American Game Options and Reflected BSDEs with Quadratic Growth

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Outline

1 Introduction

1.1 Motivation

- An American option is a contract which gives the right to its holder to exercise, i.e. to ask for the wealth, when he decides before the maturity T of the option. The liabilities of the seller of the option is to provide this wealth.
- An American game option (also called Israeli option) is a contract which gives the right to the seller of the option to recall it if he accepts to pay a money penalty.

The main reason for such options: to face *high level risky* environment/market.

See: Kifer [24], Kallsen-Kuhn [23], Baurdoux-Kyprianou [3], Hamadène [15], Kyprianou-Kuhn [26].

In order to tackle

- American options we mainly use Snell envelope of stochastic processes;
- American game options we use value functions of zero-sum Dynkin games.

In the standard Black and Scholes model, the value of the American game option is given by the value of the zero-sum Dynkin game under the risk neutral probability. Additionally a hedging strategy for the seller of the option exists (see e.g. Kifer [24], Hamadène [15], Kallsen-Kuhn [23]).

- the Ramsey model in a growth model in finance (see e.g. Amilon-Bermin [1]). Assume we have a capital from which is withdrawn a consumption and whose dynamics is given by:

$$dX_t = X_t(r - c_t)dt + X_t\sigma dB_t,\ t \le T,\ X_0 = x > 0,$$

where

- r is the spot mean-return of the capital;
- c is a proportion of the capital which is consumed.

Therefore, usually the main objective is to find an optimal consumption process with respect to an index which indicates the satisfaction of the capital holder. This index depends on $(c_t)_{t\le T}$ but also on X and many other parameters such as risk sensitiveness, utility and so on.

Consider a system on which intervene two agents a_1 and a_2 and whose dynamics is given by:

$$\begin{cases} dX_s^{\alpha,\beta} = b\left(s, X_s^{\alpha,\beta}, \alpha_s, \beta_s\right) ds + \sigma\left(s, X_s^{\alpha,\beta}, \alpha_s, \beta_s\right) dW_s,\ \ s \in [0, T]\,; \\ X_0^{\alpha,\beta} = x. \end{cases} \tag{1}$$

Here

- the stochastic processes $(\alpha_t)_{t\le T}$ and $(\beta_t)_{t\le T}$ are the intervention functions of a_1 and a_2;
- σ and τ are the stopping times of a_1, respectively a_2.

The interests of the agents are antagonistic and the payoff is given by:

$$\Gamma(\alpha,\sigma;\beta,\tau) := \mathbb{E}\Big[exp\{\int_0^{T\wedge\tau\wedge\sigma} \varphi(s, X_s^{\alpha,\beta}, \alpha_s, \beta_s)ds + h(\sigma, X_\sigma^{\alpha,\beta})1_{[\sigma\le\tau<T]} + h'(\tau, X_\tau^{\alpha,\beta})1_{[\tau<\sigma]} + g(X_T^{\alpha,\beta})1_{[\sigma=\tau=T]]}\}\Big]. \quad (2)$$

The payoff $\Gamma(\alpha,\sigma;\beta,\tau)$ is a reward for a_1 and a cost for a_2. The role of the exponential utility function is to capture the sensitiveness with respect to risk of the agents.

The objective is to study the upper and lower values of this mixed zero-sum stochastic differential game , which are defined by:

$$\sup_\alpha \inf_\beta \sup_\sigma \inf_\tau \Gamma(\alpha,\sigma;\beta,\tau) \quad \text{and} \quad \inf_\beta \sup_\alpha \inf_\tau \sup_\sigma \Gamma(\alpha,\sigma;\beta,\tau). \quad (3)$$

- When the data of this problem do not depend on the controls α and β and the criterion is of risk-neutral type, the problem reduces to the zero-sum Dynkin game;
- If $\tau = \sigma = T$ and the data do not depend on β we obtain the Ramsey model.

We are going to consider a more general setting of payoffs, namely payoffs defined by solutions of BSDEs with two reflecting barriers and continuous coefficients whose growth are quadratic with respect to the component z.

Short history

- El-Karoui et al. [13] - first studies on BSDEs with one reflecting barrier.
- Cvitanic-Karatzas introduced in [10] the notion of BSDEs with two reflecting barriers.
- The connection with zero-sum Dynkin games and American game options was analyzed by Bahlali-Hamadène-Mezerdi [2], Hamadène-Hassani [16], Hamadène-Hdhiri [17], Hamadène [15]. In [17] it is shown that if the barriers are completely separated and the coefficient of the BSDE is continuous with quadratic growth, then a minimal and a maximal solution exist for the BSDE.
- The lower and upper values of a zero-sum stochastic differential game have been investigated by Fleming-Souganidis in [14], where it's proved that they are unique solutions (in viscosity sense) of the associated Hamilton-Jacobi-Bellman-Isaacs equations.
- The link to RBSDEs have been studied by Hamadène-Lepeltier [20], Hamadène-Hdhiri [17], Hamadène-Lepeltier-Wu [22].
- In the paper [8] Buckdahn-Li considered the framework of the payoffs which are given by a solution of a controlled BSDE with two reflecting barriers whose coefficient is Lipschitz in (y, z).

1.2 Preliminaries. Notations. Hypothesis

Consider

- a finite horizon $T > 0$;
- a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined
- a standard d - dimensional Brownian motion $W = (W_t)_{t\leq T}$ whose natural filtration is denoted $\mathbb{F} = \{\mathcal{F}_t,\ 0 \leq t \leq T\}$ ($\mathcal{F}_t = \sigma\{W_s,\ s \leq t\} \vee \mathcal{N}_{\mathbb{P}}$, where $\mathcal{N}_{\mathbb{P}}$ is the set of all $\mathbb{P}$-null sets).

Denote

- $\mathcal{P}$, the σ-algebra of $\mathcal{F}_t$-progressively measurable sets on $[0, T] \times \Omega$;
- $\mathcal{L}^{2,k}$, the set of $\mathcal{P}$-measurable and $\mathbb{R}^k$-valued processes $z = (z_t)_{t\leq T}$ such that $\int_0^T |z_t|^2 dt < \infty$, $\mathbb{P}$ - a.s.; $\mathcal{H}^{2,k}$ is the subspace of $\mathcal{L}^{2,k}$, such that $\mathbb{E}[\int_0^T |z_t|^2 dt] < \infty$;
- $\mathcal{S}^2$, the set of $\mathcal{P}$-measurable and continuous processes $Y = (Y_t)_{t\leq T}$ such that $\mathbb{E}[\sup_{t\leq T} |Y_t|^2] < \infty$;
- $\mathcal{M}$, the set of continuous $\mathcal{P}$-measurable nondecreasing processes $(K_t)_{t\leq T}$ such that $K_0 = 0$ and $K_T < \infty$, $\mathbb{P}$ - a.s.

Recall now the existence result for the solutions of RBSDEs with two barriers with quadratic growth coefficient. Let us take four objects which define the equation:

- a continuous function $F : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ (also known as the *coefficient or generator* of the equation) for which there exists a constant $C > 0$ such that, $\mathbb{P}$ - a.s.,

$$|F(t,\omega,y,z)| \leq C(1+|z|^2),\ \forall\,(t,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d; \tag{4}$$

- a terminal value ξ, which is a $\mathcal{F}_T$ - measurable random variable;

- two stochastic processes $U = (U_t)_{t\leq T}$ and $L = (L_t)_{t\leq T}$ from $\mathcal{S}^2$, satisfying $L_t < U_t$, for all $t \leq T$, and $L_T \leq \xi \leq U_T$.

Definition 1 A solution for the RBSDE associated with (F, ξ, L, U) is a quadruple of $\mathcal{P}$-measurable processes $\left(Y_t, Z_t, K_t^+, K_t^-\right)_{t\leq T} \in \mathcal{S}^2 \times \mathcal{L}^{2,d} \times \mathcal{M} \times \mathcal{M}$ such that, $\mathbb{P}$ - a.s.

$$Y_t = \xi + \int_t^T F\left(s, Y_s, Z_s\right) ds + \left(K_T^+ - K_t^+\right) - \left(K_T^- - K_t^-\right) - \int_t^T Z_s dW_s,\ \forall t \leq T$$

and

$$L_t \leq Y_t \leq U_t,\ \forall t \leq T, \quad \int_0^T \left(Y_s - L_s\right) dK_s^+ = \int_0^T \left(U_s - Y_s\right) dK_s^- = 0.$$

We will also assume that U, L and ξ are bounded, i.e.,

$$\text{esssup}[|\xi| + \sup_{t \leq T} \{|U_t| + |L_t|\}] < +\infty. \tag{5}$$

We have the following results (see Hamadène-Hdhiri [17], Theorem 3.2 and Remark 3.3).

Theorem 1 Under the assumptions (4) and (5), there exists a $\mathcal{P}$-measurable process (Y, Z, K^+, K^-) solution for the RBSDE associated with (F, ξ, L, U). Moreover, the solution is maximal, i.e., if (Y', Z', K'^+, K'^-) is another solution of the above equation, then $\mathbb{P}$ - a.s., for all $t \leq T$, we have $Y'_t \leq Y_t$.

The main idea for proving the existence of a solution for RBSDE associated with (F, ξ, L, U) is to find a solution for a RBSDE associated with data obtained by an exponential transform of (F, ξ, L, U). Then, by a limiting procedure applied to a suitably constructed monotone sequence of bounded, continuous generators which approximate the driver, it is shown that the initial equation has a solution.

Proposition 1 (comparison) Let F and F' be two generators satisfying (4) and (5) such that, $\mathbb{P}$ - a.s., $F(t, \omega, y, z) \leq F'(t, \omega, y, z)$ for any t, y, z, and consider $\left(Y_t, Z_t, K_t^+, K_t^-\right)_{t \leq T}$, (resp. $\left(Y'_t, Z'_t, K'^+_t, K'^-_t\right)_{t \leq T}$) the maximal solution of the RBSDE associated with (F, ξ, L, U) (resp. (F', ξ, L, U)). Then, $\mathbb{P}$ - a.s. $Y_t \leq Y'_t$, for all $t \leq T$.

2 Main results

2.1 FBSDEs, strategies for stochastic games, viscosity solutions, Isaacs equations

Let A and B be two compact metric spaces.

Definition 2 An admissible control process $\alpha = (\alpha_s)_{s\in[t,T]}$ (resp., $\beta = (\beta_s)_{s\in[t,T]}$) for Player I (resp., Player II) on $[t,T]$ ($t<T$) is an $\mathbb{F}$-progressively measurable process taking values in A (resp., B). The set of all admissible controls on $[t,T]$ for the two players will be denoted by $\mathcal{A}_t$, respectively $\mathcal{B}_t$.

Now, for $t<T$, $\alpha(\cdot)\in\mathcal{A}_t$ and $\beta(\cdot)\in\mathcal{B}_t$, let us consider the following SDE:

$$\begin{cases} dX_s^{t,x;\alpha,\beta} = b\left(s, X_s^{t,x;\alpha,\beta}, \alpha_s, \beta_s\right) ds + \sigma\left(s, X_s^{t,x;\alpha,\beta}, \alpha_s, \beta_s\right) dW_s,\ \ s\in[t,T]\,; \\ X_s^{t,x;\alpha,\beta} = x,\ \ s\le t, \end{cases} \tag{6}$$

where the coefficients

$$b:[0,T]\times\mathbb{R}^n\times A\times B\longrightarrow\mathbb{R}^n \text{ and } \sigma:[0,T]\times\mathbb{R}^n\times A\times B\longrightarrow\mathbb{R}^{n\times d}$$

satisfy the following conditions:

$$\begin{cases} (i) & \text{for every } x \in \mathbb{R}^n,\ b(\cdot, x, \cdot, \cdot) \text{ and } \sigma(\cdot, x, \cdot, \cdot) \text{ are continuous;} \\ (ii) & \text{there exists } C_L > 0 \text{ such that, for all } t \in [0,T],\ x, x' \in \mathbb{R}^n,\ \alpha \in A,\ \beta \in B, \\ & |b(t,x,\alpha,\beta) - b(t,x',\alpha,\beta)| + |\sigma(t,x,\alpha,\beta) - \sigma(t,x',\alpha,\beta)| \le C_L \left| x - x' \right|. \end{cases} \tag{H1}$$

It is known (see Karatzas-Shreve [27]) that, under the assumptions (H1), for every

$$(\alpha(\cdot), \beta(\cdot)) \in \mathcal{A}_t \times \mathcal{B}_t$$

the SDE (6) has a unique solution. Moreover, for every $p \ge 2$, there exists $C_p > 0$ such that, for all $t \in [0,T]$, $x, x' \in \mathbb{R}^n$, $(\alpha(\cdot), \beta(\cdot)) \in \mathcal{A}_t \times \mathcal{B}_t$, we have, $\mathbb{P}$ - a.s.:

$$\mathbb{E}\left[\sup_{s \in [t,T]} |X_s^{t,x;\alpha,\beta} - X_s^{t,x';\alpha,\beta}|^p \middle| \mathcal{F}_t \right] \le C_p |x - x'|^p;$$

$$\mathbb{E}\left[\sup_{s \in [t,T]} |X_s^{t,x;\alpha,\beta}|^p \middle| \mathcal{F}_t \right] \le C_p (1 + |x|^p);$$

(one can see Buckdahn-Li [7, 8] for more details). The constant C_p depends only on the Lipschitz and the linear growth constants of b and σ.

Let us now consider the functions

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad h, h' : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad F : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times A \times B \longrightarrow \mathbb{R}\,,$$

that satisfy the following conditions:

$$\begin{cases} (i) \quad g \text{ is continuous and bounded; } h \text{ and } h' \text{ are also continuous and bounded} \\ \text{and, for any } (t,x) \in [0,T] \times \mathbb{R}^n, \quad h(t,x) < h'(t,x). \text{ Moreover, we assume that} \\ \qquad h(T,x) \le g(x) \le h'(T,x)\,,\ \forall x \in \mathbb{R}^n. \\ (ii) \quad F \text{ is continuous and has quadratic growth in } z,\ \textit{i.e.} \\ |F(t,x,y,z,\alpha,\beta)| \le C\left(1+|z|^2\right),\ \forall (t,x,y,z,\alpha,\beta) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times A \times B. \end{cases} \tag{H2}$$

Under the above hypothesis, for any $(t,x) \in [0,T] \times \mathbb{R}^n$ and $(\alpha(\cdot), \beta(\cdot)) \in \mathcal{A}_t \times \mathcal{B}_t$, there exists a maximal solution $\left(Y^{t,x;\alpha,\beta}, Z^{t,x;\alpha,\beta}, K^{+,t,x;\alpha,\beta}, K^{-,t,x;\alpha,\beta}\right)$ of the RBSDE associated with

$$\left(F\left(\cdot, X^{t,x;\alpha,\beta}, \cdot, \cdot, \alpha(\cdot), \beta(\cdot)\right), g(X_T^{t,x;\alpha,\beta}), h\left(\cdot, X^{t,x;\alpha,\beta}\right), h'\left(\cdot, X^{t,x;\alpha,\beta}\right)\right)\,, \tag{7}$$

where $X^{t,x;\alpha,\beta}$ is the solution of Eq. (6).

Definition 3 A nonanticipative strategy for Player I on $[t,T]$ is an application $S_1 : \mathcal{B}_t \longrightarrow \mathcal{A}_t$ such that, for any $\mathbb{F}$-stopping time $\tau : \Omega \longrightarrow [t,T]$ and any $\beta_1(\cdot), \beta_2(\cdot) \in \mathcal{B}_t$ satisfying $\beta_1 = \beta_2$ on $[t,\tau]$, $\mathbb{P} \otimes dt$ a.e., we have that $S_1(\beta_1) = S_1(\beta_2)$ on $[t,\tau]$, $\mathbb{P} \otimes dt$ a.e. A nonanticipative strategy on $[t,T]$ for the second player is a function $S_2 : \mathcal{A}_t \longrightarrow \mathcal{B}_t$ defined in the same manner. We will denote the sets of nonanticipative strategies for the two players by $\mathbb{A}_t$, respectively $\mathbb{B}_t$.

For any given control processes $\alpha(\cdot) \in \mathcal{A}_t$ and $\beta(\cdot) \in \mathcal{B}_t$, we consider the associated cost functional

$$J(t,x;\alpha,\beta) := Y_t^{t,x;\alpha,\beta}, \quad (t,x) \in [0,T] \times \mathbb{R}^n \tag{8}$$

and we define the lower value function of the stochastic differential game with reflection

$$\mathcal{W}(t,x) := \operatorname*{essinf}_{S_2 \in \mathbb{B}_t} \operatorname*{esssup}_{\alpha \in \mathcal{A}_t} J(t,x;\alpha,S_2(\alpha)) \tag{9}$$

and the upper value function

$$\mathcal{V}(t,x) := \operatorname*{esssup}_{S_1 \in \mathbb{A}_t} \operatorname*{essinf}_{\beta \in \mathcal{B}_t} J(t,x;S_1(\beta),\beta). \tag{10}$$

Remark 1 The essential infimum and the essential supremum exist and should be understood with respect to indexed families of random variables (see the appendix of Karatzas-Shreve [27]*, pp. 323–325).*

Let us now introduce the following two Isaacs equations with obstacles

$$\begin{cases} \min\left\{u(t,x)-h(t,x),\max\left\{-\frac{\partial u}{\partial t}(t,x)-H^{-}\left(t,x,u,Du,D^2u\right),u(t,x)-h'(t,x)\right\}\right\}=0; \\ u(T,x)=g(x), \end{cases} \tag{11}$$

$$\begin{cases} \min\left\{v(t,x)-h(t,x),\max\left\{-\frac{\partial v}{\partial t}(t,x)-H^{+}\left(t,x,v,Dv,D^2v\right),v(t,x)-h'(t,x)\right\}\right\}=0; \\ v(T,x)=g(x), \end{cases} \tag{12}$$

associated with the Hamiltonians

$$H^{-}(t,x,u,q,X) := \sup_{\alpha\in A}\inf_{\beta\in B}\left\{\frac{1}{2}Tr\left(\sigma\sigma^T(t,x,\alpha,\beta)X\right)+\langle b(t,x,\alpha,\beta),q\rangle+F(t,x,u,q\sigma(t,x,\alpha,\beta),\alpha,\beta)\right\}$$

and

$$H^{+}(t,x,u,q,X) := \inf_{\beta\in B}\sup_{\alpha\in A}\left\{\frac{1}{2}Tr\left(\sigma\sigma^T(t,x,\alpha,\beta)X\right)+\langle b(t,x,\alpha,\beta),q\rangle+F(t,x,u,q\sigma(t,x,\alpha,\beta),\alpha,\beta)\right\},$$

for all $(t,x,u,q,X)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}_n$ ($\mathbb{S}_n$ denotes the set of symmetric $n\times n$ matrices).

We show that, under suitable hypothesis, the functions $\mathcal{W}$ and $\mathcal{V}$ are the unique solutions of viscosity of the equations (11), respectively (12).

Definition 4 1. *An upper semicontinuous function* $u : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a viscosity subsolution of equation* (11) *if* $u(T,x) \le g(x)$, *for every* $x \in \mathbb{R}^n$, *and whenever* $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^n)$ *and* $(t,x) \in [0,T) \times \mathbb{R}^n$ *is a maximum point for* $u - \varphi$, *we have*

$$\min\left\{u(t,x) - h(t,x), \max\left\{-\frac{\partial \varphi}{\partial t}(t,x) - H^-\left(t,x,u,D\varphi,D^2\varphi\right), u(t,x) - h'(t,x)\right\}\right\} \le 0;$$

2. *A lower semicontinuous function* $u : [0,T) \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a viscosity supersolution of equation* (11) *if* $u(T,x) \ge g(x)$, *for every* $x \in \mathbb{R}^n$, *and whenever* $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^n)$ *and* $(t,x) \in [0,T) \times \mathbb{R}^n$ *is a minimum point for* $u - \varphi$, *we have*

$$\min\left\{u(t,x) - h(t,x), \max\left\{-\frac{\partial \varphi}{\partial t}(t,x) - H^-\left(t,x,u,D\varphi,D^2\varphi\right), u(t,x) - h'(t,x)\right\}\right\} \ge 0;$$

3. *A function* $u : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is called a viscosity solution of equation* (11) *if it is both a viscosity sub- and supersolution for this equation.*

Remark 2 (i) Of course, the definition of viscosity solutions for equation (12) is similar.

(ii) In the above definitions, one can take strict local maximum or minimum point instead of global maximum, respectively minimum point.

We recall (from Crandall-Ishii-Lions [9], pp. 49) the definition of parabolic superjet and subjet of a function defined on a locally compact set, notions which will be needed during the proof of the uniqueness.

Definition 5 For a function $u : [0,T) \times \mathbb{R}^n \longrightarrow \mathbb{R}$, the second-order parabolic superjet of u in $(t_0, x_0) \in [0,T) \times \mathbb{R}^n$, denoted by $\mathcal{P}^{2,+}u\,(t_0,x_0)$, is the set of triplets $(p,q,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$ satisfying, as $(t,x) \to (t_0,x_0)$,

$$u\,(t,x) \leq u\,(t_0,x_0) + p\,(t-t_0) + \langle q, x-x_0\rangle + \frac{1}{2}\,\langle X\,(x-x_0)\,, X\,(x-x_0)\rangle + o\left(|t-t_0| + |x-x_0|^2\right).$$

Switching the inequality sign in the above relation, we get the definition of the second-order parabolic subjet of u in (t_0,x_0), denoted by $\mathcal{P}^{2,-}u\,(t_0,x_0)$. It is clear that $\mathcal{P}^{2,-}u = -\mathcal{P}^{2,+}(-u)\,.$

One can give (see Crandall-Ishii-Lions [9]) the definition of viscosity subsolution, resp. supersolution in terms of superjets, respectively subjets, as follows:

Proposition 2 An upper semicontinuous function $u : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$*, satisfying* $u(T,\cdot) \leq g$*, is a viscosity subsolution of equation (11) if and only if for every* $(t,x) \in [0,T) \times \mathbb{R}^n$ *and every* $(p,q,X) \in \mathcal{P}^{2,+}u(t,x)$*, we have*

$$\min\left\{u(t,x) - h(t,x), \max\left\{-p - H^-(t,x,u(t,x),q,X), u(t,x) - h'(t,x)\right\}\right\} \leq 0.$$

A similar result holds for viscosity supersolutions.

Since H^- is continuous, one can replace the superjets and subjets with their closure, whose definition is given below:

Definition 6 For $u : [0,T) \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$, *we define* $\bar{\mathcal{P}}^{2,+}u(t_0,x_0)$ *as the set of triplets* $(p,q,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$ *for which there exists a sequence* $(t_n,x_n,p_n,q_n,X_n) \in [0,T) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$ *such that* $(p_n,q_n,X_n) \in \mathcal{P}^{2,+}u(t_n,x_n)$, *for all* $n \in \mathbb{N}$ *and*

$$(t_n,x_n,u(t_n,x_n),p_n,q_n,X_n) \to (t_0,x_0,u(t_0,x_0),p,q,X).$$

Similarly, we define $\bar{\mathcal{P}}^{2,-}u(t_0,x_0)$.

In order to prove the uniqueness result, we will need additional properties imposed on the generator F. We suppose that F satisfies the following assumption: there exist a constant $C > 0$ and, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$, such that for all

$$(t, x, y, z, \alpha, \beta) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times A \times B$$

we have

$$\begin{aligned} \left| \left(F + \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \right) (t, x, y, z, \alpha, \beta) \right| &\leq C \left(1 + |z|^2 \right), \\ \frac{\partial F}{\partial y} (t, x, y, z, \alpha, \beta) &\leq C_\varepsilon + \varepsilon \left| z \right|^2 . \end{aligned} \tag{H3}$$

2.2 Existence and uniqueness of a viscosity solution, Isaacs' condition

The framework is now set for the main results.

Theorem 2 (Existence) Under assumptions (H1), (H2), and (H3), the lower value function $\mathcal{W}$ *defined by* (9) *is a viscosity solution of the Isaacs equation with two barriers* (11)*, while the upper value function* $\mathcal{V}$ *defined by* (10) *is a viscosity solution of the Isaacs equation* (12)*.*

Theorem 3 (Uniqueness) Under assumptions (H1), (H2) and (H3), if u *is a bounded viscosity subsolution and* v *is a bounded viscosity supersolution of equation* (11)*, then*

$$u\left(t,x\right)\leq v\left(t,x\right),\ \forall\left(t,x\right)\in\left[0,T\right]\times\mathbb{R}^{n}.$$

The same comparison principle holds for the Isaacs equation (12)*.*

Remark 3 If, in addition, the Isaacs' condition holds, i.e.

$$H^{-}\left(t,x,u,q,X\right)=H^{+}\left(t,x,u,q,X\right),$$

for every $(t,x,u,q,X)\in[0,T]\times\mathbb{R}^{n}\times\mathbb{R}\times\mathbb{R}^{n}\times\mathbb{S}_{n}$*, then the two Isaacs equations coincide and it follows that the upper and the lower value functions are equal, which means that the corresponding reflected stochastic differential game has a value.*

2.3 Proofs

We will focus our attention on the first Isaacs equation (11), the case of the equation (12) being treated in a similar manner.

2.3.1 Uniqueness

Let us consider a subsolution u, respectively a supersolution v, of equation (11).
First, we will make a change of variable, which preserves viscosity sub- and supersolutions and which transforms the equation into a Isaacs equation whose Hamiltonian will satisfy some kind of monotonicity.
We consider $\tilde{C} := \max\left(\|u\|_\infty, \|v\|_\infty\right) + 1$ and introduce the positive, increasing function ρ used in Kobylanski [25]:

$$\rho : \mathbb{R} \longrightarrow \left(-\left(\ln \gamma\right)/\lambda, +\infty\right), \quad \rho\left(x\right) := \frac{1}{\lambda} \ln\left(\frac{e^{\lambda\gamma x} + 1}{\gamma}\right),$$

for $\gamma, \lambda > 0$ satisfying $-\left(\ln \gamma\right)/\lambda \leq \tilde{C}$. We make the change of variable

$$\bar{u} := \rho^{-1}\left(e^{Kt}\left(u - \tilde{C}\right)\right), \quad \text{with} \quad K > 0.$$

Equation (11) becomes

$$\begin{cases} \min\left\{\rho\left(\bar{u}\right)-\rho\left(\bar{h}\right),\max\left\{\rho'\left(\bar{u}\right)\left[-\frac{\partial \bar{u}}{\partial t}-\bar{H}^{-}\left(t,x,\bar{u},D\bar{u},D^{2}\bar{u}\right)\right],\rho\left(\bar{u}\right)-\rho\left(\bar{h}'\right)\right\}\right\}=0, \\ \bar{u}\left(T,x\right)=\bar{g}\left(x\right), \end{cases} \tag{13}$$

where

$$\bar{H}^{-}\left(t,x,\bar{u},\bar{q},\bar{X}\right)=\sup_{\alpha\in A}\inf_{\beta\in B}\left[\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^{T}\left(t,x,\alpha,\beta\right)\bar{X}\right)+\left\langle b\left(t,x,\alpha,\beta\right),\bar{q}\right\rangle+\bar{F}\left(t,x,\bar{u},\bar{q}\sigma\left(t,x,\alpha,\beta\right),\alpha,\beta\right)\right],$$

with $\bar{F}$ defined by

$$\bar{F}\left(t,x,\bar{u},\bar{z},\alpha,\beta\right)=\frac{\rho''\left(\bar{u}\right)}{\rho'\left(\bar{u}\right)}\left|\bar{z}\right|^{2}-K\frac{\rho\left(\bar{u}\right)}{\rho'\left(\bar{u}\right)}+\frac{e^{Kt}}{\rho'\left(\bar{u}\right)}F\left(t,x,e^{-Kt}\rho\left(\bar{u}\right)+\tilde{C},e^{-Kt}\rho'\left(\bar{u}\right)\bar{z},\alpha,\beta\right)$$

and

$$\begin{aligned} \bar{h}\left(t,x\right) &:= \rho^{-1}\left(e^{Kt}\left(h\left(t,x\right)-\tilde{C}\right)\right), \\ \bar{h}'\left(t,x\right) &:= \rho^{-1}\left(e^{Kt}\left(h'\left(t,x\right)-\tilde{C}\right)\right), \\ \bar{g}\left(x\right) &:= \rho^{-1}\left(e^{Kt}\left(g\left(x\right)-\tilde{C}\right)\right). \end{aligned}$$

The function $\bar{F}$ verifies, for γ big enough, *Lemma 1 There exist some positive constants $\tilde{K}$ and $\bar{C}$ such that for all $t \in (0,T)$, $(\alpha,\beta) \in A \times B$, $x, y \in \mathbb{R}^n$, $z, z' \in \mathbb{R}^d$, and $u, v \in \mathbb{R}$ such that $u < v$,*

$$\bar{F}(t,x,u,z,\alpha,\beta) - \bar{F}(t,y,v,z',\alpha,\beta) \le \mathcal{K}(z,z')\left(-\tilde{K}(u-v) + \bar{C}|x-y| + \bar{C}|z-z'|\right),$$

where $\mathcal{K}(z,z') := \left(1 + \frac{|z|^2}{2} + \frac{|z'|^2}{2}\right).$

By this transformation, if u (resp., v) is a subsolution (resp., a supersolution) of equation (11), then $\bar{u}$ (resp., $\bar{v}$) is one for equation (13). Therefore, we want to prove that

$$M := \sup_{(t,x)\in[0,T]\times\mathbb{R}^n} (\bar{u}(t,x) - \bar{v}(t,x)) \le 0.$$

Define also, for $h > 0$,

$$M(h) := \sup_{|x-y|\le h} |\bar{u}(t,x) - \bar{v}(t,y)| \quad \text{and} \quad M' := \lim_{h\to 0} M(h).$$

It is clear that

$$M \le M'.$$

We assume to the contrary that $M > 0$ and define for every $\varepsilon,\, \eta > 0$

$$\Psi_{\varepsilon,\eta}(t,x,y) := \bar{u}(t,x) - \bar{v}(t,y) - \frac{|x-y|^2}{\varepsilon^2} - \eta\left(|x|^2 + |y|^2\right).$$

Let us consider

$$M_{\varepsilon,\eta} := \sup_{(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n} \Psi_{\varepsilon,\eta}(t,x,y) = \max_{(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n} \Psi_{\varepsilon,\eta}(t,x,y)$$

Since the functions $\bar{u}$ and $\bar{v}$ are bounded, the supremum of $\Psi_{\varepsilon,\eta}$ is reached at some point $(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})$, which will be denoted for simplicity $\left(\hat{t},\hat{x},\hat{y}\right)$.

Let recall some notations and results from Kobylanski [25]:
For the sequence $(a_{\varepsilon,\eta})_{\varepsilon,\eta}$, we write $a = \lim\limits_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta}$ if

$$\liminf_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta} = \limsup_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta} = a,$$

where

$$\liminf_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta} = \liminf_{\eta\to 0}\left(\liminf_{\varepsilon\to 0} a_{\varepsilon,\eta}\right),$$

$$\limsup_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta} = \limsup_{\eta\to 0}\left(\limsup_{\varepsilon\to 0} a_{\varepsilon,\eta}\right).$$

The following result is the equivalent of Lemma 3.1 from Crandall-Ishii-Lions [9].

Lemma 2 Considering the above notations, we have

$$(i) \quad \lim_{\varepsilon \ll \eta \to 0} M_{\varepsilon,\eta} = M, \quad \lim_{\varepsilon \ll \eta \to 0} \bar{u}\left(\hat{t}, \hat{x}\right) - \bar{v}\left(\hat{t}, \hat{y}\right) = M;$$

$$(ii) \quad \lim_{\eta \ll \varepsilon \to 0} M_{\varepsilon,\eta} = M', \quad \lim_{\eta \ll \varepsilon \to 0} \bar{u}\left(\hat{t}, \hat{x}\right) - \bar{v}\left(\hat{t}, \hat{y}\right) = M';$$

$$(iii) \quad \lim_{\eta \ll \varepsilon \to 0} \frac{|\hat{x} - \hat{y}|}{\varepsilon} = 0, \quad \lim_{\eta \ll \varepsilon \to 0} \eta \left(|\hat{x}|^2 + |\hat{y}|^2\right) = 0.$$

By extracting a subsequence, we suppose that for every η,

$$\left(t^{\varepsilon,\eta}\right)_\varepsilon \longrightarrow t^\eta \quad \text{as} \quad \varepsilon \to 0$$

and, extracting again a subsequence, the sequences $\left(x^{\varepsilon,\eta}\right)_\varepsilon$ and $\left(y^{\varepsilon,\eta}\right)_\varepsilon$ converge to a common limit x^η.

Define the functions $\phi_1, \phi_2 : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\phi_1(t,x) = \bar{v}\left(t, y^{\varepsilon,\eta}\right) + \frac{|x - y^{\varepsilon,\eta}|^2}{\varepsilon^2} + \eta\left(|x|^2 + |y^{\varepsilon,\eta}|^2\right)$$

$$\phi_2(t,y) = \bar{u}\left(t, x^{\varepsilon,\eta}\right) - \frac{|x^{\varepsilon,\eta} - y|^2}{\varepsilon^2} - \eta\left(|x^{\varepsilon,\eta}|^2 + |y|^2\right).$$

It is obvious that

- $(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})$ is a maximum point for the function $(t,x) \longmapsto \Psi_{\varepsilon,\eta}(t,x,y^{\varepsilon,\eta}) = (\bar{u} - \phi_1)(t,x)$,
- $(t^{\varepsilon,\eta}, y^{\varepsilon,\eta})$ is a minimum point for $(t,y) \longmapsto -\Psi_{\varepsilon,\eta}(t,x^{\varepsilon,\eta},y) = (\phi_2 - \bar{v})(t,y)$.

Since $\bar{u}$ and $\bar{v}$ are viscosity subsolution, respectively supersolution, we obtain

- either $t^{\varepsilon,\eta} = T$ and then $\bar{u}(T, x^{\varepsilon,\eta}) \leq \bar{g}(x^{\varepsilon,\eta})$ and $\bar{g}(y^{\varepsilon,\eta}) \leq \bar{v}(T, y^{\varepsilon,\eta})$,
- or $t^{\varepsilon,\eta} \neq T$ and we have, in $(\hat{t}, \hat{x})$,

$$\min\left\{\rho(\bar{u}) - \rho(\bar{h}), \max\left\{\rho'(\bar{u})\left[-\frac{\partial\phi_1}{\partial t} - \bar{H}^-\left(\hat{t}, \hat{x}, \bar{u}, D\phi_1, D^2\phi_1\right)\right], \rho(\bar{u}) - \rho\left(\bar{h}'\right)\right\}\right\} \leq 0$$

and, respectively, in $(\hat{t}, \hat{y})$,

$$\min\left\{\rho(\bar{v}) - \rho(\bar{h}), \max\left\{\rho'(\bar{v})\left[-\frac{\partial\phi_2}{\partial t} - \bar{H}^-\left(\hat{t}, \hat{y}, \bar{v}, D\phi_2, D^2\phi_2\right)\right], \rho(\bar{v}) - \rho\left(\bar{h}'\right)\right\}\right\} \geq 0.$$

In the first situation, there exists a subsequence of $(t^\eta)_\eta$, supposed, without restricting the generality, to be the same, such that $t^\eta = T$ for every η. The semicontinuity of the functions $\bar{u}$ and $\bar{v}$ and the continuity of $\bar{g}$ give us that, for all η and ε sufficiently small

$$\bar{u}\left(\hat{t}, \hat{x}\right) \leq \bar{u}(T, x^\eta) + \eta \leq \bar{g}(x^\eta) + \eta \quad \text{and} \quad \bar{g}(x^\eta) - \eta \leq \bar{v}(T, x^\eta) - \eta \leq \bar{v}\left(\hat{t}, \hat{x}\right).$$

We obtain

$$\bar{u}\left(\hat{t},\hat{x}\right) \leq \bar{v}\left(\hat{t},\hat{x}\right) + 2\eta$$

and, from here, we find that

$$M_{\varepsilon,\eta} \leq 2\eta.$$

Taking the limit of ε, and after this η, to zero, it follows that

$$M \leq 0,$$

which is a contradiction. Therefore $t^{\varepsilon,\eta} \neq T$.

Now, since ρ is an increasing function, if there exists a subsequence $t^{\eta} \neq T$, and, for every η a subsequence of $\left(x^{\varepsilon,\eta}\right)_{\varepsilon}$ and one of $\left(y^{\varepsilon,\eta}\right)_{\varepsilon}$ such that the following inequalities hold

$$\bar{u}\left(t^{\varepsilon,\eta},x^{\varepsilon,\eta}\right) - \bar{h}\left(t^{\varepsilon,\eta},x^{\varepsilon,\eta}\right) \leq 0 \quad \text{and} \quad \bar{v}\left(t^{\varepsilon,\eta},y^{\varepsilon,\eta}\right) \geq \bar{h}\left(t^{\varepsilon,\eta},y^{\varepsilon,\eta}\right),$$

we obtain

$$M_{\varepsilon,\eta} \leq \bar{u}\left(t^{\varepsilon,\eta},x^{\varepsilon,\eta}\right) - \bar{v}\left(t^{\varepsilon,\eta},y^{\varepsilon,\eta}\right) \leq \bar{h}\left(t^{\varepsilon,\eta},x^{\varepsilon,\eta}\right) - \bar{h}\left(t^{\varepsilon,\eta},y^{\varepsilon,\eta}\right).$$

The continuity of $\bar{h}$ and the above Lemma leads again to a contradiction.

When, also on a subsequence like in the previous case, $\bar{u}\left(\hat{t},\hat{x}\right) \leq \bar{h}'\left(\hat{t},\hat{x}\right)$,

$$-\frac{\partial \phi_1}{\partial t}\left(\hat{t},\hat{x}\right) - \bar{H}^-\left(\hat{t},\hat{x},\bar{u}\left(\hat{t},\hat{x}\right),D\phi_1\left(\hat{t},\hat{x}\right),D^2\phi_1\left(\hat{t},\hat{x}\right)\right) \leq 0$$

and $\bar{v}\left(\hat{t},\hat{y}\right) \geq \bar{h}'\left(\hat{t},\hat{y}\right)$, we obtain also that $M \leq 0$.

For the remaining situation, we first use Theorem 8.3 from Crandall-Ishii-Lions [9]. Setting

$$\varphi\left(x,y\right) := \frac{\left|x-y\right|^2}{\varepsilon^2} + \eta\left(\left|x\right|^2 + \left|y\right|^2\right),$$

from the mentioned result, we have that there exist the matrices $\bar{X},\bar{Y} \in \mathbb{S}_n$ such that

$$\begin{cases} \left(0, D_x\varphi\left(\hat{x},\hat{y}\right),\bar{X}\right) \in \bar{\mathcal{P}}^{2,+}\bar{u}\left(\hat{t},\hat{x}\right), \\ \left(0, D_y\varphi\left(\hat{x},\hat{y}\right),-\bar{Y}\right) \in \bar{\mathcal{P}}^{2,+}\left(-\bar{v}\right)\left(\hat{t},\hat{y}\right) = -\bar{\mathcal{P}}^{2,-}\bar{v}\left(\hat{t},\hat{y}\right). \end{cases}$$

So $\left(0, -D_y\varphi\left(\hat{t},\hat{x},\hat{y}\right),\bar{Y}\right) \in \bar{\mathcal{P}}^{2,-}\bar{v}\left(\hat{t},\hat{y}\right)$.

Moreover,

$$\begin{pmatrix} \bar{X} & 0 \\ 0 & -\bar{Y} \end{pmatrix} \leq \frac{2}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\eta\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{14}$$

We have then from the definition of the sub- and superjet that

$$\begin{cases} \bar{H}^{-}\left(\hat{t},\hat{x},\bar{u}\left(\hat{t},\hat{x}\right),D_x\varphi\left(\hat{x},\hat{y}\right),\bar{X}\right)\geq 0, \\ \bar{H}^{-}\left(\hat{t},\hat{y},\bar{v}\left(\hat{t},\hat{y}\right),-D_y\varphi\left(\hat{x},\hat{y}\right),\bar{Y}\right)\leq 0, \end{cases}$$

which implies

$$\bar{H}^{-}\left(\hat{t},\hat{x},\bar{u}\left(\hat{t},\hat{x}\right),\frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}+2\eta\hat{x},\bar{X}\right)\geq \bar{H}^{-}\left(\hat{t},\hat{y},\bar{v}\left(\hat{t},\hat{y}\right),\frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}-2\eta\hat{y},\bar{Y}\right).$$

By denoting

$$\begin{aligned}\hat{\mathcal{K}}_x^{\alpha,\beta}:=&\frac{1}{2}Tr\left(\sigma\sigma^T\left(\hat{t},\hat{x},\alpha,\beta\right)\bar{X}\right)\\ &+\left\langle b\left(\hat{t},\hat{x},\alpha,\beta\right),\frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}+2\eta\hat{x}\right\rangle+\bar{F}\left(\hat{t},\hat{x},\bar{u}\left(\hat{t},\hat{x}\right),\left(\frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}+2\eta\hat{x}\right)\sigma\left(\hat{t},\hat{x},\alpha,\beta\right),\alpha,\beta\right)\end{aligned}$$

and

$$\begin{aligned}\hat{\mathcal{K}}_y^{\alpha,\beta}:=&\frac{1}{2}Tr\left(\sigma\sigma^T\left(\hat{t},\hat{y},\alpha,\beta\right)\bar{Y}\right)\\ &+\left\langle b\left(\hat{t},\hat{y},\alpha,\beta\right),\frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}-2\eta\hat{y}\right\rangle+\bar{F}\left(\hat{t},\hat{y},\bar{v}\left(\hat{t},\hat{y}\right),\left(\frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}-2\eta\hat{y}\right)\sigma\left(\hat{t},\hat{y},\alpha,\beta\right),\alpha,\beta\right)\end{aligned}$$

this can be written as

$$\sup_{\alpha\in A}\inf_{\beta\in B}\hat{\mathcal{K}}_x^{\alpha,\beta}\geq\sup_{\alpha\in A}\inf_{\beta\in B}\hat{\mathcal{K}}_y^{\alpha,\beta}. \tag{15}$$

Also, by denoting

$$\begin{cases}\hat{b}_x := b\left(\hat{t},\hat{x},\alpha,\beta\right), & \hat{b}_y := b\left(\hat{t},\hat{y},\alpha,\beta\right),\\ \hat{\sigma}_x := \sigma\left(\hat{t},\hat{x},\alpha,\beta\right), & \hat{\sigma}_y := \sigma\left(\hat{t},\hat{y},\alpha,\beta\right),\\ \bar{q}_x := \dfrac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}+2\eta\hat{x}, & \bar{q}_y := \dfrac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}-2\eta\hat{y},\end{cases}$$

the inequality (14) infers that

$$\frac{1}{2}Tr\left(\hat{\sigma}_x\hat{\sigma}_x^T\bar{X}\right)\leq\frac{1}{2}Tr\left(\hat{\sigma}_y\hat{\sigma}_y^T\bar{Y}\right)+\frac{1}{\varepsilon^2}\left|\hat{\sigma}_x-\hat{\sigma}_y\right|^2+\eta\left(\left|\hat{\sigma}_x\right|^2+\left|\hat{\sigma}_y\right|^2\right)$$

On the other hand

$$\left\langle\hat{b}_x,\bar{q}_x\right\rangle=\left\langle\hat{b}_y,\bar{q}_y\right\rangle+\left\langle\hat{b}_x-\hat{b}_y,\frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}\right\rangle+\left\langle\hat{b}_y,2\eta\hat{y}\right\rangle+\left\langle\hat{b}_x,2\eta\hat{x}\right\rangle$$

and

$$\bar{F}\left(\hat{t},\hat{x},\bar{u}\left(\hat{t},\hat{x}\right),\bar{q}_x\hat{\sigma}_x,\alpha,\beta\right) \leq \bar{F}\left(\hat{t},\hat{y},\bar{v}\left(\hat{t},\hat{y}\right),\bar{q}_y\hat{\sigma}_y,\alpha,\beta\right) \\ + \mathcal{K}\left(\bar{q}_x\hat{\sigma}_x,\bar{q}_y\hat{\sigma}_y\right)\left(-\tilde{K}\left(\bar{u}\left(\hat{t},\hat{x}\right)-\bar{v}\left(\hat{t},\hat{y}\right)\right)+\bar{C}\left|\hat{x}-\hat{y}\right|+\bar{C}\left|\bar{q}_x\hat{\sigma}_x-\bar{q}_y\hat{\sigma}_y\right|\right).$$

Adding the last three relations, we obtain, for all $(\alpha,\beta)\in A\times B$

$$\mathcal{K}_x^{\alpha,\beta} \leq \mathcal{K}_y^{\alpha,\beta} + \frac{C_L^2}{\varepsilon^2}\left|\hat{x}-\hat{y}\right|^2 + C\eta\left(1+\left|\hat{x}\right|^2+\left|\hat{y}\right|^2\right) + \frac{2C_L}{\varepsilon^2}\left|\hat{x}-\hat{y}\right|^2 \\ + \mathcal{K}\left(\bar{q}_x\hat{\sigma}_x,\bar{q}_y\hat{\sigma}_y\right)\left(-\tilde{K}M_{\varepsilon,\eta}+\bar{C}\left|\hat{x}-\hat{y}\right|+2\bar{C}\left(C_L\frac{\left|\hat{x}-\hat{y}\right|^2}{\varepsilon^2}+\eta\left(\left|\hat{\sigma}_x\hat{x}\right|+\left|\hat{\sigma}_y\hat{y}\right|\right)\right)\right).$$

Taking the $\sup_{\alpha\in A}\inf_{\beta\in B}$, and then passing to the limit as $\varepsilon\to 0$ and $\eta\to 0$ (in this order), we obtain

$$\sup_{\alpha\in A}\inf_{\beta\in B}\hat{\mathcal{K}}_x^{\alpha,\beta} \leq \sup_{\alpha\in A}\inf_{\beta\in B}\hat{\mathcal{K}}_y^{\alpha,\beta} - \tilde{K}M,$$

which contradicts (15) if M is strictly positive as we supposed. So M must be less or equal to zero and we obtain the desired comparison result. ■

2.3.2 Existence

As indicated in preliminaries, we make the following transform:

$$f(t,x,y,z,\alpha,\beta) := \begin{cases} 2Cy\left[F\left(t,x,\frac{\ln y}{2C},\frac{z}{2Cy},\alpha,\beta\right) - \frac{|z|^2}{4Cy^2}\right], & y>0 \\ 0, & y \leq 0 \end{cases}$$

for $(\alpha,\beta) \in A \times B$ and $(t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$, where C is the constant from (H3). We also set, for the simplicity of notations,

$$\bar{h} := \exp(2Ch), \quad \bar{h}' := \exp(2Ch'), \quad \bar{g} := \exp(2Cg) \quad \text{and} \quad M := \max\left\{\inf\left(1/\bar{h}\right), \sup \bar{h}'\right\}.$$

As showed in Hamadène-Hdhiri [17], under conditions (H1), (H2) and (H3), the RBSDE associated with

$$\left(f, \bar{g}(X_T^{t,x;\alpha,\beta}), \bar{h}(\cdot, X^{t,x;\alpha,\beta}), \bar{h}'(\cdot, X^{t,x;\alpha,\beta})\right)$$

has a maximal solution, denoted

$$\left(y^{t,x;\alpha,\beta}, z^{t,x;\alpha,\beta}, k^{+,t,x;\alpha,\beta}, k^{-,t,x;\alpha,\beta}\right).$$

In order to use already known results on the lower value function associated with a RBSDE with double barrier with a Lipschitz generator (see Buckdahn-Li [8]), we approximate, in a monotone manner, the function f with Lipschitz functions.

We will make use, for $p \geq 0$, of functions $\rho_p \in C^\infty(\mathbb{R})$ satisfying $\rho_p = 1$ on $[-p, p]$ and $\rho_p = 0$ on $[-(p+1), p+1]^c$. Let us consider the function

$$\tilde{f}(t, x, y, z, \alpha, \beta) := f(t, x, y, z, \alpha, \beta)\, \rho_M\left(\frac{\ln y}{2C}\right),$$

which is bounded by a constant $C' > 0$. Let, for $p \in \mathbb{N}$,

$$\tilde{f}^p(t, x, y, z, \alpha, \beta) := \tilde{f}(t, x, y, z, \alpha, \beta)\, \rho_p(|x| + |z|) + \frac{3}{2^{p+2}};$$

it is clear that the function $\tilde{f}^p$ is bounded and uniformly continuous .

Now, by a standard *mollification* procedure, we approximate $\tilde{f}^p$ by Lipschitz functions.
Let $\theta \in C^\infty\left(\mathbb{R}^{n+1+d}\right)$ satisfying $\theta \geq 0$, $\operatorname{supp}\theta \subseteq B(0;1)$ and $\int_{\mathbb{R}^{n+1+d}} \theta(a)\, da = 1$; we set

$$\tilde{f}^p_\varepsilon(t, x, y, z, \alpha, \beta) := \frac{1}{\varepsilon^{n+1+d}} \int_{\mathbb{R}^{n+1+d}} \theta\left(\frac{x - x'}{\varepsilon}, \frac{y - y'}{\varepsilon}, \frac{z - z'}{\varepsilon}\right) \tilde{f}^p(t, x, y, z, \alpha, \beta)\, dx' dy' dz'.$$

Then $\tilde{f}^p_\varepsilon$ is Lipschitz in $(x, y, z) \in \mathbb{R}^{n+1+d}$ and

$$\left|\tilde{f}^p_\varepsilon - \tilde{f}^p\right| \leq \eta_{\tilde{f}_p}(\varepsilon),$$

where $\eta_{\tilde{f}^p}$ is the modulus of uniform continuity of $\tilde{f}^p$. Hence, one can extract a sequence $\varepsilon_p \searrow 0$ such that

$$\left|\tilde{f}_{\varepsilon_p}^p - \tilde{f}^p\right| \leq 2^{-(p+2)},\ \forall p \in \mathbb{N}^*.$$

An easy calculus shows us that $\tilde{f}_{\varepsilon_p}^p$ is upper bounded by $C' + 2^{-p}$ and

$$\left(\tilde{f}_{\varepsilon_{p+1}}^{p+1} - \tilde{f}_{\varepsilon_p}^p\right)(t,x,y,z,\alpha,\beta) \leq \tilde{f}\,(t,x,y,z,\alpha,\beta)\left(\rho_{p+1} - \rho_p\right)(|x| + |z|)\,.$$

If we set, for $p \in \mathbb{N}^*$,

$$f^p\,(t,x,y,z,\alpha,\beta) := \rho_{p-1}\,(|x| + |z|)\,\tilde{f}_{\varepsilon_p}^p\,(t,x,y,z,\alpha,\beta) + \left(1 - \rho_{p-1}\,(|x| + |z|)\right)\left(C' + 2^{-p}\right),$$

then the functions f^p are still Lipschitz in $(x,y,z) \in \mathbb{R}^{n+1+d}$ and $f^p \searrow \tilde{f}$, the convergence being uniform.

Consider, for $p \in \mathbb{N}^*$, the RBSDE associated with

$$\left(f^p, \bar{g}(X_T^{t,x;\alpha,\beta}), \bar{h}(\cdot, X^{t,x;\alpha,\beta}), \bar{h}'(\cdot, X^{t,x;\alpha,\beta})\right).$$

According to Hamadène-Hassani [16], Theorem 3.7, it has a unique solution in $\mathcal{S}^2 \times \mathcal{L}^{2,d} \times \mathcal{M} \times \mathcal{M}$, denoted

$$\left(y^{p;t,x;\alpha,\beta}, z^{p;t,x;\alpha,\beta}, k^{+,p;t,x;\alpha,\beta}, k^{-,p;t,x;\alpha,\beta}\right).$$

By the comparison result, it is obvious that the sequence $\left(y^{p;t,x;\alpha,\beta}\right)_{p\in\mathbb{N}^*}$ is non-increasing.

Proposition 3 *For every* $(t,x) \in [0,T]$, $(\alpha,\beta) \in \mathcal{A}_t \times \mathcal{B}_t$, $\lim_{p\to\infty} y_s^{p;t,x;\alpha,\beta} = y_s^{t,x;\alpha,\beta}$ *for all* $s \in [t,T]$, $\mathbb{P}$ - $a.s.$

The proof of this result follows the same five steps of the proof of the Theorem 3.1 in Hamadène-Hdhiri [17], since the different approximation sequence is constituted also by generators which are bounded and continuous. So we skip the proof.

Let us consider, for any $p \in \mathbb{N}^*$, and any given control processes $\alpha(\cdot) \in \mathcal{A}_t$, $\beta(\cdot) \in \mathcal{B}_t$, the associated cost functional

$$j^p(t,x;\alpha,\beta) := y_t^{p;t,x;\alpha,\beta}, \quad (t,x) \in [0,T] \times \mathbb{R}^n$$

and define the lower value function of the approximative stochastic differential game

$$w^p(t,x) := \operatorname*{essinf}_{S_2 \in \mathbb{B}_t} \operatorname*{esssup}_{\alpha \in \mathcal{A}_t} j^p(t,x;\alpha,S_2(\alpha)).$$

It is known, from Buckdahn-Li [8], Proposition 3.1, that w^p is deterministic and is the unique viscosity solution of the equation

$$\begin{cases} \min\left\{u(t,x) - \bar{h}(t,x), \max\left\{-\frac{\partial u}{\partial t}(t,x) - H^p\left(t,x,u,Du,D^2u\right), u(t,x) - \bar{h}'(t,x)\right\}\right\} = 0; \\ u(T,x) = \bar{g}(x), \end{cases} \quad (16)$$

where

$$H^p(t,x,u,q,X) := \sup_{\alpha\in A}\inf_{\beta\in B}\left\{\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^T(t,x,\alpha,\beta)X\right)+\langle b(t,x,\alpha,\beta),q\rangle+f^p(t,x,u,q\sigma(t,x,\alpha,\beta),\alpha,\beta)\right\}.$$

On the other hand, since the processes $\left(y^{p;t,x;\alpha,\beta}\right)_{p\in\mathbb{N}^*}$ form a non-increasing sequence, the sequence $(w^p)_{p\in\mathbb{N}^*}$ is also non-increasing. By boundedness, it has a limit, $w^0:[0,T]\times\mathbb{R}^n\longrightarrow\mathbb{R}$.

Proposition 4 The function w^0 is a viscosity subsolution of the equation

$$\begin{cases}\min\left\{u(t,x)-\bar h(t,x),\max\left\{-\frac{\partial u}{\partial t}(t,x)-\bar H\left(t,x,u,Du,D^2u\right),u(t,x)-\bar h'(t,x)\right\}\right\}=0,\\ u(T,x)=\bar g(x),\end{cases}\tag{17}$$

where

$$\bar H(t,x,u,q,X) := \sup_{\alpha\in A}\inf_{\beta\in B}\left\{\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^T(t,x,\alpha,\beta)X\right)+\langle b(t,x,\alpha,\beta),q\rangle+f(t,x,u,q\sigma(t,x,\alpha,\beta),\alpha,\beta)\right\}.$$

Proof. It is clear that w^0 is an upper semicontinuous function satisfying

$$w^0(T,\cdot) \le g(\cdot).$$

Let now suppose that $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^n)$ and that $(t,x) \in [0,T) \times \mathbb{R}^n$ is a strict local maximum point for $w^0 - \varphi$. Then there exists a sequence (t_p, x_p) in $[0,T) \times \mathbb{R}^n$, converging to (t,x), such that $w^p - \varphi$ has a local maximum point in (t_p, x_p) for all $p \in \mathbb{N}^*$ and

$$\lim_{p \to \infty} w^p(t_p, x_p) = w^0(t,x).$$

Since w^p is a viscosity solution for equation (16), it follows that, for all $p \in \mathbb{N}^*$, in (t_p, x_p),

$$\min\left\{ w^p - \bar{h}, \max\left\{ -\frac{\partial \varphi}{\partial t} - H^p\left(t_p, x_p, w^p, D\varphi, D^2\varphi\right), w^p - \bar{h}' \right\} \right\} = 0.$$

Because $\bar{h}(t,x) \le w^0(t,x) \le \bar{h}'(t,x)$, we analyse just two cases:

(i) $w^0(t,x) = \bar{h}(t,x)$, hence equation (17) is trivially satisfied;

(ii) $w^0(t,x) > \bar{h}(t,x)$, which implies $w^p(t_p, x_p) > \bar{h}(t_p, x_p)$ for sufficiently large p, and so

$$-\frac{\partial \varphi}{\partial t}(t_p, x_p) - H^p\left(t_p, x_p, w^p(t_p, x_p), D\varphi(t_p, x_p), D^2\varphi(t_p, x_p)\right) \le 0.$$

Since

$$\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(t_p,x_p,\alpha,\beta)\,D^2\varphi(t_p,x_p)\right) + \langle b(t_p,x_p,\alpha,\beta), D\varphi(t_p,x_p)\rangle + f^p(t_p,x_p,w^p(t_p,x_p), D\varphi(t_p,x_p)\sigma(t_p,x_p,\alpha,\beta),\alpha,\beta)$$

converges uniformly (with respect to α and β) to

$$\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(t,x,\alpha,\beta)\,D^2\varphi(t,x)\right) + \langle b(t,x,\alpha,\beta), D\varphi(t,x)\rangle + \tilde{f}\left(t,x,w^0(t,x), D\varphi(t,x)\sigma(t,x,\alpha,\beta),\alpha,\beta\right)$$

and $w^0(t,x) \leq M$, it follows that

$$-\frac{\partial\varphi}{\partial t}(t,x) - \bar{H}\left(t,x,w^0(t,x), D\varphi(t,x), D^2\varphi(t,x)\right) \leq 0.$$

This finishes our proof. ■

One can repeat the above schema, but with lower approximation, *i.e.* we can construct (in the same manner), an increasing sequence of Lipschitz, bounded functions (f_p) converging to $\tilde{f}$. By denoting

$$\left(y_p^{t,x;\alpha,\beta}, z_p^{t,x;\alpha,\beta}, k_p^{+,t,x;\alpha,\beta}, k_p^{-,t,x;\alpha,\beta}\right)$$

the solution of the RBSDE associated with

$$\left(f_p, \bar{g}(X_T^{t,x;\alpha,\beta}), \bar{h}(\cdot, X^{t,x;\alpha,\beta}), \bar{h}'(\cdot, X^{t,x;\alpha,\beta})\right),$$

one can show that $\left(y_p^{t,x;\alpha,\beta}\right)_p$ is an increasing sequence of processes, converging to a minimal solution of the RBSDE associated with

$$\left(f, \bar{g}(X_T^{t,x;\alpha,\beta}), \bar{h}(\cdot, X^{t,x;\alpha,\beta}), \bar{h}'(\cdot, X^{t,x;\alpha,\beta})\right).$$

Let us denote

$$j_p\left(t, x; \alpha, \beta\right) := y_{p,t}^{t,x;\alpha,\beta}, \quad (t,x) \in [0,T] \times \mathbb{R}^n$$

and

$$w_p\left(t, x\right) := \operatorname*{essinf}_{S_2 \in \mathbb{B}_t} \operatorname*{esssup}_{\alpha \in \mathcal{A}_t} j_p\left(t, x; \alpha, S_2\left(\alpha\right)\right).$$

Then, setting

$$w_0 := \lim w_p,$$

we have the analogous of Proposition 4, whose proof is essentially the same:

Proposition 5 The function w_0 is a viscosity supersolution of the equation (17).

Let us now define

$$\mathcal{W}^0 := \ln \frac{w^0}{2C} \quad \text{and} \quad \mathcal{W}_0 := \ln \frac{w_0}{2C}.$$

It is straightforward to show that $\mathcal{W}^0$ and $\mathcal{W}_0$ are viscosity subsolution, respectively supersolution, of equation (11). By the comparison result (see Hamadène-Hdhiri [17], Remark 3.3), $\mathbb{P}$ - a.s.,

$$\mathcal{W}^0(t,x) \geq \mathcal{W}(t,x) \geq \mathcal{W}_0(t,x), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n,$$

since, for all $p \in \mathbb{N}^*$, $(t,x) \in [0,T] \times \mathbb{R}^n$, all $(\alpha(\cdot), \beta(\cdot)) \in \mathcal{A}_t \times \mathcal{B}_t$, a.s.,

$$y_s^{p;t,x;\alpha,\beta} \geq y_s^{t,x;\alpha,\beta} \geq y_{p,s}^{t,x;\alpha,\beta}, \ \forall s \in [t,T].$$

On the other hand, by Theorem 3, we have that $\mathcal{W}^0 \leq \mathcal{W}_0$. These inequalities imply that $\mathcal{W}$ is deterministic and is a viscosity solution of equation (11). The existence result is thus proved.

Remark 4 As one can see from the proof of the existence result, that $\mathcal{W}$ can be defined via any solution of the BSDE associated with (F, g, h, h'), not necessarily the maximal one. This could be used to deduce the uniqueness for this equation.

3 Applications

Proposition 6 Assume that $F(t,x,y,z) = \varphi(t,x) + \frac{1}{2}|z|^2$*, where* φ *is a bounded measurable function. For any stopping times* τ *and* σ *let us consider the following standard BSDE:*

$$\begin{cases} (Y^{(t,x);(\alpha,\sigma),(\beta,\tau)}, Z^{(t,x);(\alpha,\sigma),(\beta,\tau)}) \in \mathcal{S}^2 \times \mathcal{H}^{2,d} \\ Y_s^{(t,x);(\alpha,\sigma),(\beta,\tau)} = h(X_\sigma^{t,x;\alpha,\beta})1_{[\sigma\le\tau<T]} + h'(X_\tau^{t,x;\alpha,\beta})1_{[\tau<\sigma]} + g(X_T^{t,x;\alpha,\beta})1_{[\sigma=\tau=T]} \\ \qquad + \int_{s\wedge\tau\wedge\sigma}^{T\wedge\tau\wedge\sigma} \left\{ \varphi(r, X_r^{t,x;\alpha,\beta}, \alpha_r, \beta_r) + \frac{1}{2}|Z_r^{(t,x);(\alpha,\sigma),(\beta,\tau)}|^2 \right\} dr - \int_{s\wedge\tau\wedge\sigma}^{T\wedge\tau\wedge\sigma} Z_r^{(t,x);(\alpha,\sigma),(\beta,\tau)} dW_r, \quad \forall s \le T. \end{cases} \tag{18}$$

Then

$$\Gamma(\alpha,\sigma;\beta,\tau) = \exp\{Y_0^{(0,x);(\alpha,\sigma),(\beta,\tau)}\},$$

$$\operatorname*{essinf}_{S_2\in\mathbb{B}_t} \operatorname*{esssup}_{\alpha\in\mathcal{A}_t} Y_t^{t,x;\alpha,S_2(\alpha)} = \operatorname*{essinf}_{S_2\in\mathbb{B}_t} \operatorname*{esssup}_{\alpha\in\mathcal{A}_t} \operatorname*{essinf}_{\tau\in\mathcal{T}_t} \operatorname*{esssup}_{\sigma\in\mathcal{T}_t} Y_t^{(t,x);(\alpha,\sigma),(S_2(\alpha),\tau)}$$

and

$$\operatorname*{esssup}_{S_1\in\mathbb{A}_t} \operatorname*{essinf}_{\beta\in\mathcal{B}_t} Y_t^{t,x;S_1(\beta),\beta} = \operatorname*{esssup}_{S_1\in\mathbb{A}_t} \operatorname*{essinf}_{\beta\in\mathcal{B}_t} \operatorname*{esssup}_{\sigma\in\mathcal{T}_t} \operatorname*{essinf}_{\tau\in\mathcal{T}_t} Y_t^{(t,x);(S_1(\beta),\sigma),(\beta,\tau)},$$

where by $\mathcal{T}_t$ *we denoted the set of stopping times* τ *such that* $t \le \tau \le T$*.*

Proof. For the sake of simplicity we denote $Y^{(t,x);(\alpha,\sigma),(\beta,\tau)}$ by Y. First note that since the functions h, h', g and φ are bounded then through the result by Kobylanski [25], Theorem 2.3, the solution of (18) exists and is unique. Now for $t \leq T$ let us set

$$\tilde{Y}_t = \exp\left\{Y_t + \int_0^{t\wedge\tau\wedge\sigma} \varphi(s, X_s^{t,x;\alpha,\beta})ds\right\}$$

Using now Itô's formula to obtain that:

$$\begin{cases} d\tilde{Y}_t = \tilde{Z}_t dW_t,\ t \leq T \text{ and} \\ \tilde{Y}_T = \exp\left\{h(X_\sigma^{t,x;\alpha,\beta})1_{[\sigma\leq\tau<T]} + h'(\tau, X_\tau^{t,x;\alpha,\beta})1_{[\tau<\sigma]} + g(X_T^{t,x;\alpha,\beta})1_{[\sigma=\tau=T]} + \int_0^{T\wedge\tau\wedge\sigma} \varphi(s, X_s^{t,x;\alpha,\beta})ds\right\}. \end{cases}$$

It implies that $\mathbb{E}[\tilde{Y}_0] = \mathbb{E}[\tilde{Y}_T]$.
As $\tilde{Y}_0$ is deterministic since it is $\mathcal{F}_0$-measurable then $\tilde{Y}_0 = \mathbb{E}[\tilde{Y}_0] = \mathbb{E}[\tilde{Y}_T]$, i.e.

$$\exp\{Y_0\} = \Gamma(\alpha, \sigma; \beta, \tau).$$

Let us now deal with the second relations. First note (see for example El Karoui-Kapoudjian-Pardoux-Peng-Quenez [13]) that the characterization of a solution a BSDE with two reflecting barriers implies that for any α and β,

$$Y_t^{t,x;\alpha,\beta} = \operatorname*{essinf}_{\tau\geq t} \operatorname*{esssup}_{\sigma\geq t} Y_t^{(t,x);(\alpha,\sigma),(\beta,\tau)} = \operatorname*{esssup}_{\sigma\geq t} \operatorname*{essinf}_{\tau\geq t} Y_t^{(t,x);(\alpha,\sigma),(\beta,\tau)} \tag{19}$$

since

$$\exp(Y_t^{t,x;\alpha,\beta}) = \operatorname*{essinf}_{\tau\geq t} \operatorname*{esssup}_{\sigma\geq t} \exp\{Y_t^{(t,x);(\alpha,\sigma),(\beta,\tau)}\} = \operatorname*{esssup}_{\sigma\geq t} \operatorname*{essinf}_{\tau\geq t} \exp\{Y_t^{(t,x);(\alpha,\sigma),(\beta,\tau)}\}.$$

Next, for any $\alpha \in \mathcal{A}_t$ and $S_2 \in \mathbb{B}_t$, the formula (19) *implies clearly that the first equality holds. The second one is treated in the same manner.* ■

Remark 5 The following relation holds true: $\forall s \leq T$, $\forall \tau, \sigma \in \mathcal{T}_t$,

$$Y_s^{(t,x);(\alpha,\sigma),(\beta,\tau)} = \ln\Bigg(\mathbb{E}\Bigg[\exp\{h(X_\sigma^{t,x;\alpha,\beta})1_{[\sigma\leq\tau<T]} + h'(X_\tau^{t,x;\alpha,\beta})1_{[\tau<\sigma]} + g(X_T^{t,x;\alpha,\beta})1_{[\sigma=\tau=T]} + \int_s^{T\wedge\tau\wedge\sigma} \varphi(r, X_r^{t,x;\alpha,\beta}, \alpha_r, \beta_r)dr\}\Bigg| \mathcal{F}_s\Bigg]\Bigg).$$

Remark 6 A more interesting and difficult issue is when the upper and the lower values of the mixed zero-sum two-players stochastic differential game are equal respectively to

$$\inf_{(\beta,\tau)} \sup_{(\alpha,\sigma)} \Gamma(\alpha,\sigma;\beta,\tau) \quad \textit{and} \quad \sup_{(\alpha,\sigma)} \inf_{(\beta,\tau)} \Gamma(\alpha,\sigma;\beta,\tau).$$

Unfortunately, this is still an open problem.

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Thank you !