set \( A \). The k-th security is fixed supply \( n^k \in \mathbb{R} \) with price \( R_T^k = R^k \) at time \( T \in \mathbb{N} \) for \( R^k_T \in L^\infty(\mathcal{F}_T)^K \). Its price process \((R_t^k)_{t=0}^{T-1}\), will be derived endogenously by the market clearing condition of zero total excess demand.

- Each agent \( a \in A \) is **endowed** with \( H^a \in L^0(\mathcal{F}) \), which is bounded from below.
Preferences

Each agent $a \in A$ optimizes at time $t \in \{0, \ldots, T\}$ a translation invariant preference functional $U^a_t : L^0(\mathcal{F}) \to \bar{L}^0(\mathcal{F}_t)$ with the following properties:

(N) Normalization: $U^a_t(0) = 0$.

(M) Monotonicity: $U^a_t(X) \geq U^a_t(Y)$ for all $X, Y \in L^0(\mathcal{F})$ such that $X \geq Y$.

(T) Translation property: $U^a_t(X + Z) = U^a_t(X) + Z$ for all $X \in L^0(\mathcal{F})$ and $Z \in L^0(\mathcal{F}_t)$.

(C) $L^0(\mathcal{F}_t)$-concavity: $U^a_t(\lambda X + (1 - \lambda)Y) \geq \lambda U^a_t(X) + (1 - \lambda)U^a_t(Y)$ for all $X, Y \in L^0(\mathcal{F})$ and $\lambda \in L^0(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$.

Moreover, $(U^a_t)_{t=0}^T$ is time-consistent, i.e.,

$$U^a_t(X) = U^a_t(U^a_{t+1}(X))$$

for all $t = 0, \ldots, T - 1$. 
Preferences

- A convenient way of defining time-consistent preference functions is to determine them via one-time-step generators $u_t : \overline{L}^0(\mathcal{F}_{t+1}) \to \overline{L}^0(\mathcal{F}_t)$, $t = 0, \ldots, T - 1$. The generators specify the local preference structure of the time consistent utility function

$$U_t(X) := u_t \circ u_{t+1} \circ \cdots \circ u_{T-1}(X), \quad X \in L^0(\mathcal{F}).$$

- If $(u_t)_{t=0}^{T-1}$ satisfy (N), (M), (T) and (C), then the respective $(U_t)_{t=0}^{T}$ satisfy (N), (M), (T) and (C).
Examples

- Optimized certainty equivalents (OCE): (Ben-Tal, Teboulle)

\[ u_t(X) = \text{ess inf}_{\xi \in \mathcal{P}_t} \mathbb{E} [\xi X + G(\xi) \mid \mathcal{F}_t], \quad X \in L^0(\mathcal{F}_{t+1}) \]

\[ \mathcal{P}_t = \{\xi \in L^1_+(\mathcal{F}) \mid \mathbb{E} [\xi \mid \mathcal{F}_t] = 1\} \] denotes the set of \( \mathcal{F}_t \)-conditional probability densities and \( G : [0, \infty) \rightarrow \mathbb{R} \) is a convex function that is differentiable on \((0, \infty)\) and satisfies \( \lim_{x \to \infty} \frac{G(x)}{x} = +\infty \).

Entropic utility function:

\[ G(x) = \frac{1}{\gamma} x \log x \]

Monotone mean variance preferences:

\[ G(x) := \frac{1}{2\gamma}(x^2 - 1) \]
Examples

- Generalized mean-variance preferences:

\[ u_t(X) = \lambda \mathbb{E}[X | \mathcal{F}_t] - (1 - \lambda) \rho_t(X), \quad X \in L^0(\mathcal{F}_{t+1}) \]

generalize the classical mean variance preferences. \( \lambda \in (0, 1) \) is the risk aversion coefficient and \( \rho_t \) is a conditional convex risk measure in the sense of Artzner, Delbaen, Eber and Heath and Föllmer and Schied.
The agents can invest in a financial market consisting of \( J \) liquidly traded stocks \((S_t)_{t=0}^{T}\) with

\[
S_t = (S_1^t, \ldots, S_J^t) \in L^\infty(\mathcal{F}_t)^J
\]

and in \( K \) securities with terminal value

\[
R = (R^1, \ldots, R^K) \in L^\infty(\mathcal{F})^K.
\]

**Goal:** Find an adapted equilibrium price process \((R_t)_{t=0}^{T}\) for these securities with \( R_T = R \).
An equilibrium consists of a bounded, $\mathbb{R}^K$-valued, adapted process $(R_t)_{t=0}^T$ with $R_T = R$ together with predictable trading strategies $\hat{\vartheta}^a = (\hat{\vartheta}^{a,1}, \hat{\vartheta}^{a,2})$, $a \in \mathcal{A}$, such that:

**Individual optimality:**

$$U^a_t \left( H^a + \sum_{s=1}^T \hat{\vartheta}_s^{a,1} \cdot \Delta S_s + \hat{\vartheta}_s^{a,2} \cdot \Delta R_s \right)$$

$$\geq U^a_t \left( H^a + \sum_{s=1}^t \hat{\vartheta}_s^{a,1} \cdot \Delta S_s + \hat{\vartheta}_s^{a,2} \cdot \Delta R_s + \sum_{s=t+1}^T \vartheta_s^{a,1} \cdot \Delta S_s + \vartheta_s^{a,2} \cdot \Delta R_s \right)$$

for all $t = 0, \ldots, T - 1$ and possible continuation strategies $(\vartheta_s^{a,1})_{s=t+1}^T$, $(\vartheta_s^{a,2})_{s=t+1}^T$

**Market clearing:** $\sum_{a \in \mathcal{A}} \hat{\vartheta}_t^{a,2} = n$ for all $t = 1, \ldots, T$
Equilibrium

- **One-period-model, complete market**: Let $u^a : L^\infty \to \mathbb{R}$, $a \in A$ satisfy (M), (T), (C)
  
  $X = \sum_{a \in A} H^a$ is the aggregate endowment

- $(\hat{X}^a)_{a \in A}$ is Pareto optimal iff
  
  $\square_{a \in A} u^a(X) := \sup \sum_{a \in A} x^a = X \sum_{a \in A} u^a(X) = \sum_{a \in A} u^a(\hat{X}^a)$.
  
  (Jouini, Schachermayer, Touzi)

- Every Equilibrium is Pareto optimal and (up to cash-transfers) vice versa. $\mu$ is equilibrium pricing rule $\iff \mu \in \partial \square_{a \in A} u^a(X)$.
The equilibrium is constructed by **backward induction**. Suppose that $R_{t+1}, R_{t+2}, \ldots, R_T = R$ are already constructed. Define

$$H_{t+1}^a := \operatorname{ess} \sup_{\vartheta^a} U_{t+1}^a \left( H^a + \sum_{s=t+2}^{T} \vartheta_s^{a,1} \cdot \Delta S_s + \vartheta_s^{a,2} \cdot \Delta R_s \right).$$

Then define a **representative agent** with preference functional $\hat{U}_t : L^0(\mathcal{F}_{t+1}) \rightarrow \overline{L}^0(\mathcal{F}_t)$ given by

$$\hat{U}_t(X) := \operatorname{ess} \sup_{\vartheta^a \in L^0(\mathcal{F}_t)^{|\mathcal{A}|+K}} \sum_{a \in \mathcal{A}} U_t^a \left( \frac{X}{|\mathcal{A}|} + H_{t+1}^a + \vartheta_s^{a,1} \cdot \Delta S_{t+1} + \vartheta_s^{a,2} \cdot R_{t+1} \right).$$
The equilibrium is constructed by backward induction. Suppose that $R_{t+1}, R_{t+2}, \ldots, R_T = R$ are already constructed. Define

$$H_{t+1}^a := \text{ess.sup}_{\varphi^a} U_{t+1}^a \left( H_{t+2}^a + \varphi_{t+1}^{a,1} \cdot \Delta S_{t+1} + \varphi_{t+1}^{a,2} \cdot \Delta R_{t+1} \right).$$

Then define a representative agent with preference functional $\hat{U}_t : L^0(\mathcal{F}_{t+1}) \rightarrow \overline{L}^0(\mathcal{F}_t)$ given by

$$\hat{U}_t(X) := \text{ess sup}_{\varphi^a \in L^0(\mathcal{F}_t)^{J+K}} \sum_{a \in A} U_t^a \left( \frac{X}{|A|} + H_{t+1}^a + \varphi_{t+1}^{a,1} \cdot \Delta S_{t+1} + \varphi_{t+1}^{a,2} \cdot R_{t+1} \right).$$

$\sum_{a \in A} \varphi_{t+1}^{a,2} = 0$
Characterization of equilibrium

- We restrict our attention to the span of $R_{t+1}$:

$$
\hat{u}_t(x) := \hat{U}_t(x \cdot R_{t+1}) = \text{ess sup} \sum_{a \in A} U_t^a \left( H_{t+1}^a + \vartheta_{t+1}^a \cdot \Delta S_{t+1} + \vartheta_{t+1}^a \cdot R_{t+1} \right)
$$

Note that $\hat{u}_t : L^0(\mathcal{F}_t)^K \rightarrow \bar{L}^0(\mathcal{F}_t)$ is $L^0(\mathcal{F}_t)$-concave, i.e.,

$$
\hat{u}_t(\lambda x + (1 - \lambda)y) \geq \lambda \hat{u}_t(x) + (1 - \lambda)\hat{u}_t(y)
$$

for all $x, y \in L^0(\mathcal{F}_t)^K$ and $\lambda \in L^0(\mathcal{F}_t)$ with $0 \leq \lambda \leq 1$. 
Characterization of equilibrium

- We define the conditional concave conjugate 
  \( \hat{u}_t^* : L^0(\mathcal{F}_t)^K \to \overline{L}^0(\mathcal{F}_t) \) by
  \[
  \hat{u}_t^*(y) = \text{ess inf}_{x \in L^0(\mathcal{F}_t)^K} \{ x \cdot y - \hat{u}_t(x) \}
  \]

- \( y \in L^0(\mathcal{F}_t)^K \) is a conditional supergradient of \( \hat{u}_t \) at 
  \( x \in L^0(\mathcal{F}_t)^K \) (i.e. \( y \in \partial \hat{u}_t(x) \)) if
  \[
  \hat{u}_t(x) \in L^0(\mathcal{F}_t) \quad \text{and} \quad \hat{u}_t(z) - \hat{u}_t(x) \leq y \cdot (z - x) \quad \text{for all} \quad z \in L^0(\mathcal{F}_t)^K.
  \]

- Filipović, Kupper, Vogelpoth study \( L^0(\mathcal{F}_t) \)-convex duality in 
  \( L^0(\mathcal{F}_t) \)-modules.
Characterization of equilibrium

Theorem

A bounded, \( \mathbb{R}^K \)-valued, adapted process \((R_t)_{t=0}^T\) satisfying \(R_T = R\) together with trading strategies \((\hat{\vartheta}^a_t)_{t=1}^T\), form an equilibrium if and only if for all \(t = 0, \ldots, T - 1\) the following three conditions hold:

(i) \(R_t \in \partial \hat{u}_t(n)\)

(ii) \(\sum_{a \in A} U^a_t(H^a_{t+1} + \hat{\vartheta}^{a,1}_{t+1} \cdot \Delta S_{t+1} + \hat{\vartheta}^{a,2}_{t+1} \cdot R_{t+1}) = \hat{u}_t(n)\)

(iii) \(\sum_{a \in A} \hat{\vartheta}^{a,2}_{t+1} = n\)
Existence of Equilibrium

**Assumption (A):** for all $t = 0, \ldots, T - 1$, all $V^a \in L^0(\mathcal{F}_{t+1})$ bounded below and each $W \in L^\infty(\mathcal{F}_{t+1})^K$, the following essential supremum is attained:

$$\text{ess.sup}_{\vartheta^a \in L^0(\mathcal{F}_t)^{J+K}} \sum_{a \in \mathcal{A}} U^a_t \left( V^a + \vartheta^{a,1} \cdot \Delta S^j_{t+1} + \vartheta^{a,2} \cdot W \right).$$

$$\sum_{a \in \mathcal{A}} \vartheta^{a,2} = n$$

- Under Assumption (A) the convolution $\hat{u}_t$ of $(U^a_t)_{a \in \mathcal{A}}$ over the subspace $\text{span}(\Delta S_{t+1}, R_{t+1})$ is exact.
Existence of Equilibrium

Theorem

(i) Under assumption (A) an equilibrium exists.

(ii) If all $U_0^a$ are sensitive to large losses and the liquidly traded stocks $S^1, \ldots, S^J$ satisfy the (NA) condition then assumption (A) holds.

- $U_0^a$ is sensitive to large losses if for all $X \in L^0(\mathcal{F})$ with $\mathbb{P}[X < 0] > 0$ it follows $\lim_{\lambda \to \infty} U_0^a(\lambda X) = -\infty$.

- (OCE) are sensitive to large losses.
Uniqueness

$U^a_t$ satisfies (D) if there is $\nabla U^a_t(X) \in L^1(F_{t+1})$ such that

$$\lim_{\epsilon \downarrow 0} \frac{U^a_t(X + \epsilon Z) - U^a_t(X)}{\epsilon} = \mathbb{E} [\nabla U^a_t(X) Z \mid F_t]$$

for all $X \in \text{dom} \ U^a_t$ and $Z \in L^\infty(F_{t+1})$.

**Theorem**

Assume that $U^a_t$ satisfies (D) for all $a \in \mathbb{A}$. Then there can exist at most one equilibrium price process $(R_t)$. Further

$$\nabla \hat{U}_t(n \cdot R_{t+1}) = \frac{1}{|A|} \sum_{a \in \mathbb{A}} \nabla U^a_t \left( H^a_{t+1} + \hat{\vartheta}^{a,1}_{t+1} \cdot \Delta S_{t+1} + \hat{\vartheta}^{a,2}_{t+1} \cdot R_{t+1} \right)$$

and $\frac{dQ}{dP} = \prod_{t=1}^T \nabla \hat{U}_t (n \cdot R_{t+1})$ is a martingale measure for $(S,R)$. 
Random factors

- In order to compute the equilibrium we have to introduce some random factors.

- The random factors are
  - a finite number of random walks
  - a finite number of Brownian motions
which satisfy a certain predictable representation property.

**Idea:** Write the utility functions in terms of the factors → (discrete) BSDE for the utilities and the equilibrium.
Consider $d' \geq 1$ independent random walks $b^1_t, \ldots, b^{d'}_t$ and let $(\mathcal{F}_t)_{t=0}^T$ be the filtration generated by $b^1_t, \ldots, b^{d'}_t$.

$\mathcal{F}_t$ has $2^{d't}$ atoms with equal measure and there exist $b^{d'+1}_t, \ldots, b^d_t$ for $d = 2^{d'} - 1$ such that

$$
\mathbb{E}[\Delta b^i_{t+1} \Delta b^l_{t+1} | \mathcal{F}_t] = 0 \quad \text{for all} \quad i \neq l.
$$

Every $X \in L^0(\mathcal{F}_{t+1})$ can uniquely be written as

$$
X = \mathbb{E}[X | \mathcal{F}_t] + \pi_t(X) \cdot \Delta b_{t+1}
$$

for a mapping $\pi_t : L^0(\mathcal{F}_{t+1}) \to L^0(\mathcal{F}_t)^d$. 
Equilibrium in a random walk framework

For the preference functionals $U_t^a$ one has for $X \in L^0(\mathcal{F}_{t+1})$

$$U_t^a(X) = U_t^a(\mathbb{E}[X | \mathcal{F}_t] + \pi_t(X) \cdot \Delta b_{t+1}) = \mathbb{E}[X | \mathcal{F}_t] - f_t^a(\pi_t(X))$$

for the $\mathcal{F}_t$-convex function

$$f_t^a(z) := -U_t^a(z \cdot \Delta b_{t+1}) : L^0(\mathcal{F}_t)^d \to L^0(\mathcal{F}_t).$$

Hence,

$$U_{t+1}^a(X) - U_t^a(X) = f_t^a(Z_{t+1}^a) + Z_{t+1}^a \cdot \Delta b_{t+1} \quad \text{with} \quad U_T^a = X.$$
Suppose conditions (A) and (D) are satisfied. We consider one stock and one security.

Define $Z_{t+1}^S := \pi_t(S_{t+1})$, $Z_{t+1}^R := \pi_t(R_{t+1})$, $Z_{t+1}^a := \pi(H_{t+1}^a)$ and $Z_{t+1} = (Z_{t+1}^S, Z_{t+1}^R, Z_{t+1}^a, a \in \mathbb{A}) \in L^0(\mathcal{F})(2^{\mathbb{A}})^d$.

The convolution becomes

$$f_t(Z, Z_{t+1}) = \operatorname{ess inf}_{\vartheta^a, \sum_a \vartheta^{a,2} = 0} \sum_{a \in \mathbb{A}} f_t^a \left( \frac{Z}{|\mathbb{A}|} + Z_{t+1}^a + \vartheta^{a,1} Z_{t+1}^S + \vartheta^{a,2} Z_{t+1}^R \right) - \vartheta^{a,2} \mathbb{E}[\Delta S_{t+1} \mid \mathcal{F}_t].$$

$f_t$ as a function of $Z \in L^0(\mathcal{F}_t)^d$ is conditionally differentiable and $\nabla^Z f_t(nZ_{t+1}^R, Z_{t+1}) \in L^0(\mathcal{F})^d$ describes the equilibrium market prices of risks with respect to the components $\Delta b_{t+1}$. 
Equilibrium in a random walk framework

- Introduce the drivers

\[ g^S_t(Z_{t+1}) := Z^S_{t+1} \cdot \nabla^Z f_t(nZ^R_{t+1}, Z_{t+1}) \]
\[ g^R_t(Z_{t+1}) := Z^R_{t+1} \cdot \nabla^Z f_t(nZ^R_{t+1}, Z_{t+1}) \]
\[ g^a_t(Z_{t+1}) := f^a_t(Z^a_{t+1} + \hat{\varphi}_{a,1}^1 Z^S_{t+1} + \hat{\varphi}_{a,2}^R Z^R_{t+1}) - \hat{\varphi}_{t+1}^a g^S_t(Z_{t+1}) - \hat{\varphi}_{t+1}^a g^R_t(Z_{t+1}). \]

**Theorem**

The processes \((S_t), (R_t)\) and \((H^a_t)\) satisfy

\[ \Delta S_{t+1} = g^S_t(Z_{t+1}) + Z^S_{t+1} \cdot \Delta b_{t+1} \]
\[ \Delta R_{t+1} = g^R_t(Z_{t+1}) + Z^R_{t+1} \cdot \Delta b_{t+1} \]
\[ \Delta H^a_{t+1} = g^a_t(Z_{t+1}) + Z^a_{t+1} \cdot \Delta b_{t+1} \]

with terminal conditions \(R_T = R\) and \(H^a_T = H^a\).
We consider two agents $\mathbb{A} := \{a, b\}$.

$$U^i_t(X) = \frac{1}{\gamma^i} \log \mathbb{E}[\exp(-\gamma^i X) \mid \mathcal{F}_t]$$

Then, $U^i_t(X) = \mathbb{E}[X \mid \mathcal{F}_t] - hf^i_t(\pi^i_t(X))$, where $f^i_t(x) = \frac{\gamma^i}{2} \|x\|^2$. 
Example

- We consider four risk factors \((W^S, W^R, W^a, W^b)\).

- There is a "stock" with price dynamics \(\frac{\Delta S_{t+1}}{S_t} = \theta^S + \Delta W^S_{t+1}\).

- The risk bond pays \(R = h(W^R_T)\), where \(h : \mathbb{R} \to \mathbb{R}\).

- The initial endowments are of the form
  \[ H^i = h^i(\log S_T, W^R_T + W^i_T) \]
  for some idiosyncratic risk factors \(W^i, i = a, b\).
Example

- If \( h, h' \) are regular enough it is possible to aggregate the optimal utility of the agents and introduce a representative agent.

Lemma

*If the process \( \tilde{Z}^R \) does not vanish then the backward equation*

\[
\tilde{H}_t = \mathbb{E}_P \left[ \tilde{H}^{a,b}_{t+1} \mid \mathcal{F}_t \right] - \tilde{g}(\tilde{Z}_t) \quad \text{with} \quad \tilde{H}_T = \gamma \left( H^a + H^b + H \right)
\]

*with*

\[
\tilde{Z}^j_t = \pi^j_t(\tilde{H}_{t+1}) \quad \text{for} \quad j \in \{S, R, a, b\}.
\]

\[
\tilde{g}(\tilde{Z}_{t+1}) = -\frac{[\theta^S_t]^2}{2} + \theta^S_t \tilde{Z}^S_{t+1} + \frac{[\tilde{Z}^R_{t+1}]^2}{2} + \frac{[\tilde{Z}^{a,a}_{t+1}]^2}{2} + \frac{[\tilde{Z}^{b,b}_{t+1}]^2}{2}.
\]

*describes the approximate evolution of the rep. agent’s utility.*
Conclusion

- We give characterization, existence and uniqueness results for equilibria in incomplete markets for translation invariant preferences.

- The equilibrium can be constructed by backward recursion and can be represented in terms of (discrete) BSDEs.
Thank you!