Credit Risk.

Finance and Insurance - Stochastic Analysis and Practical Methods

Spring School

Jena, March 2009

III. Hazard process Approach

1. The Model
2. ($\mathcal{H}$)-Hypothesis
3. Enlargement of Filtration
4. Partial information
5. Intensity approach

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We denote by $\mathcal{G}_t \stackrel{def}{=} \mathcal{F}_t \vee \mathcal{H}_t$.

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Let X be an $\mathcal{F}_T$-measurable integrable r.v. Then,

$$\mathbb{E}(X 1\!\!1_{T<\tau} | \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \frac{\mathbb{E}(X 1\!\!1_{\{\tau > T\}} | \mathcal{F}_t)}{\mathbb{E}(1\!\!1_{\{\tau > t\}} | \mathcal{F}_t)} = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}(X e^{-\Gamma_T} | \mathcal{F}_t).$$

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Let h be an $\mathbb{F}$-predictable process. Then,

$$\mathbb{E}(h_\tau \mathbb{1}_{\tau<T}|\mathcal{G}_t) = h_\tau \mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}\left(\int_t^T h_u dF_u | \mathcal{F}_t \right).$$

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is a $\mathbb{G}$*-martingale.*

Proofs: *The process* $L_t = (1 - H_t)e^{\Gamma_t}$ *is a* $\mathbb{G}$-***martingale***.

From the key lemma, for $t > s$

$$\mathbb{E}(L_t|\mathcal{G}_s) \quad = \quad \mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{F}_s)$$

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&= 1\!\!1_{\{\tau>s\}}e^{\Gamma_s} = L_s
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$$M_t \stackrel{def}{=} \int_{]0,t]} dH_u - \int_{]0,t]} (1 - H_u) d\Gamma_u = -\int_{]0,t]} e^{-\Gamma_u} dL_u$$

and is a $\mathbb{G}$-martingale since L is $\mathbb{G}$-martingale.

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Let $s < t$. We give the proof in two steps, using the Doob-Meyer decomposition of F as $F_t = Z_t + A_t$.

First step: we prove

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

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In a second step, we prove that, setting $\Lambda_u = \int_0^u \frac{dA_s}{G_s}$,

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Let t fixed. From the key formula, for $h_u = \Lambda_{t\wedge u}$:

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\end{aligned}$$

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hence

$$\begin{aligned}\int_s^t \Lambda_u dF_u + \Lambda_t G_t &= -\Lambda_t G_t + \Lambda_s G_s + A_t - A_s + \Lambda_t G_t \\ &= \Lambda_s G_s + A_t - A_s\end{aligned}$$

From

$$\mathbb{E}(\Lambda_{t\wedge\tau}|\mathcal{G}_s) = \Lambda_{s\wedge\tau}\mathbb{1}_{\tau\le s} + \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(\int_s^t \Lambda_u dF_u + \Lambda_t G_t|\mathcal{F}_s\right)$$

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From

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + \mathbb{1}_{s<\tau} \frac{1}{G_s} \mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

and

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we deduce

$$\mathbb{E}(H_t - \Lambda_{t\wedge\tau}|\mathcal{G}_s) = H_s - \Lambda_{s\wedge\tau}$$

If A is absolutely continuous w.r.t. the Lebesgue measure, there exists an $\mathbb{F}$-adapted process γ, called the intensity such that the process

$$H_t - \int_0^{t\wedge\tau} \gamma_u du = H_t - \int_0^t (1 - H_u)\gamma_u du$$

is a $\mathbb{G}$-martingale. The process γ satisfies

$$\gamma_t = \lim_{h\to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t + h|\mathcal{F}_t)}{\mathbb{P}(t < \tau|\mathcal{F}_t)}$$

Exercise 1:

Let $\widetilde{V}$ and R be $\mathbb{F}$-predictable processes. The process

$$V_t \stackrel{def}{=} \widetilde{V}_t \mathbb{1}_{\{t<\tau\}} + R_\tau \mathbb{1}_{\{\tau \leq t\}}$$

is a $\mathbb{G}$-martingale if and only if the process

$$\widetilde{V}_t e^{-\Gamma_t} + \int_0^t R_u e^{-\Gamma_u} d\Gamma_u$$

is an $\mathbb{F}$-martingale

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Hint: The direct part comes from the fact that if V is a $\mathbb{G}$-martingale, then $\mathbb{E}_{\mathbb{Q}}(V_t|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale. The converse is an immediate application of the Key Lemma

Exercise 2:

Let P be the ex-dividend price process of a claim which delivers R_τ at default time and pays a cumulative coupon C till the default time, i.e. the discounted cum-dividend process

$$\beta_t P_t + 1\!\!1_{\{\tau \le t\}} \beta_\tau R_\tau + \int_0^{t\wedge\tau} \beta_u dC_u$$

is a $\mathbb{G}$-martingale. Let $\widetilde{P}_t$ be the predefault price of the process P, i.e., $\tilde{P}$ is $\mathbb{F}$-predictable and $P_t = 1\!\!1_{\{t<\tau\}} \widetilde{P}_t$. Then the process

$$P_t^* = \alpha_t \widetilde{P}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u \alpha_u \, d\Gamma_u$$

is an $\mathbb{F}$-martingale.

Conversely, if $\widetilde{V}$ is an $\mathbb{F}$-predictable process such that the process $\alpha_t \widetilde{V}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u \alpha_u \, d\Gamma_u$ is an $\mathbb{F}$-martingale, then (the discounted cum-dividend) process

$$\beta_t \widetilde{V}_t \mathbb{1}_{\{t<\tau\}} + \mathbb{1}_{\{\tau \le t\}} \beta_\tau R_\tau + \int_0^{t\wedge\tau} \beta_u dC_u$$

is a $\mathbb{G}$-martingale.

Hint: This is an application of the previous Lemma .

Computation in a restricted filtration

Let $\widetilde{\mathbb{F}} \subset \mathbb{F}$ and $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$.

From

$$F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$$

we deduce

$$\widetilde{F}_t = \mathbb{P}(\tau \leq t | \widetilde{\mathcal{F}}_t) = \mathbb{E}(F_t | \widetilde{\mathcal{F}}_t)$$

The computation of the intensity is more difficult, the $\widetilde{\mathbb{F}}$- intensity in the restricted filtration is not the conditional expectation of the $\mathbb{F}$-intensity

($\mathcal{H}$) Hypothesis

Conditional independence

We now introduce the notion of conditional independence. Let $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ be three σ-algebra. The σ-algebra $\mathcal{G}$ and $\mathcal{H}$ are said to be conditionally independent with respect to $\mathcal{F}$ if

$$\mathbb{E}(\xi\,\eta\,|\,\mathcal{F}) = \mathbb{E}(\xi\,|\,\mathcal{F})\,\mathbb{E}(\eta\,|\,\mathcal{F})$$

for any bounded, $\mathcal{G}$-measurable random variable ξ and bounded, $\mathcal{H}$-measurable random variable η.

Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations with $\mathbb{F} \subset \mathbb{G}$. *The σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ if and only if one of the following conditions holds*

(i) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ:
$\mathbb{E}(\xi \,|\, \mathcal{G}_t) = \mathbb{E}(\xi \,|\, \mathcal{F}_t)$.

(ii) For any $t \in \mathbb{R}_+$, and any bounded, $\mathcal{G}_t$-measurable random variable η:
$\mathbb{E}(\eta \,|\, \mathcal{F}_t) = \mathbb{E}(\eta \,|\, \mathcal{F}_\infty)$.

PROOF: (a) Let us assume that $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$. Note that $\mathbb{E}(\xi|\mathcal{F}_t)$ is $\mathcal{F}_t$ hence $\mathcal{G}_t$-measurable. To establish (i), we shall prove that

$$E(\eta_t \mathbb{E}(\xi|\mathcal{G}_t)) = \mathbb{E}(\eta_t \mathbb{E}(\xi \,|\, \mathcal{F}_t)) \; \forall \xi \in \mathcal{F}_\infty \, \forall \eta_t \in \mathcal{G}_t$$

or equivalently

$$E(\xi\eta_t) = \mathbb{E}(\eta_t \mathbb{E}(\xi \,|\, \mathcal{F}_t))$$

The rules of conditional expectation yield to the equalities

$$\begin{aligned} E(\xi\eta_t) &= \mathbb{E}\{\mathbb{E}(\xi\eta_t|\mathcal{F}_t)\} = \mathbb{E}\{\mathbb{E}(\xi|\mathcal{F}_t)\mathbb{E}(\eta_t|\mathcal{F}_t)\} \\ &= \mathbb{E}\{\mathbb{E}[\eta_t \mathbb{E}(\xi|\mathcal{F}_t)|\mathcal{F}_t]\} = \mathbb{E}\{\mathbb{E}[\eta_t \mathbb{E}(\xi|\mathcal{F}_t)]\} = \mathbb{E}(\eta_t \mathbb{E}(\xi|\mathcal{F}_t)) \end{aligned}$$

(b) Let us prove that (i) implies (ii). Note that $\mathbb{E}(\eta_t|\mathcal{F}_t)$ is $\mathcal{F}_t$ hence $\mathcal{F}_\infty$-measurable. From the definition of conditional expectation (ii) is equivalent to: for any bounded $\mathcal{F}_\infty$-measurable r.v. ξ

$$E(\xi\mathbb{E}(\eta_t \mid \mathcal{F}_t)) = \mathbb{E}(\xi\eta_t)$$

From (i)

$$\mathbb{E}(\xi\eta_t) = \mathbb{E}(\eta_t\mathbb{E}(\xi|\mathcal{G}_t)) = \mathbb{E}(\eta_t\mathbb{E}(\xi|\mathcal{F}_t)) = \mathbb{E}(\mathbb{E}(\eta_t|\mathcal{F}_t)\mathbb{E}(\xi|\mathcal{F}_t)) = \mathbb{E}(\xi\mathbb{E}(\eta_t|\mathcal{F}_t))$$

It remains to prove that (ii) implies the conditional independence. Let ξ be any bounded $\mathcal{F}_\infty$-measurable random variable and η any bounded $\mathcal{G}_t$-measurable r.v. Then

$$\mathbb{E}(\xi\eta|\mathcal{F}_t) = \mathbb{E}(\xi\eta|\mathcal{G}_t|\mathcal{F}_t) = \mathbb{E}(\xi\mathbb{E}(\eta|\mathcal{F}_t)|\mathcal{F}_t) = \mathbb{E}(\xi|\mathcal{F}_t)\mathbb{E}(\eta|\mathcal{F}_t)$$

Note that (i) is equivalent to : any bounded $\mathbb{F}$-martingale is a bounded $\mathbb{G}$-martingale.

Definition and Properties of immersion

We shall now examine the immersion property (or $(\mathcal{H})$-hypothesis) which reads:

$(\mathcal{H})$ **Every $\mathbb{F}$ square-integrable martingale is a $\mathbb{G}$ square integrable martingale.**

This hypothesis implies that the $\mathbb{F}$-Brownian motion remains a Brownian motion in the enlarged filtration and that every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale .

Assume that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is an arbitrary filtration and $\mathbb{H}$ is generated by the process $H_t = \mathbb{1}_{\{\tau \leq t\}}$. Then the following conditions are equivalent to the hypothesis $(\mathcal{H})$.

(i) For any $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_\infty).$$

(ii) For any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{P}$, that is,

$$\mathbb{E}_{\mathbb{P}}(\xi\,\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)\,\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t)$$

for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and bounded, $\mathcal{G}_t$-measurable random variable η.

(iii) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ: $\mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)$.

Complete model case

Let S be a semi-martingale on $(\Omega, \mathcal{G}, \mathbb{P})$ such that there exists a **unique** probability $\mathbb{Q}$, equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, where $\mathcal{F}_t = \mathcal{F}_t^S = \sigma(S_s, s \leq t)$ such that $(\widetilde{S}_t = S_t R_t, 0 \leq t \leq T)$ is an $\mathbb{F}^S$-martingale under the probability Q.
We assume that there exists a probability $\tilde{\mathbb{Q}}$, equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ such that $(\widetilde{S}_t, 0 \leq t \leq T)$ is a $\mathbb{G}$-martingale under the probability $\tilde{\mathbb{Q}}$.
Then, **any $(\mathbb{F}, \mathbb{Q})$-martingale is a $(\mathbb{G}, \tilde{\mathbb{Q}})$-martingale** and the restriction of $\tilde{\mathbb{Q}}$ to $\mathcal{F}_T$ is equal Q.

Change of a probability measure

Kusuoka shows, by means of a counter-example, that the hypothesis $(\mathcal{H})$ is not invariant with respect to an equivalent change of the underlying probability measure, in general.

Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in \mathbb{R}_+$, with the associated Radon-Nikodým density process η. If the **density process η is $\mathbb{F}$-adapted** then we have

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t)$$

for every $t \in \mathbb{R}_+$. Hence, the hypothesis $(\mathcal{H})$ is also valid under $\mathbb{Q}$ and the $\mathbb{F}$-intensities of τ under $\mathbb{Q}$ and under $\mathbb{P}$ coincide.

PROOF:

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)}$$

$$= \frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_\infty).$$

Stochastic Barrier

Suppose that

$$P(\tau \leq t|\mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing $\mathbb{F}$-adapted process. There exists a random variable Θ, independent of $\mathcal{F}_\infty$, with exponential law of parameter 1, such that $\tau \overset{law}{=} \inf\{t \geq 0 : \Gamma_t > \Theta\}$. In fact $\Theta \overset{def}{=} \Gamma_\tau$.

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Let us set $\Theta \stackrel{def}{=} \Gamma_\tau$. Then

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$.

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$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$. Therefore

$$P(\Theta > u|\mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability law of Θ and its independence of the σ-field $\mathcal{F}_\infty$. Furthermore, $\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$.

Representation theorem

In the case of a Brownian reference filtration, and under immmersion property, Kusuoka establishes that any $\mathbb{G}$-square integrable martingale admits a representation as the sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M.

Suppose that hypothesis $(\mathcal{H})$ holds under $\mathbb{P}$ and that any $\mathbb{F}$-martingale is continuous. Then, the martingale $M_t^h = \mathbb{E}_{\mathbb{P}}(h_\tau | \mathcal{G}_t)$, where h is an $\mathbb{F}$-predictable process such that $\mathbb{E}(h_\tau) < \infty$, admits the following decomposition

$$M_t^h = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u^h + \int_{]0,t\wedge\tau]} (h_u - J_u)\, dM_u,$$

where m^h is the continuous $\mathbb{F}$-martingale

$$m_t^h = \mathbb{E}_{\mathbb{P}}\Big(\int_0^\infty h_u dF_u \,|\, \mathcal{F}_t\Big),$$

$J_t = e^{\Gamma_t}(m_t^h - \int_0^t h_u dF_u)$ and M is the discontinuous $\mathbb{G}$-martingale $M_t = H_t - \Gamma_{t\wedge\tau}$.

PROOF: : We know that

$$\begin{aligned} M_t^h &= \mathbb{E}(h_\tau \,|\, \mathcal{G}_t) \\ &= \mathbb{1}_{\{\tau \le t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}\Big(\int_t^\infty h_u dF_u \,\Big|\, \mathcal{F}_t \Big) \end{aligned}$$

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$$
\begin{aligned}
M_t^h &= \mathbb{E}(h_\tau \,|\, \mathcal{G}_t) \\
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&= \mathbb{1}_{\{\tau \le t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \Big(m_t^h - \int_0^t h_u dF_u \Big) \\
&= \int_0^t h_u dH_u + \mathbb{1}_{\{\tau > t\}} J_t \,.
\end{aligned}
$$

PROOF: : We know that

$$
\begin{aligned}
M_t^h &= \mathbb{E}(h_\tau \,|\, \mathcal{G}_t) \\
&= 1\!\!1_{\{\tau \le t\}} h_\tau + 1\!\!1_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}\Big(\int_t^\infty h_u dF_u \,\Big|\, \mathcal{F}_t \Big) \\
&= 1\!\!1_{\{\tau \le t\}} h_\tau + 1\!\!1_{\{\tau > t\}} e^{\Gamma_t} \Big(m_t^h - \int_0^t h_u dF_u \Big) \\
&= \int_0^t h_u dH_u + 1\!\!1_{\{\tau > t\}} J_t \,.
\end{aligned}
$$

From the facts that Γ is an increasing process

m^h a continuous martingale

and using the integration by parts formula, we deduce that

$$dJ_t = e^{\Gamma_t} dm_t^h + (J_t - h_t) \frac{dF_t}{G_t}$$

Enlargement of filtration

In a general setting, we need a condition that ensures that $\mathbb{F}$-martingales are $\mathbb{G}$-semi-martingales

In the litterature, results are known under the hypothesis that τ is honest

Honest times

The random time τ is **honest** if τ is equal, on $\{\tau < t\}$ to an $\mathcal{F}_t$-measurable random variable. In particular, τ is $\mathcal{F}_\infty$-measurable.

Example: if X is a transient diffusion, the last passage time Λ_a is honest.

A key point is the following description of $\mathbb{F}^\tau$-predictable processes: if τ is honest, and if Z is an $\mathbb{F}^\tau$-predictable process, then there exist two $\mathbb{F}$-predictable processes z and $\widetilde{z}$ such that

$$Z_t = z_t 1\!\!1_{\{\tau > t\}} + \widetilde{z}_t 1\!\!1_{\{\tau \leq t\}} \,.$$

Initial times

The positive random time τ is called an **initial time** if it satisfies Jacod's criterion, i.e.,

$$G_t(\theta) = \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty f_t(u)\eta(du) \,.$$

From $G_s(\theta) = \mathbb{E}(G_t(\theta)|\mathcal{F}_s)$ for any $s \leq t$, it follows that for any $u \geq 0$, $(f_t(u))_t$ is a non-negative $\mathbb{F}$-martingale.

The Doob-Meyer decomposition of $G_t(t)$ is

$$G_t = G_t(t) = G_0(t) + \int_0^t g_s(s) dW_s - \int_0^t f_s(s)\eta(ds)$$

where $G_t(\theta) = G_0(\theta) + \int_0^t g_s(\theta) dW_s$.

• Under the condition that the initial time τ avoids the $\mathbb{F}$-stopping times, there is equivalence between $\mathbb{F}$ is immersed in $\mathbb{G}$ and for any $u \geq 0$, the martingale $(f_t(u), t \geq 0)$ is constant after u.

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• Let $(K_t(u))_{t\geq 0}$ be a family of $\mathbb{F}$-predictable processes indexed by $u \geq 0$. Then

$$\mathbb{E}\left(K_t(\tau) \middle| \mathcal{F}_t \right) = \int_0^\infty K_t(u) f_t(u) \eta\,(du) \qquad (*)$$

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• If X is an $\mathbb{F}$-martingale

$$\widehat{X}_t := X_t - \int_0^{t \wedge \tau} \frac{d\left\langle X, G \right\rangle_s}{G_s} - \int_{t \wedge \tau}^t \frac{d\left\langle X, f.(\theta) \right\rangle_s}{f_s(\theta)} \Bigg|_{\theta = \tau} \in \mathcal{M}(\mathbb{G}).$$

Partial information

As pointed out by Jamshidian, *“one may wish to apply the general theory perhaps as an intermediate step, to a subfiltration that is not equal to the default-free filtration. In that case,* $\mathbb{F}$ *rarely satisfies hypothesis* $(\mathcal{H})$*”*.

Information at discrete times

Assume that

$$dV_t = V_t(\mu dt + \sigma dW_t),\ V_0 = v$$

i.e., $V_t = ve^{\sigma(W_t+\nu t)} = ve^{\sigma X_t}$. The default time is assumed to be the first hitting time of α with $\alpha < v$, i.e.,

$$\tau = \inf\{t : V_t \le \alpha\} = \inf\{t : X_t \le a\}$$

where $a = \sigma^{-1}\ln(\alpha/v)$.

Here, $\mathbb{F}$ is the filtration of the observations of V at discrete times $t_1, \cdots t_n$ where $t_n \leq t < t_{n+1}$, i.e.,

$$\mathcal{F}_t = \sigma(V_{t_1}, \cdots, V_{t_n}, t_i \leq t)$$

The process $F_t = P(\tau \leq t|\mathcal{F}_t)$ F is **continuous and increasing in** $[t_i, t_{i+1}[$ but is **not increasing**.

Lemma 0.1 *The process ζ defined by*

$$\zeta_t = \sum_{i, t_i \leq t} \Delta F_{t_i}.$$

is an $\mathbb{F}$-martingale.

The Doob-Meyer decomposition of F is

$$F_t = \zeta_t + (F_t - \zeta_t),$$

where ζ is an $\mathbb{F}$-martingale and $F_t - \zeta_t$ is a predictable increasing process.

From

$$P(\inf_{s\le t} X_s > z) = \Phi(\nu, t, z)\,,$$

where

$$\begin{aligned}\Phi(\nu,t,z) &= \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z}\mathcal{N}\left(\frac{z+\nu t}{\sqrt{t}}\right), && \text{for } z<0,\ t>0,\\ &= 0, \qquad \text{for } z \ge 0,\ t\ge 0, \\ \Phi(\nu,0,z) &= 1, \qquad\qquad \text{for } z<0\end{aligned}$$

we obtain (we skip the parameter ν in the definition of Φ) for $t_1 < t < t_2$ and $X_{t_1} > a$

$$F_t = 1 - \Phi(t-t_1, a - X_{t_1})\left[1 - \exp\left(-\frac{2\,a}{t_1}\,(a - X_{t_1})\right)\right].$$

The case $X_{t_1} \le a$ corresponds to default: for $X_{t_1} \le a$, $F_t = 1$.

Another example, related with Parisian stopping times is presented in Çetin et al.

Delayed information

Guo et al. suggested to start form a structural model with delayed information, i.e. the reference filtration is $\mathcal{F}_t = \sigma(S_u, u \le t - \delta)$. In that case, (H) hypothesis is not satisfied.

Intensity approach

In the so-called intensity approach, the default time τ is a $\mathbb{G}$-stopping time. The intensity is defined as any non-negative process λ, such that

$$M_t \stackrel{def}{=} H_t - \int_0^{t \wedge \tau} \lambda_s ds$$

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The intensity exists only if τ is a totally inaccessible stopping time.

We emphasize that, in that setting the intensity is not well defined after time τ, i.e., if λ is an intensity, for any non-negative predictable process g the process $\tilde{\lambda}_t = \lambda_t 1\!\!1_{t\leq\tau} + g_t 1\!\!1_{\{t>\tau\}}$ is also an intensity.

If the process $Y_t = \mathbb{E}\left(X \exp\left(-\int_0^T \lambda_u du\right)|\mathcal{G}_t\right)$ ***is continuous at time*** τ, *then, setting* $L_t = 1\!\!1_{\{t<\tau\}} e^{\Gamma_t}$

$$\mathbb{E}(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \mathbb{E}\left(X \exp\left(-\int_t^T \lambda_u du\right)|\mathcal{G}_t\right) = L_t Y_t$$

If Y *is not continuous*

$$\mathbb{E}(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = L_t Y_t - \mathbb{E}(\Delta Y_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t)\,.$$

It can be mentioned that the continuity of the process depends on the choice of λ after time τ.

If the process Y is not continuous, then setting

$$U_t = L_t Y_t = 1\!\!1_{t<\tau} \exp\left(\int_0^t \lambda_s ds\right) \mathbb{E}\left(X \exp\left(-\int_0^T \lambda_u du\right) |\mathcal{G}_t\right)$$

we have $U_T = X 1\!\!1_{\{T<\tau\}}$ and

$$dU_t = L_{t-} dY_t + Y_{t-} dL_t + d[L, Y]_t = L_{t-} dY_t + Y_{t-} dL_t + \Delta L_t \Delta Y_t$$

and

$$\mathbb{E}(U_T|\mathcal{G}_t) = \mathbb{E}(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = U_t - \mathbb{E}(e^{\Lambda_\tau} \Delta Y_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) \,.$$

Then, for any $X \in \mathcal{G}_T$:

$$\mathbb{E}(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\tau>t} \left(e^{\Lambda_t}\mathbb{E}(e^{-\Lambda_T} X|\mathcal{G}_t) - \mathbb{E}(e^{\Lambda_\tau} \Delta Y_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t)\right)$$

where $Y_t = \mathbb{E}\left(X \exp\left(-\Lambda_T\right)|\mathcal{G}_t\right)$ and $\Lambda_t = \int_0^t \lambda_u du$

CDS Price, General case

The ex-dividend price of a credit default swap, with a rate process κ and a protection payment δ_τ at default, equals, for every $t \in [0,T]$

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}\Big(\int_t^T B_u^{-1} G_u (\delta_u \lambda_u - \kappa)\, du \,\Big|\, \mathcal{F}_t \Big)$$

and thus the cumulative price of a CDS equals, for any $t \in [0,T]$,

$$S_t^{\text{cum}}(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}\Big(\int_t^T B_u^{-1} G_u (\delta_u \lambda_u - \kappa)\, du \,\Big|\, \mathcal{F}_t \Big) + B_t \int_{]0,t]} B_u^{-1}\, dD_u.$$

The dividend process $D(\kappa, \delta, T, \tau)$ of a CDS equals

$$D_t = \int_{]0,t\wedge T]} \delta_u\, dH_u - \kappa \int_{]0,t\wedge T]} (1 - H_u)\, du = \delta_\tau \mathbb{1}_{\{\tau \le t\}} - \kappa (t \wedge T \wedge \tau).$$

We now assume that **(H) hypothesis holds** between $\mathbb{F}$ and $\mathbb{G}$, that is $\mathbb{F}$-martingales are $\mathbb{G}$-martingales. Then, F is increasing and the process

$$M_t = H_t - \int_0^{t \wedge \tau} \gamma_u \, du,$$

with $\gamma_t dt = \frac{dF_t}{G_t}$ is a $\mathbb{G}$-martingale.

The dynamics of the ex-dividend price $S_t(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1-H_t)B_t G_t^{-1}\, dm_t + (1-H_t)(r_t S_t(\kappa) + \kappa - \delta_t \gamma_t)\, dt,$$

where m is the $(Q, \mathbb{F})$-martingale given by

$$m_t = \mathbb{E}_Q \left(\int_0^T B_u^{-1} \delta_u G_u \gamma_u \, du - \kappa \int_0^T B_u^{-1} G_u \, du \,\middle|\, \mathcal{F}_t \right).$$

Hedging defaultable claims

Our aim is to hedge

$$Y = 1\!\!1_{\{T \geq \tau\}} Z_\tau + 1\!\!1_{\{T < \tau\}} X.$$

using two CDS with maturities T_i, rates κ_i and protection payment δ^i. We assume $r = 0$. Let ζ_t^i defined as

$$m_t^i = \mathbb{E}_Q \left(\int_0^T \delta_u^i G_u \gamma_u \, du - \kappa_i \int_0^T G_u \, du \,\middle|\, \mathcal{F}_t \right) \, , \; dm_t^i = \zeta_t^i dW_t$$

and

$$m_t^Z = \mathbb{E}_Q(-\int_0^\infty Z_u dG_u + G_T X | \mathcal{F}_t), \; dm_t^Z = \zeta_t^Z dW_t$$

Assume that there exist $\mathbb{F}$-predictable processes ϕ^1, ϕ^2 such that

$$\sum_{i=1}^{2} \phi_t^i \big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big) = Z_t - \widetilde{y}_t, \quad \sum_{i=1}^{2} \phi_t^i \zeta_t^i = \zeta_t,$$

where $\widetilde{y}$ is given by

$$\widetilde{y}_t = \frac{1}{G_t} \mathbb{E}_Q \left(- \int_t^T Z_u \, dG_u + G_T X \,\Big|\, \mathcal{F}_t \right).$$

Let $\phi_t^0 = V_t(\phi) - \sum_{i=1}^{2} \phi_t^i S_t^i(\kappa_i)$, where the process $V(\phi)$ is given by

$$dV_t(\phi) = \sum_{i=1}^{2} \phi_t^i \big(dS_t^i(\kappa_i) + dD_t^i \big)$$

with the initial condition $V_0(\phi) = \mathbb{E}_Q(Y)$. Then the self-financing trading strategy $\phi = (\phi^0, \phi^1, \phi^2)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$.