Models of financial markets with asymmetric information: additional utility and entropy

Stefan Ankirchner, Steffen Dereich, Peter Imkeller

Humboldt-Universität zu Berlin
http://wws.mathematik.hu-berlin.de/∼imkeller

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1 The model

(J. Amendinger, P.I., M. Schweizer, SPA ’98; P.I., M. Pontier, F. Weisz, SPA ’00; P.I. ’01; J.-M. Corcuera, P.I., A. Kohatsu-Higa, D. Nualart ’03; S. Ankirchner, P.I. ’04; S. Ankirchner, S. Dereich, P.I. ’04)

Financial market with d risky assets, time horizon 1; to simplify: $d = 1$

price process

$$\frac{dX_t}{X_t} = dW_t + \alpha_t dt, \quad W \text{ 1-dim. Wiener}$$
$$\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le 1}, \quad \mathcal{F}_t = \sigma(W_s : s \le t), \quad \alpha \quad \mathbb{F} - \text{adapted}$$

2 information levels

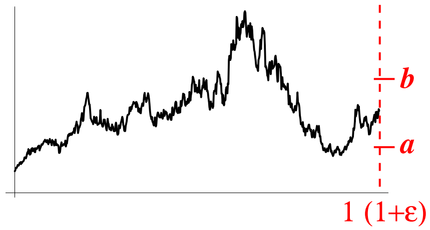
regular trader	$\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le 1}$
insider	$\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le 1}$

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G),$$

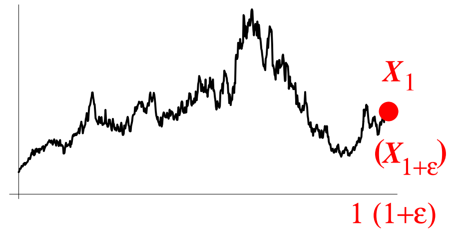
G $\mathcal{F}_1$–measurable r.v., *additional information*

(Lit: F. Baudoin, L. Denis, A. Grorud, M. Jeanblanc, J. Karatzas, I. Pikovsky,. . .)

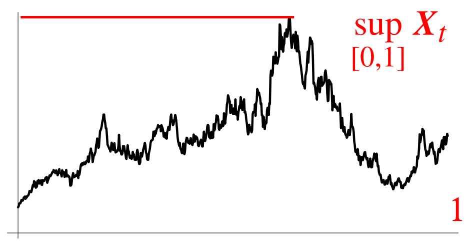
2 Examples



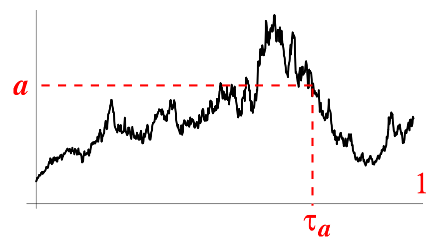
price interval at time $1(1+\epsilon)$



price of asset at time $1(1+\epsilon)$



maximal price before time 1



last passage of level a before time 1

3 expected utility

development of portfolio

$$V_0 = 1, \quad \frac{dV_t}{V_t} = \begin{matrix} \pi_t \\ \pi_t \end{matrix} \quad \frac{dX_t}{X_t},$$

where

$\pi \ \mathbb{F} -$ adapted,
$\pi \ \mathbb{G} -$ adapted,
$$\int_0^1 \pi_s^2 ds < \infty \ P - a.s.$$

utility function

logarithmic utility $U(x) = \ln(x), \quad x > 0$

Pb: compute *expected additional utility*

regular trader: $N_{\mathbb{F}} = \max_{\pi \mathbb{F}-\text{portfolio}} E(\ln V_1)$

insider: $N_{\mathbb{G}} = \max_{\pi \mathbb{G}-\text{portfolio}} E(\ln V_1)$

$$\Delta N = N_{\mathbb{G}} - N_{\mathbb{F}}$$

3 utility increment: $N_{\mathbb{F}}$

$$dV_t = V_t \cdot \begin{matrix} \pi_t \\ \pi_t \end{matrix} \quad [dW_t + \alpha_t dt]$$

$$\Longrightarrow \quad V_t = \exp[\int_0^t \begin{matrix} \pi_s \\ \pi_s \end{matrix} \quad \alpha_s ds + \int_0^t \begin{matrix} \pi_s \\ \pi_s \end{matrix} \quad dW_s - \frac{1}{2} \int_0^t \begin{matrix} \pi_t^2 \\ \pi_t^2 \end{matrix} \quad ds]$$

$$\Longrightarrow \quad N_{\mathbb{F}} \quad = \quad \max_{\pi \mathbb{F}-\text{portfolio}} E[\int_0^1 \pi_s \alpha_s ds - \frac{1}{2} \int_0^1 \pi_s^2 ds]$$

$$= \quad \frac{1}{2} E(\int_0^1 \alpha_s^2 ds)$$

argument same as for maximizing

$$T : \mathbb{R} \ni \pi \mapsto \alpha\pi - \frac{1}{2}\pi^2, \quad \alpha \in \mathbb{R}$$

$$T'(\pi) = \alpha - \pi = 0$$

for $\alpha = \pi$

3 utility increment: $N_{\mathbb{G}}$

Pb when calculating $N_{\mathbb{G}}$:

$$\int_0^\cdot \pi_s dW_s \text{ is no martingale !}$$

Basic problem of *grossissement de filtrations*:

$$W \quad \text{in} \quad \mathbb{G} \quad \text{semimartingale?}$$

$$\begin{array}{ccccc} W & = & \tilde{W} & + & \int_0^\cdot \mu_s^G ds \\ \mathbb{F}-\text{martingale} & & \mathbb{F}-\text{martingale} & & \textit{information drift} \end{array}$$

$$\Longrightarrow \quad N_{\mathbb{G}} = \frac{1}{2} E\left(\int_0^1 [\alpha_s + \mu_s^G]^2 ds\right)$$

$$\begin{aligned} \Delta N &= N_{\mathbb{G}} - N_{\mathbb{F}} \\ &= \frac{1}{2} E\left(\int_0^1 (\mu_s^G)^2 ds\right) \end{aligned}$$

$$\left[E\left(\int_0^1 \alpha_s \mu_s^G ds\right) = E\left(\int_0^1 \alpha_s [dW_s - d\tilde{W}_s]\right) = 0\right]$$

4 The information drift as a function of G

(H. Föllmer, P.I. ’94; Jacod ’85; Yor, Jeulin ’81-’85)

Pb: description of μ^G

$$\tilde{W} = W - \int_0^\cdot \mu_u^G du \quad \mathbb{G}- \quad \text{martingale}$$

$$\Longleftrightarrow$$

$$E([W_t - W_s] 1_A \, 1_C(G)) \quad = \quad E(\int_s^t \mu_s^G du \, 1_A \, 1_C(G)),$$

$0 \leq s < t \leq 1$, $A \in \mathcal{F}_s$, C Borel.

Condition

$$(\mathbf{H}) \qquad P^G \qquad \text{equivalent} \quad P_t^G(\omega, \cdot) \quad P - a.e. \quad \omega,$$

law of G — cond. law of G given $\mathcal{F}_t$

$$p_t(\cdot, l) = \frac{dP_t(\cdot, \cdot)}{dP^G}(l)$$

4 calculation of information drift μ^G

simplify: $A = \Omega$

$$E([W_t - W_s]1_C(G)) = E(\int_s^t \mu_u^G du\, 1_C(G))$$
$$= E(\int_C [W_t - W_s] P_t^G(\cdot, dl))$$
$$= \int_C E([W_t - W_s] \cdot (p_t - p_s)(\cdot, l))\, P^G(dl)$$

$$p_t(\cdot, l)\, \mathbb{F} - \text{martingale}, \quad p_t(., l) = \int_0^t k_s^l dW_s$$

$$\Longrightarrow \quad E([W_t - W_s] \cdot (p_t - p_s)(\cdot, l))$$
$$= E(\int_s^t k_u^l du)$$
$$= E(\int_s^t \frac{k_u^l}{p_u(\cdot, l)} du \cdot p_t(\cdot, l))$$

$$\Longrightarrow \quad \mu_t^l = \frac{k_u^l}{p_u(\cdot, l)} = \frac{\frac{d}{dt}\langle p(\cdot, l), W\rangle_t}{p_t(\cdot, l)}$$
$$\mu^G = \mu^l|_{l=G}$$

5 Examples

Example 1: $G = W_{1+\epsilon} \quad (\epsilon \geq 0)$

A Borel; then

$$\begin{aligned} P(W_{1+\epsilon} \in A) | \mathcal{F}_t) &= P(W_{1+\epsilon} - W_t \in A - W_t) \\ &= N(0, 1+\epsilon - t)(A - W_t) \end{aligned}$$

$\Longrightarrow \quad (\mathbf{H})$

Example 2: $G = \sup_{0 \leq t \leq 1} W_t, \quad G_t = \sup_{0 \leq s \leq t} W_s$

$$G = G_t \vee (W_t + \underbrace{\sup_{t \leq s \leq 1}(W_s - W_t))}_{=: \tilde{G}_{1-t}}$$

$$\Longrightarrow \quad P_t^G = \underbrace{F_{1-t}(G_t - W_t)}_{\text{distribution function of } \tilde{G}_{1-t}} \cdot \; \delta_{G_t} + \ldots$$

$\Longrightarrow \quad (\mathbf{H})$not valid

Amendinger (thesis):

(H) $\Longrightarrow$ (NFLVR) $\Longrightarrow$ no arbitrage for insider

6 Extension of Jacod's framework

Need new interpretation of $\mu_t^G = \frac{k_u^l}{p_u(\cdot,l)} = \frac{\frac{d}{dt}\langle p(\cdot,l),W\rangle_t}{p_t(\cdot,l)}|_{l=G}$!

a) Malliavin's calculus

$$p_t(\cdot,l) \quad = \quad \int_0^t k_u^l dW_u \quad = \quad \int_0^t D_s p_s(\cdot,l) dW_s$$

Clark-Ocone

$$\Longrightarrow \quad \frac{d}{dt}\langle P(\cdot,l),W\rangle_t \quad = \quad D_t p_t(\cdot,l)$$

$$\Longrightarrow \quad \mu_t^G \quad = \quad D_t \ln p_t(\cdot,l)|_{l=G}$$

b) R-N derivatives

$$\frac{D_t p_t(\cdot,l)}{p_t(\cdot,l)} \quad = \quad \frac{D_t \frac{dP_t^G}{dP^G}(l)}{\frac{dP_t^G}{dP^G}(l)} \qquad \text{(assume } D_t \leftrightarrow \frac{d}{dP^G}\text{)}$$

$$= \quad \frac{D_t P_t^G(\cdot,dl)}{P_t^G(\cdot,dl)}(l)$$

Needs Malliavin calculus for measure valued objects! (P.I., M. Pontier, F. Weisz)

Replace **(H)** with

(AC) $D_t P_t^G(\cdot,dl) \ll P_t^G(\cdot,dl)$!

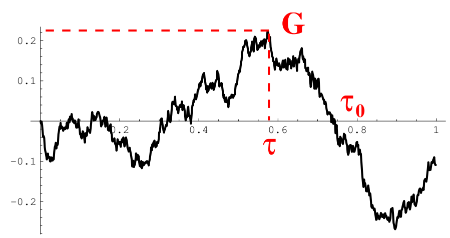
7 Examples

Back to Example 2: $G = \sup_{0\le t\le 1} W_t$

$$P_t^G = F_{1-t}(G_t - W_t)\cdot\ \delta_{G_t} + \dots$$

$$\Longrightarrow \quad \mu_t^G = -\frac{f_{1-t}(G_t-W_t)}{F_{1-t}(G_t-W_t)}\Big|_{G_t=G} + \dots$$

$$\cong -\frac{1}{G-W_t} \quad \text{near } \tau$$



excursion theory $\Longrightarrow$ $(G - W_t : \tau \le t \le \tau_0)$ is **BES(3)**

trajectorial properties of BES(3) $\Longrightarrow$

$$\int_0^1 |\mu_s^G| ds < \infty \quad P - \text{a.s.}$$

$$\int_0^1 (\mu_s^G)^2 ds = \infty \quad \text{on a set of positive probability}$$

$\Longrightarrow$ W semimartingale in $\mathbb{G}$; no equivalent martingale measure w.r.t. $\mathbb{G}$; arbitrage!

8 structure of information drift, general enlargement

Pb: $S = M + \alpha \cdot \langle M, M \rangle$ w.r.t. $\mathbb{G}$, structure of α?

Hypothesis: Ω standard Borel $\Longrightarrow$ $\mathcal{G}_t$ countably generated, regular cond. probabilities exist

reg. cond. prob. of $\mathcal{F}$ given $\mathcal{F}_t$: $P_t(\cdot, \cdot)$

$$P_t(\cdot, A) = P(A) + \int_0^t k_s(\cdot, A) dM_s + L_t^A, \quad L^A \text{ orthogonal to } M$$

$$(\mathbf{ACL}) \qquad k_t(\cdot, \cdot) \text{ signed measure,}$$

$$k_t(\omega, \cdot)\Big|_{\mathbb{G}_{t-}} \ll P_t(\omega, \cdot)\Big|_{\mathbb{G}_{t-}}$$

$$\gamma_t(\omega, \omega') = \frac{dk_t(\omega, \cdot)}{dP_t(\omega, \cdot)}\Big|_{\mathcal{G}_{t-}}$$

Thm 2: **(ACL)** $\Longrightarrow \alpha_t(\omega) = \gamma_t(\omega, \omega)$ gives information drift of $\mathbb{G}$ relative to $\mathbb{F}$.

Pb: under which conditions is **(ACL)** satisfied?

8 structure of information drift ($L = 0$)

Crucial observation: $0 \leq s < t$, $\mathcal{P} = \{A_1, \ldots, A_n\}$ partition of Ω in $\mathcal{G}_s$, Itô's formula:

$$(*) \quad \sum_{k=1}^{n} \left[1_{A_k} \ln P_s(\cdot, A_k) - 1_{A_k} \ln P_t(\cdot, A_k)\right]$$

$$= \sum_{k=1}^{n} \Big[- \int_s^t \frac{1}{P_u(\cdot, A_k)} 1_{A_k} \, dP_u(\cdot, A_k) + \int_s^t \frac{1}{2} \frac{1}{P_u(\cdot, A_k)^2} 1_{A_k} \, d\langle P(\cdot, A_k), P(\cdot, A_k)\rangle_u \Big]$$

$$= \mathbb{G} - \text{mart.} + \sum_{k=1}^{n} \int_s^t [-\frac{k_u}{P_u}(\cdot, A_k)\alpha_u + \frac{1}{2}(\frac{k_u}{P_u})^2(\cdot, A_k)] \, 1_{A_k} \, d\langle M, M\rangle_u$$

Jensen, Cauchy-Schwarz: $\frac{1}{2} E \int_s^t \sum_{k=1}^{n} \left(\frac{k_u}{P_u}\right)^2 (\cdot, A_k) \, 1_{A_k} \, d\langle M, M\rangle_u$

$$\leq E \left(\int_s^t \sum_{k=1}^{n} \left(\frac{k_u}{P_u}\right)^2 (\cdot, A_k) \, 1_{A_k} \, d\langle M, M\rangle_u \right)^{\frac{1}{2}} E \left(\int_s^t \alpha_u^2 \, d\langle M, M\rangle_u \right)^{\frac{1}{2}}$$

$\Longrightarrow$

$$E \int_s^t \sum_{k=1}^{n} \left(\frac{k_u}{P_u}\right)^2 (\cdot, A_k) \, 1_{A_k} \, d\langle M, M\rangle_u \leq 4E \left(\int_s^t \alpha_u^2 \, d\langle M, M\rangle_u \right).$$

8 structure of information drift, general enlargement

Apply martingale convergence along nested sequence of partitions in $\mathcal{G}_s$, let $s \uparrow t$:

Thm 3: $\mathcal{G}_{t-} \rightarrow \mathbb{R}, A \mapsto k_t(\cdot, A)$ can be chosen as signed random measure, **(ACL)** satisfied.

Summary:

Thm 4: Equivalent:

i) $k_t(\omega, \cdot)$ signed measure,

$$k_t(\omega, \cdot)\Big|_{\mathbb{G}_{t-}} \ll P_t(\omega, \cdot)\Big|_{\mathbb{G}_{t-}} \quad \text{for } P_M = P \otimes \langle M, M \rangle\text{–a.e.}(\omega, t),$$

R-N process

$$\gamma_t(\omega, \omega') = \frac{dk_t(\omega, \cdot)}{dP_t(\omega, \cdot)}\Big|_{\mathcal{G}_{t-}} \quad \text{square integrable on diagonal } \omega = \omega' \text{ for } P_M$$

ii) M is $\mathbb{G}$–semimartingale; Doob-Meyer decomposition

$$M = \tilde{M} - \alpha \cdot \langle M, M \rangle, \quad \alpha \text{ square integrable for } P_M$$

In this case:

$$\alpha_t(\omega) = \gamma_t(\omega, \omega), \text{ for } P_M \text{ –a.e. } (\omega, t).$$

9 Relationship with entropy: initial enlargement

$$p_t(\cdot, l) = \int_0^t k_u^l dW_u \quad \mathbb{F}-\text{martingale};$$

Itô's formula $\Longrightarrow$

$$\ln p_t(\cdot, l) = \int_0^t \frac{1}{p_u(\cdot, l)} k_u^l dW_u - \frac{1}{2}\int_0^t (\mu_u^l)^2 du$$

Thm 6: **(H)** $\Longrightarrow$ $\frac{1}{p_t(\cdot, G)}$ $\mathbb{G}$-martingale

$\Longrightarrow$ (analogously)

Thm 7: $\Delta N = \frac{1}{2} E(\int_0^1 (\mu_s^G)^2 ds) = E(\ln p_1(\cdot, G))$

additional log utility = entropy of additional information

10 explicit results under (H)

Case 1: G discrete

$$p_i = P(G = x_i),\ i \in \mathbb{N},\ \sum_{i\in\mathbb{N}} p_i = 1$$

$$p_t(\cdot, G) = \sum_{i\in\mathbb{N}} \frac{P(G = x_i|\mathcal{F}_t)}{p_i} 1_{\{G=x_i\}}$$

$$E(\ln p_1(\cdot, G)) = \sum_{i\in\mathbb{N}} p_i \ln \frac{1}{p_i}$$

Thm 8: G discrete; then $\Delta N < \infty \iff$ entropy of G finite

Case 2: G not purely atomic

there is a Borel set B with $c = P^G(B) > 0$, and for each $n \in \mathbb{N}$ a partition $(B_i^n)_{1\leq i\leq n}$ with $P^G(B_i^n) = \frac{c}{n}$

$$G_n = \sum_{i=1}^n i 1_{B_i^n}(G) \quad \Longrightarrow$$

$$\begin{array}{ccc} \text{entropy of } G & \geq & \text{entropy of } G_n \\ E(\ln p_1(\cdot, G)) & \geq & \sum_{i=1}^n \ln(\frac{n}{c}) \cdot \frac{c}{n} \to \infty \end{array}$$

Thm 9: G not purely atomic $\Longrightarrow \Delta N = \infty$.

11 Examples

Example 3: $\mathbb{R} \ni a < b \in \mathbb{R}; G = 1_{[a,b]}(W_1)$

$$\begin{aligned} p_1 &= P(W_1 \in [a,b]) \\ p_2 &= 1 - p_1 \\ \Longrightarrow \Delta N &= p_1 \ln \tfrac{1}{p_1} + p_2 \ln \tfrac{1}{p_2} \end{aligned}$$

Example 4: $G = W_{1+\epsilon} \quad (\epsilon > 0)$

$\Longrightarrow$ (H), (NFLVR), $\Delta N \to \infty$ as $\epsilon \to 0$

Example 5: (P.I. '01)

progressive enlargement by τ_a, the last passage time through level a by Wiener process or regular diffusion

$\Longrightarrow$ arbitrage

12 Utility increment in general enlargement

(S. Ankirchner, P.I. '03; S. Ankirchner, S. Dereich, P.I '04)

General setting: price process S adapted w.r.t. $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le 1}$, $\mathbb{G} = (\mathcal{G}_t)_{0\le t\le 1}$,

$$\mathcal{F}_t \subset \mathcal{G}_t$$

agent's information horizon $\mathbb{G}$, $S = M + \alpha \cdot \langle M, M\rangle$, M continuous, θ *x-admissible* for $x > 0$ if $\mathbb{G}$–predictable, S–integrable, $x + (\theta \cdot S)_t > 0$

$$u_{\mathbb{G}}(x) = \sup_{\theta\ x-\text{admissible}} E(\log(x + (\theta \cdot S)_T) = \ln x + \frac{1}{2} E(\int_0^1 \alpha_s^2 d\langle M, M\rangle_s)$$

$$\mathbb{G} = (\mathcal{G}_t)_{0\le t\le 1}, \quad \mathbb{H} = (\mathcal{H}_t)_{0\le t\le 1} \quad \text{finite utility filtrations,} \quad \mathcal{G}_t \subset \mathcal{H}_t$$

$$\mathbb{G}: \quad S = M + \alpha \cdot \langle M, M\rangle \qquad \mathbb{H}: \quad S = N + \beta \cdot \langle M, M\rangle$$

information drift $\mathbb{G} \rightarrow \mathbb{H}$ $\quad \mu = \beta - \alpha$

$$E(\int_0^1 \alpha(\beta - \alpha) d\langle M, M\rangle) = E(\int_0^1 \alpha(d(N - M))) = 0$$

Thm 1: $u_{\mathbb{H}}(x) - u_{\mathbb{G}}(x) = \frac{1}{2} E(\int_0^1 \mu^2 d\langle M, M\rangle)$

13 additional utility and entropy

$\mathcal{G}$ sub-$\sigma-$algebra of $\mathcal{F}$, R, Q probability measures on $\mathcal{F}$;

relative entropy of R with respect to Q on $\mathcal{G}$:

$$\mathcal{H}_{\mathcal{G}}(R\|Q) = \begin{cases} \int \log \frac{dR}{dQ}\Big|_{\mathcal{G}} \, dR, & \text{if } R \ll Q \\ \infty, & \text{else.} \end{cases}$$

additional information of $\mathcal{G}$ relative to $(\mathcal{F}_r)$ on $[s,t]$ $(0 \leq s < t \leq T)$:

$$H_{\mathcal{G}}(s,t) = \int \mathcal{H}_{\mathcal{G}}(P_t(\omega,\cdot)\|P_s(\omega,\cdot))\, dP(\omega).$$

$\mathbb{G}$ enlargement of $\mathbb{F}$, information drift μ, $\Delta_n = \{0 = t_0^n < \cdots < t_{k_n}^n = 1\}, n \in \mathbb{N}$, sequence of partitions of $[0,1]$, mesh $|\Delta_n| \to 0$

consequence of (*):

Thm 5:

$$H_{\mathbb{G}|\mathbb{F}} = \lim_{n\to\infty} H_{\Delta_n} = \sum_{i=0}^{n-1} H_{\mathcal{G}_{t_i^n}}(t_i^n, t_{i+1}^n) = \frac{1}{2} E \int_0^1 \mu_u^2 \, d\langle M, M\rangle_u.$$

additional log utility = additional information

14 imperfect dynamical information

(J.M. Corcuera, P.I., A. Kohatsu-Higa, D. Nualart '02)

Example 6: $G = W_1$

Thm 9 $\Longrightarrow$ $\Delta N = \infty$, arbitrage

Pb: can additional noise on G make $\Delta N < \infty$?

Example 7: $L_t = W_1 + \tilde{W}_{g(1-t)}$, $\tilde{W}$ independent of W

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L_s : s \leq t)$$

Thm 10: $g(t) = Kt^p$;

$$\begin{aligned} p < 1 &\quad \Longrightarrow \quad \Delta N < \infty, \\ p \geq 1 &\quad \Longrightarrow \quad \Delta N = \infty. \end{aligned}$$