University of Jena
ITN Marie Curie Spring School

Multidimensional stochastic bridges: a study via SDEs

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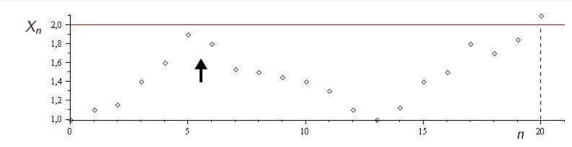
Motivation

Let $X_n, n = 0, 1, 2, \ldots$ be a time serie, and let α be a boundary.
Which is the first time passage trought the boundary α?



$\Longrightarrow$ the first time passage could be not recognised and it's always overestimated.
$\Longrightarrow$ it's important to study conditional processes forced to assume certain values at certain given future time-points: stochastic bridges.

Problem

Given a process X_t such that it fulfills the following SDE

$$dX_t = f(t, X_t)dt + G(t, X_t)dW_t, \qquad X_0 = c$$

we want to write an SDE for the bridge, i.e. an SDE of the form

$$d\tilde{X}_t = \tilde{f}(t, \tilde{X}_t)dt + \tilde{G}(t, \tilde{X}_t)dW_t, \qquad \tilde{X}_0 = c.$$

The condition $\{\tilde{X}_T = \theta\}$ must be satisfied.

Outline

- Setting
- Stochastic differential equations: a survey
- Bridges
- The main theorem
- Application to the Integrated Ornstein-Uhlenbeck process

Setting

In the following we will consider **stochastic differential equations** of the type

$$\begin{aligned} dX_t &= f(t, X_t)dt + G(t, X_t)dW_t \\ X_{t_0} &= \bar{X}_{t_0} \end{aligned}$$

where W_t stands for the Brownian Motion.

Note: in general the BM is m-dimensional while the process X_t is d-dimensional.

▷ In this case $G(t, x)$ would be a $(d \times m)$-dimensional matrix while $f(t, x)$ is a d-dimensional vector.

Of great intrest for us are the **stochastic linear differential equations** i.e. SDEs of the type

$$dX_t = (A(t)X_t + a(t))dt + \sum_{i=1}^{m}(B_i(t)X_t + b_i(t))dW_t^i$$
$$X_{t_0} = \bar{X}_{t_0}$$

where W_t stands for the Brownian Motion.

- When $a(t) = b_1(t) = \ldots = b_m(t) \equiv 0$ the equation is called *homogeneous*.
- When $B_1(t) = \ldots = B_m(t) \equiv 0$ the equation is called *linear in the narrow sense*.

Recall now the **Ornstein-Uhlenbeck** process.
This is a diffusion process that can be defined in different ways, and the definition via SDEs is the following:

let W_t be a 1-dim BM and let be $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$. Then define the process as

$$dX_t = -\alpha X_t dt + \beta dW_t$$
$$X_0 = c.$$

It can be shown that the Ornstein-Uhlenbeck process defined above is the same as

$$X_t = e^{-\alpha t} c + \beta \int_0^t e^{-\alpha(t-s)} dW_s$$

where the integral is of the Ito type.

SDEs: a survey

It is well known that under suitable conditions over f and G, the stochastic differential equation

$$dX_t = f(t, X_t)dt + G(t, X_t)dW_t$$

has a unique solution that is continuous on $[t_0, T]$ given any initial condition $X_{t_0} = c$ indipendent from the increment $W_t - W_{t_0}$ with $t \geq t_0$.

Attention

Using just this theorem we are not able to write down the explicit form of such a solution!

$\longrightarrow$ consider stochastic differential equations linear in the narrow sense.

In this special case we can write the solution explicitly.

Given the stochastic differential equation linear in the narrow sense

$$dX_t = (A(t)X_t + a(t))\,dt + G(t)dW_t$$
$$X_{t_0} = c$$

with $t \in [t_0, T]$, then the solution (under the existence and uniqueness conditions) is given by

$$X(t) = \Phi(t)\left(c + \int_{t_0}^{t} \Phi^{-1}(s)a(s)\,ds + \int_{t_0}^{t} \Phi^{-1}(s)G(s)\,dW_s\right)$$

where $\Phi(t)$ is a matrix and it's the solution of the associated homogeneous system $\dot{\Phi}(T) = A(t)\Phi(t)$, $\Phi(t_0) = I$.
Such matrix is called foundamental matrix.

Example: the integrated Ornstein-Uhlenbeck process

This process is a 2-dimensional process defined as follows:

$$\begin{cases} dX_t^{(1)} = X_t^{(2)} dt \\ dX_t^{(2)} = -\alpha X_t^{(2)} dt + \beta dW_t. \end{cases}$$

Writing it into the matrix form we obtain

$$dX_t = AX_t dt + bdW_t$$

where $X_t = \begin{bmatrix} X_t^{(1)} \\ X_t^{(2)} \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}$ e $b = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$.

We can observe that this is a SDE linear in the narrow sense, so we can write the solution using the previous theorem.

The solution of the following

$$dX_t = AX_t dt + bdW_t$$
$$X_0 = c$$

is of the form

$$X_t = \Phi(t)c + \Phi(t)\int_0^t \Phi(s)^{-1} b\, dW_s$$

where the foundamental matrix is simply
$\Phi(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = \begin{bmatrix} 1 & \frac{e^{-\alpha t}-1}{\alpha} \\ 0 & e^{-\alpha t} \end{bmatrix}.$

If we write the solution component by component we have

$$X_t^{(1)} = c_1 + \frac{c_2}{\alpha}\left(-e^{-\alpha t} + 1\right) + \frac{\beta}{\alpha}\int_0^t \left(-e^{-\alpha(t-s)} + 1\right)\, dW_s$$

$$X_t^{(2)} = e^{-\alpha t} c_2 + \beta \int_0^t e^{-\alpha(t-s)}\, dW_s.$$

Bridges

Consider now a d-dimensional stochastic process $\{X_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition

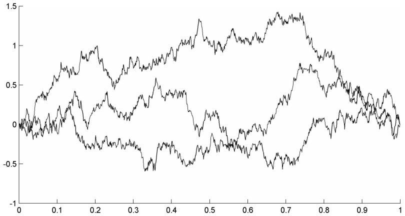
We say that the **bridge** associated to X_t conditioned to the event $\{X_T = \theta\}$ is the process

$$\{\tilde{X}_t, 0 \leq t \leq T\} = \{X_t, 0 \leq t \leq T \, : \, X_T = \theta\}$$

where T is a deterministic fixed time and $\theta \in \mathbb{R}^d$ is fixed too.

Explaining:
we are considering only the trajectories of X ending in θ at time T.

Example: the brownian bridge



We can see in the picture an example of Brownian bridge obtained conditioning a Brownian Motion to the event $E = \{W_1 = 0\}$.

There are different representations of the brownian bridge, and it can be proved that they identify the same diffusion process in law.

- $dY_t = -\frac{Y_t}{1-t}dt + dW_t$;
- $Y_t = (1-t)\int_0^t \frac{dW_s}{1-s}$;
- $\hat{W}_t = W_t - tW_1$.

Here there is another definition (more precise) of the bridge

Definition

Let $E = \{X_T = \theta\} \in \mathscr{F}$. Then the process $\{\tilde{X}_t, 0 \leq t \leq T\}$ is called **bridge** of the process $\{X_t, t \geq 0\}$ associated to the event E if

$$\mathcal{L}(\tilde{X}; \mathbb{P}) = \mathcal{L}(X; \tilde{\mathbb{P}})$$

where $\mathcal{L}(X; \mathbb{P})$ is the law of the process X defined on $(\Omega, \mathscr{F})$ equipped with the probability measure $\mathbb{P}$.
The new measure $\tilde{\mathbb{P}}$ is defined by means of the conditional expectation with respect to the σ-algebra generated by X_T.
Then we consider a particular value of the random variable X_T i.e. we condition to the event $X_T = \theta$ and define $\tilde{\mathbb{P}}$ as

$$\tilde{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot | X_T = \theta) = \mathbb{E}(\mathbb{I}_{\cdot} | X_T = \theta)$$

The main theorem

We want now to describe the bridge using the informations about the unconditioned process.

Given a process X_t such that it fulfills the following SDE

$$dX_t = f(X_t)dt + G(X_t)dW_t, \qquad X_0 = c$$

for every $t \in [0, T)$, we want to write the new SDE for the bridge, i.e. an SDE of the form

$$d\tilde{X}_t = \tilde{f}(t, \tilde{X}_t)dt + \tilde{G}(t, \tilde{X}_t)dW_t, \qquad \tilde{X}_0 = c$$

for every $t \in [0, T)$.

Theorem

Let X be a regular and time-homogeneous diffusion process, with values in $\mathbf{R}^d$ and with infinitesimal coefficients $f(x)$ and $G(x)$. Suppose that there exists the tpd function p of X_t. Then the bridge associated to the event $\tilde{X}_T = \theta$ satisfies the following sde

$$d\tilde{X}_t = \tilde{f}(t, \tilde{X}_t)dt + \tilde{G}(\tilde{X}_t)dW_t, \qquad \tilde{X}_0 = c$$

for every $t \in [0, T)$ where

$$\left(\tilde{f}(t,x)\right)_i = f_i(x) + \frac{1}{2}\sum_{k=1}^{d}\left(\alpha_{ik}(x) + \alpha_{ki}(x)\right)\frac{\partial}{\partial x_k}\ln p(\theta, T \mid x, t)$$

$$\left(\tilde{G}(x)\right)_i = \left(G(x)\right)_i, \quad \text{setting } \alpha := GG^* \in M(d,d).$$

for $i = 1, \dots, d$ and for each $t \in [0, T)$.

In order to prove this result we use the relation between the transition probability density functions of X and $\tilde{X}$.

Let $p(x,t|y,s)$ and $\tilde{p}(x,t|y,s)$ be the transition probability density function of the free process and of the bridge respectively, defined for every $x,y \in \mathbb{R}^d$ and $s < t \in [0,t_1)$.
Then the following holds

$$\tilde{p}(x,t|y,s) = \frac{p(x,t|y,s)p(b,t_1|x,t)}{p(b,t_1|y,s)}$$

for every $s < t \in [0,t_1)$.

Example

Example: the Integrated Ornstein-Uhlenbeck bridge

STEP 1: find the infinitesimal coefficients of the unconditioned process

Recall that the IOU is defined by the SDE

$$dX_t = AX_t dt + bdW_t$$
$$X_0 = c$$

and the solution process has the form

$$X_t = \Phi(t)c + \Phi(t)\int_0^t \Phi(s)^{-1} b\, dW_s$$

where $\Phi(t) = e^{tA} = \begin{bmatrix} 1 & \frac{e^{-\alpha t}-1}{\alpha} \\ 0 & e^{-\alpha t} \end{bmatrix}$.

STEP 2: find the transition probability density functions of X
In order to apply the previous result we need to compute the transition probability density function $p(x, t|y, s)$ of X.
Therefore we use the following known fact:

> A stochastic process solution of an SDE linear in the narrow sense is a Gaussian process iff the initial condition is a Gaussian random variable or is a constant.

$\longrightarrow$ The IOU process is a *Gaussian process*.

Since the process is Gaussian we compute just the mean and the covariance matrix.

We have the following formulas:

- Mean: $m_t = e^{At}c = \begin{bmatrix} c_1 + \frac{c_2}{\alpha}(1 - \exp(-\alpha t)) \\ e^{-\alpha t} c_2 \end{bmatrix}$
- Covariance matrix:

$$k(s,t) = \mathbb{E}[(X_s - \mathbb{E}X_s)(X_t - \mathbb{E}X_t)^*] = \int_0^{t \wedge s} e^{A(s-u)} bb^* e^{A^*(t-u)}\, du$$

Note

The covariance matrix is not written in the explicit form yet, and in order to do that we need to compute the exponential matrix $e^{A(s-u)}$ and then we need to evaluate the integral componentwise.

We finally write the transition probability density funcion of X:

$$p(x,t\,|\,y,s) =$$

$$= \frac{1}{2\pi\sqrt{\Delta}} \exp\Big\{ -\frac{\beta^2}{4\alpha^3\Delta}\Big[\alpha^2\left(1-e^{-2\alpha(t-s)}\right)\left(x_1-y_1-\frac{y_2}{\alpha}\left(1-e^{-\alpha(t-s)}\right)\right)^2$$

$$-2\alpha\left(1-e^{-\alpha(t-s)}\right)^2\left(x_2-y_2e^{-\alpha(t-s)}\right)\left(x_1-y_1-\frac{y_2}{\alpha}\left(1-e^{-\alpha(t-s)}\right)\right)$$

$$+\left(x_2-y_2e^{-\alpha(t-s)}\right)^2\left(-3+4e^{-\alpha(t-s)}-e^{-2\alpha(t-s)}+2\alpha(t-s)\right)\Big]\Big\}.$$

Remember:

$$\left(\tilde{f}(t,x)\right)_i = f_i(x) + \frac{1}{2}\sum_{k=1}^{d}\left(\alpha_{ik}(x)+\alpha_{ki}(x)\right)\frac{\partial}{\partial x_k}\ln p(\theta, T\,|\,x,t)$$

STEP 3: find the infinitesimal coefficient of the new process
We now apply the Theorem explained before and we compute the new drift (the diffusion coefficient is the same as the one of the free process).
The result is

$$\tilde{f}(t,x) = [\tilde{f}_1, \tilde{f}_2]^* \qquad \tilde{G}(t,x) = G(x) = [0, \beta]^*$$

with

$$\begin{aligned}
\tilde{f}_1(t,x) =& x_2 \\
\tilde{f}_2(t,x) =& x_1 C\alpha \left(1 - e^{-\alpha(T-t)}\right)^2 \\
& + x_2 \left[\alpha + C\left(1 - 4e^{-\alpha(T-t)} + (3 - 2\alpha(T-t))e^{-2\alpha(T-t)}\right)\right] \\
& - \theta_1 C\alpha \left(1 - e^{-\alpha(T-t)}\right)^2 + \theta_2 C\left(1 - e^{-2\alpha(T-t)} - 2\alpha(T-t)e^{-\alpha(T-t)}\right).
\end{aligned}$$

Observe that the new coefficient $\tilde{f}(t,x)$ can be written in the form

$$\tilde{f}(t,x) = \tilde{A}(t)x + \tilde{a}(t)$$

where $\tilde{A}(t) = \begin{bmatrix} 0 & 1 \\ h_1(t) & h_2(t) \end{bmatrix}$ and $\tilde{a}(t) = \begin{bmatrix} 0 \\ k(t) \end{bmatrix}$.
In conclusion we have the following equation governing the IOU bridge

$$d\tilde{X}_t = (\tilde{A}(t)X_t + \tilde{a}(t))dt + bdW_t$$
$$\tilde{X}_0 = c.$$

Thank you.