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# Variational inequalities driven by Lévy processes

**Adrian Zălinescu**

“O. Mayer” Institute of Mathematics Iaşi

e-mail: `adrian.zalinescu@googlemail.com`

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# Lévy processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A $d$-dimensional stochastic process $(L_t)_{t\geq 0}$ is called a *Lévy process* if:

1. it is càdlàg, *i.e.* $t \mapsto L_t$ is right continuous and has finite left limits a.s.;
2. it is stochastically continuous, *i.e.* $t \mapsto L_t$ is continuous in probability;
3. the random variables $L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}$ are independent, for every $n \in \mathbb{N}^*$ and $0 \leq t_0 < t_1 < \cdots < t_n$;
4. $L_{t+s} - L_t$ has the same law as $L_s$, for every $t, s \geq 0$; in particular, $L_0 = 0$ a.s.

The law of a Lévy process $(L_t)_{t\geq 0}$ is determined by the *characteristic exponent* of $L$, *i.e.* the unique continuous function $\Psi : \mathbb{R}^d \to \mathbb{C}$ such that $\Psi(0) = 0$ and

$$\mathbb{E}\exp\left(i\left\langle \lambda, L_t\right\rangle\right) = \exp\left(-t\Psi\left(\lambda\right)\right),\ t \geq 0,\ \lambda \in \mathbb{R}^d.$$

By the Lévy-Khintchine formula for infinitely divisible distributions, $\Psi$ has the following form:

$$\Psi\left(\lambda\right) = i\left\langle a, \lambda\right\rangle + \frac{1}{2}Q\left(\lambda\right) + \int_{\mathbb{R}^d\setminus\{0\}}\left(1 - e^{i\langle\lambda,x\rangle} + i\left\langle\lambda, x\right\rangle \mathbf{1}_{\{|x|<1\}}\right)\nu\left(dx\right), \tag{1}$$

where $a \in \mathbb{R}^d$, $Q$ is a positive semi-definite quadratic form on $\mathbb{R}^d$, and $\nu$ is a measure on $\mathbb{R}^d\setminus\{0\}$ such that

$$\int\left(1 \wedge |x|^2\right)\nu\left(dx\right) < +\infty.$$

The measure $\nu$ is called the Lévy measure associated to $L$.

For every function $\Psi$ given by the formula (1), there exists a Lévy process with characteristic exponent $\Psi$.

## Examples:

- if $\Psi(\lambda) := \frac{1}{2}|\lambda|^2$, then $L$ is a Brownian motion;
- if $\Psi(\lambda) := c\left(1 - \mathrm{e}^{i\lambda}\right)$, then $L$ is a Poisson process of intensity $c > 0$;
- A generalization of a Poisson process is the following:

Let $\xi_1, \ldots, \xi_n, \ldots$ be independent random variables with the same distribution $\nu$ on $\mathbb{R}^d \setminus \{0\}$, and

$$S(n) := \xi_1 + \cdots + \xi_n$$

the corresponding random walk. If $(N_t)_{t \geq 0}$ is a Poisson process of intensity $c > 0$, then the process

$$S \circ N_t = \sum_{i=1}^{N_t} \xi_i$$

is a Lévy process, called a compound Poisson process with Lévy measure $c\nu$.

The characteristic exponent of $S \circ N_t$ is

$$\psi(\lambda) = c \int_{\mathbb{R}^d} \left(1 - e^{i\langle \lambda, x \rangle}\right) \nu(dx);$$

so, every Lévy process with a finite Lévy measure can be represented as the sum of a Brownian motion and an independent compound Poisson process.

**Definition.** Let $\nu$ be a $\sigma$-finite measure on $\mathbb{R}^d$. A random measure $N(\omega, dt, dz)$ is a *homogeneous Poisson random measure* with intensity $\nu$, if:

i) for each $\omega \in \Omega$, $N(\omega, ., .)$ is a measure on $\mathbb{R}_+ \times \mathbb{R}^d$;

ii) for each set $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ with $(dt \otimes \nu)(B) < +\infty$, the random variable $N(\cdot, B)$ is Poisson with parameter $(dt \otimes \nu)(B)$;

iii) if $B_1, \ldots, B_n$ are disjoint Borel sets of $\mathbb{R}_+ \times \mathbb{R}^d$, then $N(\cdot, B_1), \ldots, N(\cdot, B_n)$ are independent.

The *compensated random measure* of $N$

$$\tilde{N}(dt, dz) := N(dt, dz) - dt \otimes \nu(dz)$$

has the property that $t \longmapsto \tilde{N}([0, t], A)$ is a martingale for every $A \in \mathcal{B}(\mathbb{R}^d)$ with $\nu(A) < +\infty$.

If $L$ is a Lévy process on $\mathbb{R}^d$ with Lévy measure $\nu$, then its *jump counting measure*, defined by

$$N(t, A) := \sum_{0<s\leq t} \mathbf{1}_A(\Delta L_s), \ t > 0, \ A \in \mathcal{B}(\mathbb{R}^d) \text{ with } 0 \notin \overline{A},$$

is a homogeneous Poisson random measure with intensity $\nu$, and the following holds:

$$L_t = bt + \sigma W_t + \int_{\{|z|<1\}} z \tilde{N}_t(dz) + \int_{\{|z|\geq 1\}} z N_t(dz), \tag{2}$$

where $b \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^{d\times d}$, and $W$ is a Brownian motion independent of $N$.

# Jump-diffusions

Jump-diffusions are generalizations of SDEs driven by Lévy processes. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space endowed with a right-continuous, complete filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, $W$ a $d'$-dimensional $\mathbb{F}$-Brownian motion, and $N$ a Poisson random measure with intensity $\nu$, $\mathbb{F}$-adapted and independent of $W$. Let us consider $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d'}$, $\gamma : \mathbb{R}^n \times \left(\mathbb{R}^d \backslash \{0\}\right) \to \mathbb{R}^n$ satisfying the following assumptions:

**(H1)** $b$ and $\sigma$ are Lipschitz functions;

**(H2)** $\gamma$ is a measurable function with

$$\int |\gamma(0,z)|^2 \nu(dz) < +\infty \text{ and } \int |\gamma(x,z) - \gamma(x',z)|^2 \nu(dz) \leq L\,|x - x'|^2 .$$

**Theorem** (Gihman, Skorohod, 1972). Under the above assumptions, equation

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t + \int_{\mathbb{R}^d \backslash \{0\}} \gamma(X_{t-}, z)\,d\tilde{N}_t(dz)$$

has a unique solution starting at $\xi \in L^2(\Omega, \mathcal{F}_0, P)$.

# Reflected jump-diffusions

Let $O$ be an open, convex and bounded subset of $\mathbb{R}^n$. Under assumptions (H1),

**(H2)'** for every $p \geq 2$, we have

$$\int |\gamma(0,z)|^p \, \nu(dz) \leq C_p \text{ and } \int |\gamma(x,z) - \gamma(x',z)|^p \, \nu(dz) \leq C_p \, |x - x'|^p \, .$$

and

**(H3)** $x + \gamma(x,z) \in \overline{O}, \ \forall x \in \overline{O},$

Menaldi, Robin (1985) proved that equation

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t + \int_{\mathbb{R}^d \setminus \{0\}} \gamma(X_{t-}, z)\, d\tilde{N}_t(dz) - dK_t,$$

where $K$ is the reflecting process of $X$ on the boundary of $O$, admits a unique solution with given initial starting point $x \in \overline{O}$.

# Variational inequalities

Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, l.s.c., convex function with $\operatorname{int}(\operatorname{Dom}\varphi) \neq \emptyset$. The subdifferential of $\varphi$ is defined by

$$\partial\varphi(x) := \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle + \varphi(x) \leq \varphi(y),\ \forall y \in \mathbb{R}^n\}.$$

In the case $\varphi \equiv I_{\overline{O}} : x \mapsto \begin{cases} 0, & x \in \overline{O}; \\ +\infty, & x \notin \overline{O}, \end{cases}$ the subdifferential is given by

$$\partial I_{\overline{O}}(x) = \begin{cases} \{0\}, & x \in O; \\ N_{\overline{O}}(x), & x \in \operatorname{bd} O; \\ \emptyset, & x \notin \overline{O}. \end{cases}$$

We consider the following equation

$$dX_t + \partial\varphi(X_t)\,dt \ni b(X_t)\,dt + \sigma(X_t)\,dW_t + \int_{\mathbb{R}^d\setminus\{0\}} \gamma(X_{t-},z)\,d\tilde{N}_t(dz), \tag{3}$$

We denote by $D([0,T];\mathbb{R}^n)$ the class of $\mathbb{R}^n$-valued, càdlàg functions on $[0,T]$, endowed with the uniform convergence topology. We say that $(X,K) \in L^2_{\text{ad}}(\Omega; D([0,T];\mathbb{R}^n)) \times L^2_{\text{ad}}(\Omega; C([0,T];\mathbb{R}^n))$ is a solution of (3) if:

- $\varphi(X) \in L^1(\Omega \times [0,T])$;
- $K \in L^1(\Omega; BV_0([0,T];\mathbb{R}^n))$;
- $X_t + K_t = \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \gamma(X_{s-},z)\,d\tilde{N}_s(dz)$;
- $\int_0^T \langle y(r) - X_r, dK_r\rangle + \int_0^T \varphi(X_r)\,dr \le \int_0^T \varphi(y(r))\,dr$, $\forall y \in D([0,T];\mathbb{R}^n)$.

Asiminoaiei, Răşcanu (1997): case $\gamma \equiv 0$.

**Theorem** (Uniqueness). Under assumptions (H1), (H2), equation (3) has at most one solution starting from $x \in \operatorname{Dom}\varphi$.

For the proof, we consider two solutions $(X, K)$ and $(\tilde{X}, \tilde{K})$, and a well chosen $C^2$-function $\beta : \mathbb{R}^n \to \mathbb{R}_+$ such that $\beta(x) = |x|^2$ for $|x| \leq 1$ and $\beta(x) = O(x)$ as $|x| \to +\infty$. We also denote

$$\mathcal{D}_\beta^0(x; a) := \beta(x + a) - \beta(x); \ \mathcal{D}_\beta^1(x; a) := \beta(x + a) - \beta(x) - \langle \nabla\beta(x), a\rangle .$$

We apply Itô's formula to $\beta(X_t - \tilde{X}_t)$:

$$\begin{aligned}
&\beta(X_t - \tilde{X}_t) + \int_0^t \langle \nabla\beta(X_s - \tilde{X}_s), dK_s - d\tilde{K}_s\rangle \\
&\leq \int_0^t \langle \nabla\beta(X_s - \tilde{X}_s), b(X_s) - b(\tilde{X}_s)\rangle ds + \int_0^t \langle \nabla\beta(X_s - \tilde{X}_s), \sigma(X_s) - \sigma(\tilde{X}_s) dW_s\rangle \\
&+ \int_0^t |\sigma(X_s) - \sigma(\tilde{X}_s)|^2 ds \\
&+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \mathcal{D}_\beta^0\left(X_{s-} - \tilde{X}_{s-}; \gamma(X_{s-}, z) - \gamma(\tilde{X}_{s-}, z)\right) d\tilde{N}_s(dz) \\
&+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \mathcal{D}_\beta^1\left(X_s - \tilde{X}_s; \gamma(X_s, z) - \gamma(\tilde{X}_s, z)\right) \nu(dz)\, ds.
\end{aligned}$$

For the existence result, we impose the condition ($q \geq 1$)

**(H3)'** $\varphi\left(x+\gamma\left(x,z\right)\right) \leq \varphi\left(x\right) + C_{\gamma}\left(1+\left|\gamma\left(x,z\right)\right|^{q}\right), \ \forall\left(x,z\right) \in \mathbb{R}^{n} \times \left(\mathbb{R}^{d}\backslash\left\{0\right\}\right).$

**Theorem** (Existence). Under assumptions (H1), (H2)', (H3)', equation (3) has a unique solution starting from $x_0 \in \operatorname{Dom}\varphi$.

The proof of this result uses the penalization method. We consider the Yosida regularization of $\varphi$

$$\varphi_{\varepsilon}\left(x\right) := \inf\left\{\frac{1}{2\varepsilon}\left|x-y\right|^{2} + \varphi\left(y\right) \mid y \in \mathbb{R}^{n}\right\}, \ \varepsilon > 0,$$

which is a $C^1$, convex function on $\mathbb{R}^n$, with $\nabla\varphi_{\varepsilon}$ a Lipschitz function with Lipschitz constant equal to $1/\varepsilon$. Moreover, by (H3)',

$$\varphi_{\varepsilon}\left(x+\gamma\left(x,z\right)\right) \leq \varphi_{\varepsilon}\left(x\right) + C_{\gamma}\left(1+\left|\gamma\left(x,z\right)\right|^{q}\right);$$

also, for simplicity, we can assume that $\varphi\left(x\right) \geq \varphi\left(0\right) = 0, \ \forall x \in \mathbb{R}^{n}$ and $0 \in \operatorname{int}\left(\operatorname{Dom}\varphi\right).$

We consider the jump-diffusion $X^{\varepsilon}$ given by

$$dX_{t}^{\varepsilon} + \nabla\varphi_{\varepsilon}\left(X_{t}^{\varepsilon}\right)dt = b\left(X_{t}^{\varepsilon}\right)dt + \sigma\left(X_{t}^{\varepsilon}\right)dW_{t} + \int_{\mathbb{R}^{d}\backslash\{0\}}\gamma\left(X_{t-}^{\varepsilon},z\right)d\tilde{N}_{t}\left(dz\right).$$

We will show that $X^{\varepsilon}$ and $K_{t}^{\varepsilon} := \int_{0}^{t}\nabla\varphi_{\varepsilon}\left(X_{s}^{\varepsilon}\right)ds$ converge to $X$ and $K$.

## I. Boundedness of $(X^\varepsilon)$ and $(K^\varepsilon)$

Itô's formula for $|X_t^\varepsilon|^2$

$$
\begin{aligned}
|X_t^\varepsilon|^2 + 2\int_0^t \langle X_s^\varepsilon, \nabla\varphi_\varepsilon(X_s^\varepsilon)\rangle ds &= |x_0|^2 + \int_0^t [2\langle X_s^\varepsilon, b(X_s^\varepsilon)\rangle + |\sigma(X_s^\varepsilon)|^2] ds \\
&+ \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} |\gamma(X_s^\varepsilon, z)|^2 \nu(dz)\, ds + 2\int_0^t \langle X_s^\varepsilon, \sigma(X_s^\varepsilon)\, dW_s\rangle \\
&+ \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \left[|X_{s-}^\varepsilon + \gamma\left(X_{s-}^\varepsilon, z\right)|^2 - \left|X_{s-}^\varepsilon\right|^2\right] d\tilde{N}_s(dz)\,.
\end{aligned}
$$

We obtain, since $\varphi_\varepsilon(x) \le \langle \nabla\varphi_\varepsilon(x), x\rangle,\ \forall x \in \mathbb{R}^n$,

$$
\mathbb{E}\sup_{t\in[0,T]} |X_t^\varepsilon|^4 \vee \mathbb{E}\left(\int_0^T \varphi_\varepsilon(X_s^\varepsilon)\, ds\right)^2 \le C\left(1 + |x_0|^4\right).
$$

Also, $\exists r_0 > 0,\ \exists M_0 > 0 : r_0\,|\nabla\varphi_\varepsilon(x)| \le \langle x, \nabla\varphi_\varepsilon(x)\rangle + M_0,\ \forall x \in \mathbb{R}^n$; it follows that

$$
\mathbb{E}\,\|K^\varepsilon\|^2_{BV([0,T];\mathbb{R}^n)} = \mathbb{E}\left(\int_0^T |\nabla\varphi_\varepsilon(X_s^\varepsilon)|\, ds\right)^2 \le C\left(1 + |x_0|^4\right).
$$

## II. Estimate for $\mathbb{E}\left[\sup_{t\in[0,T]}\left|\nabla\varphi_{\varepsilon}\left(X_t^{\varepsilon}\right)\right|^4\right]$

Itô's formula for $\varphi_{\varepsilon}^2\left(X_t^{\varepsilon}\right)$:

$$\begin{aligned}
\varphi_{\varepsilon}^2\left(X_t^{\varepsilon}\right)+\int_0^t\varphi_{\varepsilon}\left(X_s^{\varepsilon}\right)\left|\nabla\varphi_{\varepsilon}\left(X_s^{\varepsilon}\right)\right|^2ds \le{}& \varphi_{\varepsilon}\left(x_0\right)^2+2\int_0^t\varphi_{\varepsilon}\left(X_s^{\varepsilon}\right)\left\langle\nabla\varphi_{\varepsilon}\left(X_s^{\varepsilon}\right),b\left(X_s^{\varepsilon}\right)\right\rangle ds\\
&+\int_0^t\left|\nabla\varphi_{\varepsilon}\left(X_s^{\varepsilon}\right)\right|^2\left|\sigma\left(X_s^{\varepsilon}\right)\right|^2ds+\frac{1}{\varepsilon}\int_0^t\varphi_{\varepsilon}\left(X_s^{\varepsilon}\right)\left|\sigma\left(X_s^{\varepsilon}\right)\right|^2ds\\
&+\int_0^t\varphi_{\varepsilon}\left(X_s^{\varepsilon}\right)\left\langle\nabla\varphi_{\varepsilon}\left(X_s^{\varepsilon}\right),\sigma\left(X_s^{\varepsilon}\right)dW_s\right\rangle\\
&+\int_0^t\int_{\mathbb{R}^d\backslash\{0\}}\mathcal{D}^0_{\varphi_{\varepsilon}^2}\left(X_{s-}^{\varepsilon};\gamma\left(X_{s-}^{\varepsilon},z\right)\right)d\tilde{N}_s\left(dz\right)\\
&+\int_0^t\int_{\mathbb{R}^d\backslash\{0\}}\mathcal{D}^1_{\varphi_{\varepsilon}^2}\left(X_s^{\varepsilon};\gamma\left(X_s^{\varepsilon},z\right)\right)\nu\left(dz\right)ds.
\end{aligned}$$

We have, since $\left|\nabla\varphi_{\varepsilon}\left(x\right)\right|^{2}\leq\frac{2}{\varepsilon}\varphi_{\varepsilon}\left(x\right),\ \forall x\in\mathbb{R}^{n}$,

$$\begin{aligned}
\left|\mathcal{D}^{0}_{\varphi_{\varepsilon}^{2}}\left(X^{\varepsilon}_{s-};\gamma\left(X^{\varepsilon}_{s-},z\right)\right)\right| &\leq \sup_{\mu\in[0,1]}2\left|\nabla\varphi_{\varepsilon}\left(X^{\varepsilon}_{s-}+\mu\gamma\left(X^{\varepsilon}_{s-},z\right)\right)\right|\left|\varphi_{\varepsilon}\left(X^{\varepsilon}_{s-}+\mu\gamma\left(X^{\varepsilon}_{s-},z\right)\right)\right|\left|\gamma\left(X^{\varepsilon}_{s-},z\right)\right| \\
&\leq \frac{2\sqrt{2}}{\varepsilon^{1/2}}\sup_{\mu\in[0,1]}\left|\varphi_{\varepsilon}\left(X^{\varepsilon}_{s-}+\mu\gamma\left(X^{\varepsilon}_{s-},z\right)\right)\right|^{3/2}\left|\gamma\left(X^{\varepsilon}_{s-},z\right)\right| \\
&\leq \frac{2\sqrt{2}}{\varepsilon^{1/2}}\left|\left[\varphi_{\varepsilon}(X^{\varepsilon}_{s-})+C\left(1+\left|\gamma\left(X^{\varepsilon}_{s-},z\right)\right|^{q}\right)\right]^{3/2}\left|\gamma\left(X^{\varepsilon}_{s-},z\right)\right|;\right. \\
\left|\mathcal{D}^{1}_{\varphi_{\varepsilon}^{2}}\left(X^{\varepsilon}_{s};\gamma\left(X^{\varepsilon}_{s},z\right)\right)\right| &\leq \left(\sup_{\mu\in[0,1]}\left[\left|\nabla\varphi_{\varepsilon}\left(X^{\varepsilon}_{s}+\mu\gamma\left(X^{\varepsilon}_{s-},z\right)\right)\right|^{2}\right]+\frac{1}{\varepsilon}\varphi_{\varepsilon}\left(X^{\varepsilon}_{s}\right)\right)\left|\gamma\left(X^{\varepsilon}_{s-},z\right)\right|^{2} \\
&\leq \frac{2}{\varepsilon}\sup_{\mu\in[0,1]}\left[\varphi_{\varepsilon}\left(X^{\varepsilon}_{s}+\mu\gamma\left(X^{\varepsilon}_{s-},z\right)\right)\right]\left|\gamma\left(X^{\varepsilon}_{s-},z\right)\right|^{2} \\
&\leq \frac{2}{\varepsilon}\left[\varphi_{\varepsilon}(X^{\varepsilon}_{s})+C\left(1+\left|\gamma\left(X^{\varepsilon}_{s-},z\right)\right|^{q}\right)\right]\left|\gamma\left(X^{\varepsilon}_{s-},z\right)\right|^{2}.
\end{aligned}$$

This will give the following estimate:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\nabla\varphi_{\varepsilon}\left(X^{\varepsilon}_{t}\right)\right|^{4}\right]\leq\frac{C}{\varepsilon^{7/2}}\left(1+\left|x_{0}\right|^{q'}+\varphi\left(x_{0}\right)^{2}\right).$$

## III. Cauchy sequences argument

Itô's formula for $\left|X^{\varepsilon}-X^{\delta}\right|^{2}$:

$$\begin{aligned}
&\left|X_t^{\varepsilon}-X_t^{\delta}\right|^{2}+2\int_0^t\langle X_s^{\varepsilon}-X_s^{\delta},\nabla\varphi_{\varepsilon}\left(X_s^{\varepsilon}\right)-\nabla\varphi_{\delta}\left(X_s^{\delta}\right)\rangle ds\\
&=\int_0^t[2\langle X_s^{\varepsilon}-X_s^{\delta},b\left(X_s^{\varepsilon}\right)-b\left(X_s^{\delta}\right)\rangle+\left|\sigma\left(X_s^{\varepsilon}\right)-\sigma\left(X_s^{\delta}\right)\right|^{2}]ds\\
&+\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}|\gamma\left(X_s^{\varepsilon},z\right)-\gamma\left(X_s^{\delta},z\right)|^{2}\nu\left(dz\right)ds\\
&+2\int_0^t\langle X_s^{\varepsilon}-X_s^{\delta},(\sigma\left(X_s^{\varepsilon}\right)-\sigma\left(X_s^{\delta}\right))dW_s\rangle\\
&+\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}\left[|X_{s-}^{\varepsilon}-X_{s-}^{\delta}+\gamma\left(X_{s-}^{\varepsilon},z\right)-\gamma\left(X_{s-}^{\delta},z\right)|^{2}-\left|X_{s-}^{\varepsilon}-X_{s-}^{\delta}\right|^{2}\right]d\tilde{N}_s\left(dz\right).
\end{aligned}$$

We use the fact that

$$\langle\nabla\varphi_{\varepsilon}\left(x\right)-\nabla\varphi_{\delta}\left(y\right),x-y\rangle\geq-\left(\varepsilon+\delta\right)\langle\nabla\varphi_{\varepsilon}\left(x\right),\nabla\varphi_{\delta}\left(y\right)\rangle$$

to obtain that $\mathbb{E}\sup_{t\in[0,T]}\left|X_t^{\varepsilon}-X_t^{\delta}\right|^{4}\to 0$ as $\delta,\varepsilon\to 0$ and $\mathbb{E}\sup_{t\in[0,T]}\left|K_t^{\varepsilon}-K_t^{\delta}\right|^{2}\to 0.$

## IV. Passing to the limit

There exist $X \in L^4_{\text{ad}}(\Omega; D([0,T];\mathbb{R}^n))$ and $K \in L^2_{\text{ad}}(\Omega; C([0,T];\mathbb{R}^n))$ such that

$$\mathbb{E} \sup_{t\in[0,T]} \left[ |X^\varepsilon_t - X_t|^4 + |K^\varepsilon_t - K_t|^2 \right] \to 0$$

Since $(K^\varepsilon)$ is also bounded in $L^2(\Omega; BV_0([0,T];\mathbb{R}^n))$, it converges weakly to $K$. It is now a standard argument to show that $(X, K)$ is a solution of equation (3).

# Perspectives

- relaxing the conditions on $\gamma$;
- replacing $\partial\varphi$ by a general maximal monotone operator;
- backward variational inequations;
- link with integro-differential variational inequations via optimal control or via Feynman-Kac formula.

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Thank you for your attention!