On martingale optimality, BSDE and cross hedging energy risk

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1 Basis risk: definition and examples

Basis = price of hedged asset - price of hedging instrument

Problem of basis risk: uncertainties of processes describing the evolution of prices of asset and hedging instrument not identical, only highly correlated.

**Example 1:** weather derivatives

hedged asset: heating oil sales, hedging instrument: HDD derivative

HDD derivative: contract paying a premium in case HDD above a critical threshold.

**Example 2:** commodity markets

hedged asset: power spot price, hedging instrument: power futures

futures: contract to deliver amount of commodity at pre-fixed price.

Hedge spot price fluctuations on time slots not coinciding with futures delivery dates.
2 a toy example

**Aim:** show problems with **hedging basis risk**, given very high correlation

**Abbildung 1: Daily Spot Prices**

airline company, **hedged asset:** jet fuel spot price, **hedging instrument:** heating oil futures

**diagram indicates** **high correlation** between jet fuel spot price and heating oil spot price
2 Cross hedging principle: correlation

simplest caricature of hedging problem:

static situation: $Y$ hedged asset, $X$ hedging instrument, both standard Gaussian, possibly strongly correlated

\[
\rho = \frac{\text{E}(XY)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}
\]

(decomposition of $Y$ into part parallel to $X$ and independent standard Gaussian part $Z$:

\[
Z = \frac{1}{\sqrt{1 - \rho^2}}[Y - \rho X]
\]

then

\[
\sqrt{1 - \rho^2}\text{E}(XZ) = \text{E}(XY) - \rho\text{E}(X^2) = 0,
\]

hence $Z$ independent of $X$, and

\[
Y = \rho X + Y - \rho X = \rho X + \sqrt{1 - \rho^2}Z.
\]
2 Cross hedging principle: mean variance

What quantity $a$ of position $X$ would agent hold to optimally hedge position $Y$? quality of hedging: minimize quadratic error

$$E((Y-aX)^2) = E((\rho-a)X + \sqrt{1-\rho^2}Z)^2) = (\rho-a)^2 + (1-\rho^2)$$ minimal,

i.e.

$$a = \rho, \quad \text{Hedging error: } \sqrt{1-\rho^2}Z$$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sqrt{1-\rho^2}$</th>
<th>% uncertainty hedged</th>
</tr>
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<td>0.05</td>
<td>95</td>
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<td>0.9</td>
<td>0.44</td>
<td>56</td>
</tr>
</tbody>
</table>
3 Mean variance hedging of jet fuel by heating oil

simple model for price processes (better: geometric BM) (J. C. Hull 2008)

jet fuel

\[ J_t = J_0 + \mu t + \sigma Y_t, \quad t \geq 0 \]

heating oil

\[ H_t = H_0 + \nu t + \beta X_t, \quad t \geq 0 \]

\( \mu, \nu, \sigma, \beta \in \mathbb{R}, X \) and \( Y \) correlated BM, to be estimated from data; \( T = 1 \)
delivery date, observation days \( 0 = t_0 < t_1 < \ldots < t_N = 1 \)

ML estimator for \( \sigma \)

\[
\hat{\sigma} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{t_i - t_{i-1}} (J_{t_i} - J_{t_{i-1}})^2 - \frac{1}{N}(J_1 - J_0)^2,
\]

yields ML estimates \( \hat{\sigma} \approx 3.998, \hat{\beta} \approx 3.835; \)

ML estimator for correlation \( \rho \) between \( \hat{\sigma}Y \) and \( \hat{\beta}X \) requires estimate of quadratic variation of \( Y \) and \( X \) and yields \( \hat{\rho} = 0.897 \)
3 Mean variance hedging of jet fuel by heating oil

decomposition of the jet fuel price

\[ J_t = J_0 + \mu t + 0.897\hat{\sigma}X_t + 0.443\hat{\sigma}Z_t, \]

\( Z \) BM independent of \( X \).

airline aims at hedging increasing fuel prices by buying heating oil futures; suppose \( K = E[H_1] = H_0 + \nu \) is price of heating oil futures at time 0; quantity of futures \( a \) the airline has to hold to minimize quadratic error determined by

\[
E((J_1 - J_0) - a(H_1 - K))^2 = \mu^2 + (0.897\hat{\sigma} - a\hat{\beta})^2 + 0.321\hat{\sigma}^2,
\]

i.e. \( a = 0.897\frac{\hat{\sigma}}{\hat{\beta}} \).

hedging error at time 1

\[ I_1 = 0.443\hat{\sigma}Z_1. \]

correlation between spot prices almost 0.9; only 56\% of standard deviation of price change can be hedged!
4 Conclusions: mean variance vs. utility based hedging

- *Even if correlation very high, hedging error large!*

- correlation high: small change in correlation entails big change in percentage of basis risk relative to total risk

- correlation low: small change in correlation entails essentially no change in percentage of basis risk relative to total risk

- downside part of basis risk has to be properly respected

- *utility based approach* does this
5 Our approach of utility based hedging

Aims:

- present a purely probabilistic approach, combining martingale optimality and BSDE
- determine utility indifference price
- determine explicit derivative hedge, i.e. optimal cross hedging strategy
- clarify role of correlation in hedging
- describe reduction of risk by cross hedging
6 The financial market model

Index process, e.g. temperature, spot price

\[ dR_t = \sigma(t, R_t) dW_t + b(t, R_t) dt, \]

\( b : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m, \sigma : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m \times d} \)

deterministic functions, globally Lipschitz and of sublinear growth.  

Markov process, \( R_t^{s,r} \): start at \( t \) in \( r \)

Hedged asset: liability or derivative \( F(R_T) \), \( F : \mathbb{R}^m \to \mathbb{R} \) bounded

Hedging instrument: correlated financial market, \( k \) risky assets with price process:

\[ \frac{dS_t^i}{S_t^i} = \beta_i(t, R_t) dW_t + \alpha_i(t, R_t) dt = \beta_i(t, R_t) [dW_t + \theta_t dt], \quad i = 1, \ldots, k, \]

\( \alpha : [0, T] \times \mathbb{R}^m \to \mathbb{R}^k, \beta : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{k \times d}, \theta = \beta^* [\beta \beta^*]^{-1} \alpha. \)

\( W \) \( d \)-dimensional Brownian motion, correlation expressed by \( \beta \) and \( \sigma \).
7 The optimal investment problem

(N. El Karoui, R. Rouge ’00; J. Sekine ’02; J. Cvitanic, J. Karatzas ’92, Kramkov, Schachermayer ’99,...)

investment strategy $\lambda$: value of portfolio fraction invested in risky assets

wealth gain on $[0, s]$ (here $\beta_t = \beta(t, \cdot)$ etc.)

$$G^\lambda_s = \sum_{i=1}^{k} \int_0^s \lambda^i u dS^i_u = \int_0^s \lambda_u \beta_u [dW_u + \theta_u du],$$

utility function: $U(x) = -e^{-\eta x}$ $(0 < \eta$ risk aversion); maximal expected utility from terminal wealth without and with derivative:

$$V^0(v) = \sup_{\lambda \in \tilde{C}} EU(v + G^\lambda_T), \quad V^F(v) = \sup_{\lambda \in \tilde{C}} EU(v + G^\lambda_T - F(R_T))$$

utility indifference

utility indifference price

derivative hedge

$V^F(v^F) = V^0(v^0)$, $\lambda^0$ resp. $\lambda^F$ optimal strategies

$\Delta v = v^F - v^0 = p = p(r) = p(t, r)$

$\Delta \lambda = \lambda^F - \lambda^0$
8 Optimization under non-convex constraints

interpretation as maximization problem with convex constraints

\[ \tilde{C} \subset \mathbb{R}^k \text{ convex}, \quad \lambda \in \tilde{C} \]

\[ \pi_t = \lambda_t \beta_t \in C_t = \tilde{C} \beta_t \]

\[ C_t \text{ convex} \]

**Aim:** construct solution combining *martingale optimality* with BSDE, even for non-convex constraints

(N. El Karoui, R. Rouge ’00 for convex constraints)

\[ \tilde{C} \subset \mathbb{R}^k \text{ closed}, \quad \lambda \in \tilde{C} \]
8 Optimization under non-convex constraints

\( F = F(R_T) \) hedged asset

First formulation:
Find

\[
V(v) = \sup_{\lambda \in \tilde{C}} E(U(G_T^\lambda - F)) = \sup_{\lambda \in \tilde{C}} E(U(v + \int_0^T \lambda_s \beta_s [dW_s + \theta_s ds] - F)).
\]

For simplicity:

\[
\pi = \lambda \beta, \quad C = \tilde{C} \beta.
\]

\[
G_t^\pi = v + \int_0^t \pi_s [dW_s + \theta_s ds], \quad t \in [0, T]
\]

Second formulation:
Find

\[
V(v) = \sup_{\pi \in C} E(U(G_T^\pi - F)) = \sup_{\pi \in C} E(-\exp(-\eta(v + \int_0^T \pi_s [dW_s + \theta_s ds] - F))).
\]
9 Martingale optimality and BSDE

Idea: Construct family of processes $Q^{(\pi)}$ such that

$$
\begin{align*}
Q_0^{(\pi)} &= \text{constant}, \\
Q_T^{(\pi)} &= -\exp(-\eta(G_T^{\pi} - F)), \\
Q^{(\pi)} &= \text{supermartingale}, \quad \pi \in C, \\
Q^{(\pi^*)} &= \text{martingale, for (exactly) one } \pi^* \in C.
\end{align*}
$$

Then

$$
E(-\exp(-\eta[G_T^{\pi} - F])) = E(Q_T^{(\pi)}) \\
\leq E(Q_0^{\pi}) \\
= V(v) \\
= E(Q_0^{(\pi^*)}) \\
= E(-\exp(-\eta[G_T^{(\pi^*)} - F])).
$$

Hence $\pi^*$ optimal strategy.
9 Martingale optimality and BSDE

Introduction of BSDE into problem

Find generator $f$ of BSDE

\[ Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad Y_T = F, \]

such that with

\[ Q^{(\pi)}_t = -\exp(-\eta [G^\pi_t - Y_t]), \quad t \in [0, T], \]

we have

\[ Q^{(\pi)}_0 = -\exp(-\eta (v - Y_0)) = \text{constant}, \quad (\text{fulfilled}) \]

form 2

\[ Q^{(\pi)}_T = -\exp(-\eta (G^\pi_T - F)) \quad (\text{fulfilled}) \]

\( Q^{(\pi)} \) supermartingale, \( \pi \in C \),
\( Q^{(\pi^*)} \) martingale, for (exactly) one \( \pi^* \in C \).

This gives solution of valuation problem.
10 Construction of generator of BSDE

How to determine $f$:

Suppose $f$ generator of BSDE. Then

$$Q_t^{(\pi)} = -\exp(-\eta[G_t^\pi - Y_t])$$

$$= -\exp(-\eta[v - Y_0]) \cdot \exp(-\eta[\int_0^t (\pi_s - Z_s)dW_s - \int_0^t [f(s, Z_s) - \pi_s \theta_s]ds])$$

$$= \exp(-\eta[v - Y_0]) \cdot \exp(-\eta \int_0^t (\pi_s - Z_s)dW_s - \frac{\eta^2}{2} \int_0^t (\pi_s - Z_s)^2ds)$$

$$\cdot -\exp(\int_0^t [\eta f(s, Z_s) - \eta \pi_s \theta_s + \frac{\eta^2}{2} (\pi_s - Z_s)^2]ds)$$

$$= M_t^{(\pi)} \cdot A_t^{(\pi)},$$

with $M^{(\pi)}$ nonnegative martingale. $Q^{(\pi)}$ satisfies (form 2) iff for

$$q(\cdot, \pi, z) = f(\cdot, z) - \pi \theta + \frac{\eta}{2}(\pi - z)^2, \quad \pi \in C, z \in \mathbb{R},$$

we have
10 Construction of generator of BSDE

**form 3**

\[
q(\cdot, \pi, z) \geq 0, \quad \pi \in C \quad \text{(supermartingale cond.)}
\]

\[
q(\cdot, \pi^*, z) = 0, \quad \text{for (exactly) one} \quad \pi^* \in C \quad \text{(martingale cond.)}.
\]

Now

\[
q(\cdot, \pi, z) = f(\cdot, z) - \pi \theta + \frac{\eta}{2} (\pi - z)^2
\]

\[
= f(\cdot, z) + \frac{\eta}{2} (\pi - z)^2 - (\pi - z) \cdot \theta + \frac{1}{2\eta} \theta^2 - z\theta - \frac{1}{2\eta} \theta^2
\]

\[
= f(\cdot, z) + \frac{\eta}{2} [\pi - (z + \frac{1}{\eta})]^2 - z\theta - \frac{1}{2\eta} \theta^2.
\]

Under **non-convex constraint** \( \pi \in C \):

\[
[\pi - (z + \frac{1}{\eta})]^2 \geq d^2(C, z + \frac{1}{\eta}).
\]

with **equality** for at least one possible choice of \( p^* \) due to **closedness** of \( C \).

Hence **(form 3)** is solved by the choice
10 Construction of generator of BSDE

form 4

\[ f(\cdot, z) = -\frac{\eta}{2}d^2(C, z) + \frac{1}{\eta} z \cdot \theta + \frac{1}{2\eta} \theta^2 \]  

(supermartingale)

such that

\[ d(C, z + \frac{1}{\eta} \theta) = d(\pi^*, z + \frac{1}{\eta} \theta) \]  

(martingale).

Problem: Let

\[ \Pi_C(v) = \{ \pi \in \mathbb{R}^d : d(C, v) = d(\pi, v) \} \]

Find measurable selection \( \pi_t^* \) from

\[ \Pi_C(t \cdot (Z_t + \frac{1}{\eta} \theta_t)) \].

Solved by classical measurable selection method.
11 Main result

Thm 1

\( (Y, Z) \) unique solution of BSDE

\[
Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad t \in [0, T],
\]

with

\[
f(t, Z_t) = -\frac{\eta}{2} d^2(C_t, Z_t + \frac{1}{\eta} \theta_t) + Z_t \cdot \theta_t + \frac{1}{2\eta} \theta_t^2.
\]

Then value function of utility optimization problem under constraint \( \pi \in \mathcal{A} \) given by

\[
V(v) = -\exp(-\eta[v - Y_0]).
\]

There exists an (non-unique) optimal trading strategy \( \pi^* \in \mathcal{A} \) such that

\[
\pi^*_t \in \Pi C_t(Z_t + \frac{1}{\eta} \theta_t), \quad t \in [0, T].
\]

Proof:

- existence, uniqueness for BSDE with quadratic non-linearity in \( z \)
  (M. Kobylanski '00)
- measurable selection theorem for \( \Pi C_t(Z_t + \frac{1}{\eta} \theta_t) \)
- BMO properties of the martingales \( \int Z_s dW_s, \int \pi_s^* dW_s \)
  for uniform integrability of exponentials (regularity of coefficients)
12 Calculation of derivative hedge

generalization to \([t, T]\) instead of \([0, T]\), cond. on \(R_t = r\):

\((Y^{t,r}, Z^{t,r}), \pi^{t,r}\) (without \(F\)) resp. \((\hat{Y}^{t,r}, \hat{Z}^{t,r}), \hat{\pi}^{t,r}\) (with \(F\)) instead of \((Y, Z), \pi\)
yields

\[V^0(t, v, r) = -\exp(-\eta(v - Y^{t,r}_t)), \quad V^F(t, v, r) = -\exp(-\eta(v - \hat{Y}^{t,r}_t)),\]

instead of \(V(v) = -\exp(v - Y_0)\).

due to linearity of \(C(t, r)\) projections unique and linear, hence

\[\pi^{t,r}_s = \Pi_C(t, r)[Z^{t,r}_s + \frac{1}{\eta} \theta(s, R^{t,r}_s)], \quad \hat{\pi}^{t,r}_s = \Pi_C(t, r)[\hat{Z}^{t,r}_s + \frac{1}{\eta} \theta(s, R^{t,r}_s)],\]

and so

\[(\Delta \lambda \beta)(s, R^{t,r}_s) = \Pi_C(t, r)[\hat{Z}^{t,r}_s - Z^{t,r}_s].\]
13 Markov property and its consequences

Markov property of $R$ implies (Kobylanski ’00, El Karoui, Peng, Quenez ’97):

**Thm 2**
There are measurable (deterministic) functions $u$ and $\hat{u}$ such that

$$Y_{s}^{t,r} = u(s, R_{s}^{t,r}), \quad \hat{Y}_{s}^{t,r} = \hat{u}(s, R_{s}^{t,r}).$$

There are measurable (deterministic) functions $v$ and $\hat{v}$ such that

$$Z_{s}^{t,r} = v\sigma(s, R_{s}^{t,r}), \quad \hat{Z}_{s}^{t,r} = \hat{v}\sigma(s, R_{s}^{t,r}).$$

**Corollary 1**

$$p(t, r) := Y_{t}^{t,r} - \hat{Y}_{t}^{t,r} = u(t, r) - \hat{u}(t, r)$$

is the **indifference price**, i.e. $V^{F}(t, v - p(t, r), r) = V^{0}(t, v, r)$.

$p$ depends only on $R$, not on $S$

**Aim:** Explicit description of $\Delta_{\lambda}$
14 Differentiability

**Thm 3 (Parameter Differentiability)** smoothness conditions on $F, f$
There exists a version of $(\hat{Y}_{s}^{t,r}, \hat{Z}_{s}^{t,r})$ such that a.s.
- $\hat{Y}_{s}^{t,r}$ is continuous in $s$ and cont. differentiable in $r$ (classical sense)
- $\hat{Z}_{s}^{t,r}$ is differentiable in a weak sense (norm topology)
- $(\nabla_{r}\hat{Y}_{s}^{t,r}, \nabla_{r}\hat{Z}_{s}^{t,r})$ solves the BSDE

\[
\nabla_{r}\hat{Y}_{t}^{r} = \nabla_{r}F(R_{s}^{t,r})\nabla_{r}R_{s}^{t,r} - \int_{t}^{T} \nabla_{r}\hat{Z}_{s}^{t,r} dW_{s} + \int_{t}^{T} \left[ \nabla_{r}f(s, R_{s}^{t,r}, \hat{Z}_{s}^{t,r})\nabla_{r}R_{s}^{t,r} + \nabla_{z}f(s, R_{s}^{t,r}, \hat{Z}_{s}^{t,r})\nabla_{r}\hat{Z}_{s}^{t,r} \right] ds.
\]

Proof uses norm inequalities, and inverse Hölder inequalities, based on BMO properties of the stochastic integral processes of $\hat{Z}_{s}^{t,r}$

**Thm 4 (Malliavin Differentiability)**

\[
D_{\varnothing}\hat{Y}_{s}^{t,r} = \nabla_{r}\hat{u}(s, R_{s}^{t,r})D_{\varnothing}R_{s}^{t,r}
\]
and
\[
\hat{Z}_{s}^{t,r} = D_{s}\hat{Y}_{s}^{t,r} = \nabla_{r}\hat{u}(s, R_{s}^{t,r})\sigma(s, R_{s}^{t,r})
\]
15 Explicit description of derivative hedge

Properties of the BSDEs

Thm 5
The indifference price $p(t, r) = Y_{t}^{t,r} - \hat{Y}_{t}^{t,r}$ is differentiable in $r$.

Thm 6
The derivative hedge $\Delta_{\lambda}$ at time $t$ depends only on $R_{t}$, and

$$
\Delta_{\lambda}(t, r) \beta(t, r) = \Pi_{C(t,r)} [\hat{Z}_{t}^{t,r} - Z_{t}^{t,r}]
= \Pi_{C(t,r)} [\nabla_{r} (\hat{Y}_{t}^{t,r} - Y_{t}^{t,r}) \sigma(t, r)]
= -\Pi_{C(t,r)} [\nabla_{r} p(t, r) \sigma(t, r)].
$$

Remarks:

- complete case: $\Delta_{\lambda} =$ 'delta hedge'
- where is the risk aversion $\eta$?
16 Example: Heating degree days

- common underlying of weather derivatives

- $T_i = \text{average of the maximum and the minimum temperature on day } i \text{ at a specific location}$

- $HDD_i = \max(0, 18 - T_i)$

Cumulative heating degree days

$$cHDD_t = \sum_{i=1}^{30} HDD_{t-i}$$

Derivatives:

- Option: $(cHDD - K)^+$

- Swap: $b(cHDD - K)$
16 Example: Heating degree days

\( cHDD \):

- statistical analysis shows: cHDDs are log-normally distributed (M. Davis ’01)
- \( cHDD \) can be modeled as a geometric Brownian motion

\[
dX_t = \mu X_t dt + \nu X_t dW_t
\]

(moving average)

Other indices: cooling degree days

\[
CDD_i = \min (0, 18 - T_i)
\]
16 Example: Heating degree days

- $R = \text{cHDDs (geometric Brownian Motion)}$
- $d = 2$
- 1-dim market + index: $k = m = 1$
- index volatility: $\sigma = \begin{pmatrix} \alpha & 0 \end{pmatrix}$
- price volatility: $\beta = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$ with $\alpha, \beta_1, \beta_2 \in \mathbb{R} \setminus \{0\}$

Then

$$\Delta_\lambda(t, r) = -\alpha \frac{\partial p(t, r)}{\partial r} \frac{\beta_1}{\beta_1^2 + \beta_2^2}.$$
17 Example: Heating degree days; diversification pressure

derivative hedge:

\[ \Delta_\lambda(t, r) = -\alpha \frac{\partial p(t, r)}{\partial r} \frac{\beta_1}{\beta_1^2 + \beta_2^2}. \]

Call option: \( F(R_T) = (R_T - K)^+ \)

\[ \Rightarrow \frac{\partial p(t, r)}{\partial r} > 0 \]

Comparison of the optimal strategies:

- \( \beta_1 \alpha < 0 \) (negative correlation)
  \[ \Rightarrow F(R_T) \text{ diversifies portfolio} \Rightarrow \Delta_\lambda > 0 \]
  \[ \Rightarrow \hat{\pi} > \pi \]

- \( \beta_1 \alpha > 0 \) (positive correlation)
  \[ \Rightarrow F(R_T) \text{ amplifies portfolio} \Rightarrow \Delta_\lambda < 0 \]
  \[ \Rightarrow \hat{\pi} < \pi \]