Introduction to $G$-Expectation, $G$-Brownian motion and $G$-Backward SDEs

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The aim of the talk

- *Introduction to G-stochastic analysis theory* (S. Peng).
- *Part of my work in this research field.*
1. Non linear expectations: a general framework

Setting

\[ C_{\text{Lip}}(\mathbb{R}^n) : = \{ \varphi : \mathbb{R}^n \to \mathbb{R} \mid \exists C_\varphi \in \mathbb{R}^+, m \in \mathbb{N} \text{ s.t.} \]
\[ \varphi(x) - \varphi(y) \leq C_\varphi(1 + |x|^m + |y|^m)|x - y| \} \]

\[ \text{Lip}_b(\mathbb{R}^n) : = \{ f : \mathbb{R}^n \to \mathbb{R} \mid \exists M_f, C_f \in \mathbb{R}^+ \text{ s.t.} \]
\[ |f(x)| \leq M_f, |f(x) - f(y)| \leq C_f|x - y| \} \]

Let \( \Omega = \mathbb{R} \) and let \( \mathcal{H} \) be a linear space of real functions s.t.

\[ X_1, \cdots, X_n \in \mathcal{H} \Rightarrow \varphi(X_1, \cdots, X_n) \in \mathcal{H}, \forall \varphi \in C_{\text{Lip}}(\mathbb{R}^n). \]
Definition

A non-linear expectation $\hat{E}$ is a functional $\mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties:

- **Monotonicity**: $X, Y \in \mathcal{H}$ and $X \geq Y \Rightarrow \hat{E}[X] \geq \hat{E}[Y]$.
- **Preservation of constants**: $\hat{E}[c] = c$, $\forall c \in \mathbb{R}$.

A sub-linear expectation in addition satisfies:

- **Subadditivity**: $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y], \forall X, Y \in \mathcal{H}$.
- **Positive homogeneity**: $\hat{E}[\lambda X] = \lambda \hat{E}[X], \forall \lambda \geq 0, X \in \mathcal{H}$. 


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Remark

Norm on $\mathcal{H}$: $\|X\| := \hat{E}[|X|], \ X \in \mathcal{H}$.

$(\mathcal{H}, \| \cdot \|)$ is a normed space.

We denote its completion by $([\mathcal{H}], \| \cdot \|)$, or simply $[\mathcal{H}]$. 
2. *G*-Normal distributions

Let \(0 \leq \sigma_1^2 \leq \sigma_2^2\), \(X \in \mathcal{H}\); \(G(a) = \frac{1}{2}(\sigma_2^2 a^+ - \sigma_1^2 a^-)\), \(\forall a \in \mathbb{R}\).

**Classical case**

\(\sigma^2 = \sigma_1^2 = \sigma_2^2\).
\(X \sim \mathcal{N}(0, \sigma^2)\) can be characterized by
\(u(t, x) = \mathbb{E}[\varphi(x + \sqrt{t}X)]\), for \(\varphi \in C_{l.\text{Lip}}(\mathbb{R})\), which satisfies

\[
\partial_t u(t, x) = \frac{1}{2} \sigma^2 \partial_{xx}^2 u(t, x), \quad u(0, x) = \varphi(x).
\]

**G-case (Definition)**

\(X \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2])\) is characterized by
\(u(t, x) = \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]\), for \(\varphi \in C_{l.\text{Lip}}(\mathbb{R})\),
which is the unique viscosity solution of

\[
\partial_t u - G(\partial_{xx}^2 u) = 0, \quad u(0, x) = \varphi(x).
\]
2. $G$-Normal distributions

**Proposition**

If $X \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$, we have the following property:

- For each convex $\varphi \in C_{l.Lip}(\mathbb{R})$,

\[
    u(t,x) = \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)] = \frac{1}{2\sqrt{\pi t} \sigma_2} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma_2^2 t}} \varphi(x) dx;
\]

- For each concave $\varphi \in C_{l.Lip}(\mathbb{R})$, for $\sigma_1 > 0$,

\[
    u(t,x) = \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)] = \frac{1}{2\sqrt{\pi t} \sigma_1} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma_1^2 t}} \varphi(x) dx.
\]
Definition

A random vector \( Y = (Y_1, \cdots, Y_n) , Y_i \in \mathcal{H} \), is said to be independent of \( X = (X_1, \cdots, X_m) , X_i \in \mathcal{H} \), if for each \( \varphi \in C_{1,Lip}(\mathbb{R}^m \times \mathbb{R}^n) \),

\[
\hat{\mathbb{E}}[\varphi(X,Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x,Y)]_{x=X}].
\]
Remark

Unlike the classical case, the independence under \( \hat{E} \) is not symmetric.

Example

Let \( X, Y \in \mathcal{H} \) be s.t. \( \sigma^2_2 := \hat{E}[Y^2] > \sigma^2_1 := -\hat{E}[-Y^2] > 0 \), \( \hat{E}[X] = \hat{E}[-X] = 0 \), and \( \hat{E}[|X|] > 0 \). Then if \( Y \) is independent of \( X \):

\[
\hat{E}[XY^2] = \hat{E}[\hat{E}[xY^2]|_{x=X}] = \hat{E}[(x^+\hat{E}[Y^2] + x^-\hat{E}[-Y^2])|_{x=X}]
\]

\[
= \hat{E}[X^+\sigma^2_2 - X^-\sigma^2_1] = \hat{E}[X^+](\sigma^2_2 - \sigma^2_1) > 0.
\]

But if \( X \) is independent of \( Y \):

\[
\hat{E}[XY^2] = \hat{E}[\hat{E}[Xy^2]|_{y=Y}] = \hat{E}[(y^2\hat{E}[X]|_{y=Y}] = \hat{E}[\hat{E}[X]Y^2] = 0.
\]
2. \( G \)-Normal distributions

**Definition**

\( X, Y \) are said to be *identically distributed* \((X \sim Y, \text{ or } X \text{ is a copy of } Y)\), if \( \hat{E}[\varphi(X)] = \hat{E}[\varphi(Y)], \forall \varphi \in C_{l,Lip}(\mathbb{R}). \)

**Definition**

A sequence \( \{\eta_i\}_{i=1}^{\infty} \) in \( \mathcal{H} \) is said to *converge in law* to \( \eta \in \mathcal{H} \) under \( \hat{E} \) if

\[
\lim_{i \to \infty} \hat{E}[\varphi(\eta_i)] = \hat{E}[\varphi(\eta)],
\]

for each \( \varphi \in Lip_b(\mathbb{R}). \)
Central Limit Theorem (Peng)

Let \( \{X_i\}_{i=1}^{\infty} \) in \((\Omega, \mathcal{H}, \hat{E})\) satisfy \( X_i \sim X_1 \), each \( X_{i+1} \) is independent of \((X_1, \cdots, X_i)\),

\[
\hat{E}[|X_1|^{2+\alpha}] < \infty, \text{ for some } \alpha > 0, \text{ and } \hat{E}[X_1] = \hat{E}[-X_1] = 0.
\]

\( S_n := X_1 + \cdots + X_n \). Then \( S_n/\sqrt{n} \) converges in law to \( \mathcal{N}(0; [\sigma_1^2, \sigma_2^2]) \):

\[
\lim_{n \to \infty} \hat{E}[\varphi(S_n/\sqrt{n})] = \hat{E}[\varphi(X)], \forall \varphi \in Lip_b(\mathbb{R}),
\]

where \( X \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2]) \), \( \sigma_1^2 = -\hat{E}[-X_1^2], \sigma_2^2 = \hat{E}[X_1^2] \).
3. \( G \)-Brownian motion under \( G \)-expectation

Setting

Let \( \Omega = C_0(\mathbb{R}^+) \) the space of continuous paths \((\omega_t)_{t \in \mathbb{R}^+}\) with \(\omega_0 = 0\); distance on \(\Omega\):

\[
\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} \left[ (\max_{t \in [0,i]} |\omega_t^1 - \omega_t^2|) \land 1 \right].
\]

Coordinate process on \(\Omega\): \(B_t(\omega) = \omega_t, t \geq 0, \omega \in \Omega\).

For \(t \in [0, +\infty)\), we set

\[
\mathcal{H}_t := \{ \varphi(\omega_{t_1}, \cdots, \omega_{t_n}) : \forall n \in \mathbb{N}, t_1, \cdots, t_n \in [0, t], \\
\forall \varphi \in C_{l.Lip}(\mathbb{R}^n) \}
\]

\[
\mathcal{H} := \bigcup_{n=1}^{\infty} \mathcal{H}_n.
\]
3. $G$-Brownian motion under $G$-expectation

**Definition**

The coordinate process $B$ on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is a $G$-Brownian motion if:

1. $B_0 = 0$;
2. $B_{t+s} - B_t$ is $\mathcal{N}(0; [\sigma_1^2 s, \sigma_2^2 s])$ - distributed, $\forall t, s \geq 0$;
3. For each $n \geq 2$, $0 \leq t_1 \leq \cdots \leq t_n$, $B_{t_n} - B_{t_{n-1}}$ is independent of $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$. 

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Theorem (Peng)

Let $B_t(\omega)_{t \geq 0}$ be a process on $(\Omega, \mathcal{H}, \hat{E})$ such that

1. For each $0 \leq t_1 \leq \cdots \leq t_n$, $B_{t_n} - B_{t_{n-1}}$ is independent of $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$;
2. $B_t$ has the same distribution as $B_{t+s} - B_s$, $\forall t, s \geq 0$;
3. $\lim_{t \downarrow 0} \hat{E}[|B_t|^3]t^{-1} = 0$,

then $B$ is a $G$-Brownian motion, for

$$
\sigma_1^2 = -\hat{E}[-B_1^2], \quad \sigma_2^2 = \hat{E}[B_1^2].
$$

For what follows: for simplicity, we choose $\sigma_2^2 = 1$. 

3. \( G \)-Brownian motion under \( G \)-expectation

**Definition**

The related *conditional expectation* of
\( X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}}) \) under \( \mathcal{H}_t \) is defined by

\[
\hat{E}[X|\mathcal{H}_t] = \psi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_j} - B_{t_{j-1}}),
\]

where \( \psi(x_1, \cdots, x_j) = \hat{E}[\varphi(x_1, \cdots, x_j, B_{t_{j+1}} - B_{t_j}, \cdots, B_{t_m} - B_{t_{m-1}})] \).

**Definition**

A process \( (M_t)_{t \geq 0} \) is called a *\( G \)-martingale* (resp. *\( G \)-submartingale, \( G \)-supermartingale*), if \( \forall t \in [0, +\infty) \), \( M_t \in \mathcal{H}_t \) and for \( s \in [0, t] \),

\[
\hat{E}[M_t|\mathcal{H}_s] = M_s \text{ (resp. } \leq M_s, \; \geq M_s \text{) a.s. in } \mathcal{H}.
\]
4. Itô’s integral of $G$-Brownian motion

**Setting**

$L^2_G(\mathcal{H}_t) := \{ \xi \in \mathcal{H}_t : \hat{E}[|\xi|^2] < \infty \}.$

$L^2_G(0, T) := \{ \eta(\omega) \mid \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})}(t), \}

where $\xi_j \in L^2_G(\mathcal{H}_{t_j}), 0 = t_0 < t_1 < \cdots < t_N = T \}.$

**Definition**

For each $(\eta_t)_{0 \leq t \leq T} \in L^2_G(0, T)$, we define its Itô’s integral as

$$I(\eta) = \int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$
Lemma

We have

$$\hat{\mathbb{E}} \left[ \int_0^T \eta(s) dB_s \right] = 0,$$

and

$$\hat{\mathbb{E}} \left[ (\int_0^T \eta(s) dB_s)^2 \right] \leq \int_0^T \hat{\mathbb{E}}[(\eta_s)^2] \, ds.$$

We denote by $L^2_G(0, T)$ the completion of $L^2_G(0, T)$ under the norm

$$\| \eta \|_{L^2_G(0, T)} = \left\{ \int_0^T \hat{\mathbb{E}}[|\eta_t|^2] \, dt \right\}^{1/2}.$$

Hence the stochastic integral can be extended to $L^2_G(0, T)$. 
We denote: $\langle B \rangle_t = B_t^2 - 2 \int_0^t B_s dB_s$, then

$$\hat{E}[|\langle B \rangle_t - \sum_{k=0}^{N-1} (B_{t_{k+1}} - B_{t_k})^2|^2] \to 0, \; N \to +\infty.$$ 

$\langle B \rangle$ is an increasing process; it is called the quadratic variation of $B$.

**Proposition**

$$\hat{E}[\langle B \rangle_t] = t, \quad \text{but} \quad \hat{E}[-\langle B \rangle_t] = -\sigma^2_t t.$$ 

$$\hat{E}[(\int_0^T h(s) dB_s)^2] = \hat{E}[\int_0^T h^2(s) d\langle B \rangle_s], h \in L^2_G(0, T).$$
4. Itô’s integral of $G$-Brownian motion

Consider

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s \langle B \rangle_s + \int_0^t \beta_s dB_s.$$ 

Itô’s formula (Peng)

Let $\alpha, \beta, \eta$ be bounded processes of $L^2_G(0,T)$. Then for each $t \geq s \geq 0$ and $\Phi(X_t) \in L^2_G(\mathcal{H}_t)$, we have

$$\Phi(X_t) = \Phi(X_s) + \int_s^t \Phi_x(X_u) \beta_u dB_u + \int_s^t \Phi_x(X_u) \alpha_u du$$

$$+ \int_s^t [\Phi_x(X_u) \eta_u + \frac{1}{2} \Phi_{xx}(X_u) \beta_u^2] d\langle B \rangle_u.$$
Now we can consider the following $G$-SDE:

$$X_t = X_0 + \int_0^t b(X_s) \, ds + \int_0^t h(X_s) \, d\langle B \rangle_s + \int_0^t \sigma(X_s) \, dB_s, \quad t > 0.$$ 

where $X_0 \in \mathbb{R}$, $b, h, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions.

**Theorem (Peng)**

There exists a unique solution $X \in L^2_G(0, T)$ of the above $G$-SDE.
Recall

Classical BSDE (under a linear expectation):

\[
\begin{aligned}
\text{d}Y_t &= -f(t, Y_t, Z_t)\text{d}t + Z_t\text{d}B_t, \\
Y_T &= \xi \in L^2(\Omega, \mathcal{F}_T, P);
\end{aligned}
\]

solution \((Y_t, Z_t)_{0 \leq t \leq T}\); existence and uniqueness for \(f\) Lipschitz in \((y, z)\) and linear growth. For \(f \equiv 0\): the martingale representation theorem.

Remark

No martingale representation theorem in \(G\)-theory up to now!
Unlike the classical case, we only consider the $G$-BSDE of the following type:

$$Y_t = \hat{\mathbb{E}}[\xi + \int_t^T f(s, Y_s)ds \mid \mathcal{H}_t], \quad t \in [0, T],$$

where $\xi \in L^2_G(\mathcal{H}_T), f(t, y) \in L^2_G(0, T), y \in \mathbb{R}$, is a given Lipschitz function with respect to $y$.

**Theorem (Peng)**

*There exists a unique solution $(Y_t)_{t \in [0, T]} \in L^2_G(0, T).$*
5. SDE and BSDE driven by a $G$-Brownian motion

$$Y_t^i = \hat{E}[\xi^i + \int_t^T f_i(s, Y_s^i)ds | \mathcal{H}_t], \quad t \in [0, T],$$

where $\xi^i \in L^2_G(\mathcal{H}_T), f_i(t, y) \in L^2_G(0, T), y \in \mathbb{R}$, for $i = 1, 2$.

**Comparison Theorem (Jing)**

Suppose that $\xi^1 \geq \xi^2, f_1(t, y) \geq f_2(t, y), \forall (t, y), f_1 \uparrow$ or $f_2 \downarrow$ in $y$, both $f_1$ and $f_2$ are Lipschitz in $y$ and bounded, then $Y_t^1 \geq Y_t^2, \forall t \in [0, T]$. 

Proposition

Let \( x \in \mathbb{R}, \ Z \in L^2_G(0,T), \) and \( \eta \in L^2_G(0,T), \) then the process

\[
M_t = x + \int_0^t Z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds
\]

is a \( G \)-martingale.

Recall

If \( \sigma_1 = 1, \ \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds = 0. \)

Question

Is the stochastic integral w. r. t. this \( G \)-martingale still a \( G \)-martingale?
6. A property of $G$-martingales

Remark

$\{M_t\}$ being a $G$-martingale does not always imply that $\{-M_t\}$ is a $G$-martingale.

Example

$\{B_t\}$ and $\{-B_t\}$ are both $G$-martingales. $\{\langle B \rangle_t - t\}$ is a $G$-martingale, but $\{-\langle B \rangle_t + t\}$ is not a $G$-martingale.
Theorem (Jing)

Let $0 \leq \sigma_1 < 1$ and $\xi \in L^2_G(0, T)$. Put

$$M_t = x + \int_0^t Z_s dB_s + \int_0^t \eta_s dB_s - \int_0^t 2G(\eta_s) ds,$$

and

$$N_t = \int_0^t \xi_s dM_s, \quad t \geq 0.$$

Then $\{N_t\}$ is a $G$-martingale w. r. t. $\{\mathcal{H}_t\}$ if and only if

$$\begin{cases} 
\text{sgn}(\xi_s) \geq 0, & \text{on } \{s : \eta_s \neq 0\}; \\
\text{sgn}(\xi_s) \text{ can be arbitrary}, & \text{on } \{s : \eta_s = 0\}.
\end{cases}$$
This talk is mainly based on


Thank you!

谢谢！