Credit Risk: Recent Developments in Valuation and Risk Management for CDOs

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Overview

A. Standard approaches for CDO-valuation and risk-management

B. Markov-chain models for credit portfolios

C. Dynamic hedging of credit derivatives

A. Standard approaches for CDO-valuation and risk-management

Overview

- Synthetic CDO tranches
- Factor copula models
- Correlation skew
- Risk management of CDOs in standard models

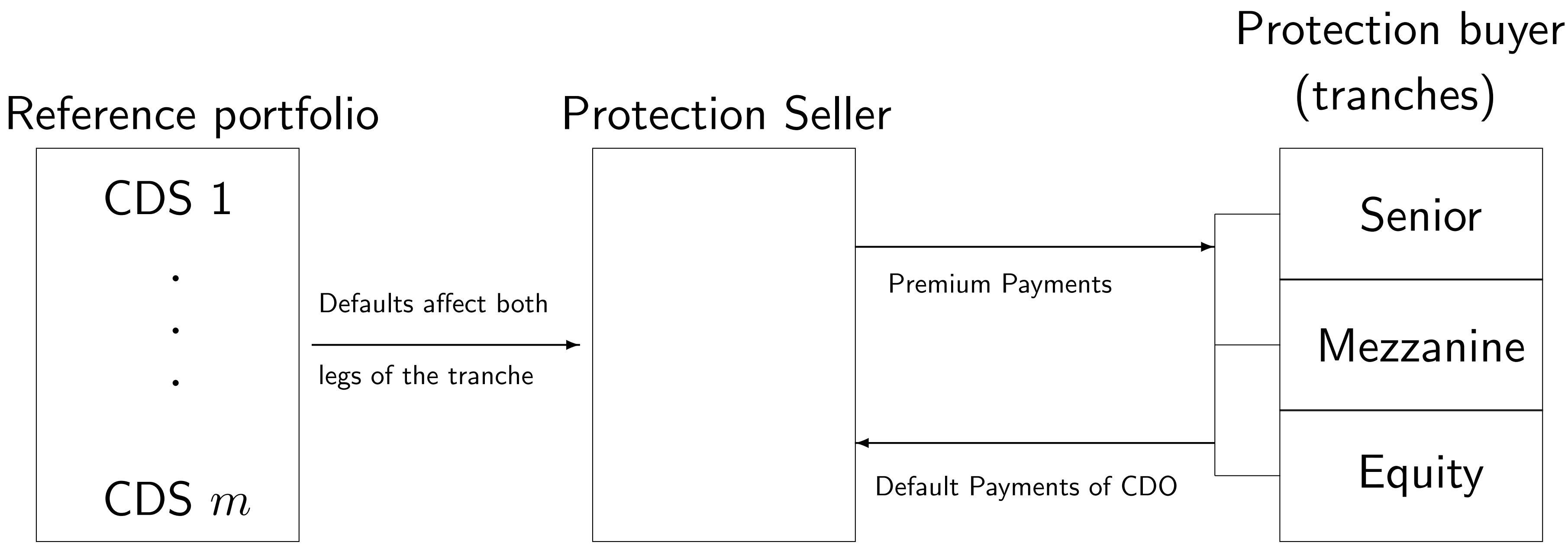
A1. Synthetic CDO tranches

Basic Notation

Consider m firms with default times τ_i, $1 \leq i \leq m$ and default indicator process $\mathbf{Y}_t = (Y_{t,1}, \ldots, Y_{t,m})$ with $Y_{t,i} = 1_{\{\tau_i \leq t\}}$.

- $\overline{F}_i(t) = P(\tau_i > t)$ survival function of obligor i; joint survival function: $\overline{F}(t_1, \ldots, t_m) = P(\tau_1 > t_1, \ldots, \tau_m > t_m)$.
- Ordered default times denoted by $T_0 < T_1 < \ldots < T_m$. $\xi_n \in \{1, \ldots, m\}$ gives identity of the firm defaulting at time T_n
- Cumulative loss of the portfolio in t given by $L_t = \sum_{i=1}^{m} \delta_i Y_{t,i}$, δ_i LGD of firm i.

Synthetic CDOs - Basic Structure



Payments in a synthetic CDO structure.

Payment Description.

Consider synthetic CDO-tranche with attachment points $0 \le l < u \le 1$ (in percent of the overall notional m). Define the notional of the tranche at t by the following put-spread

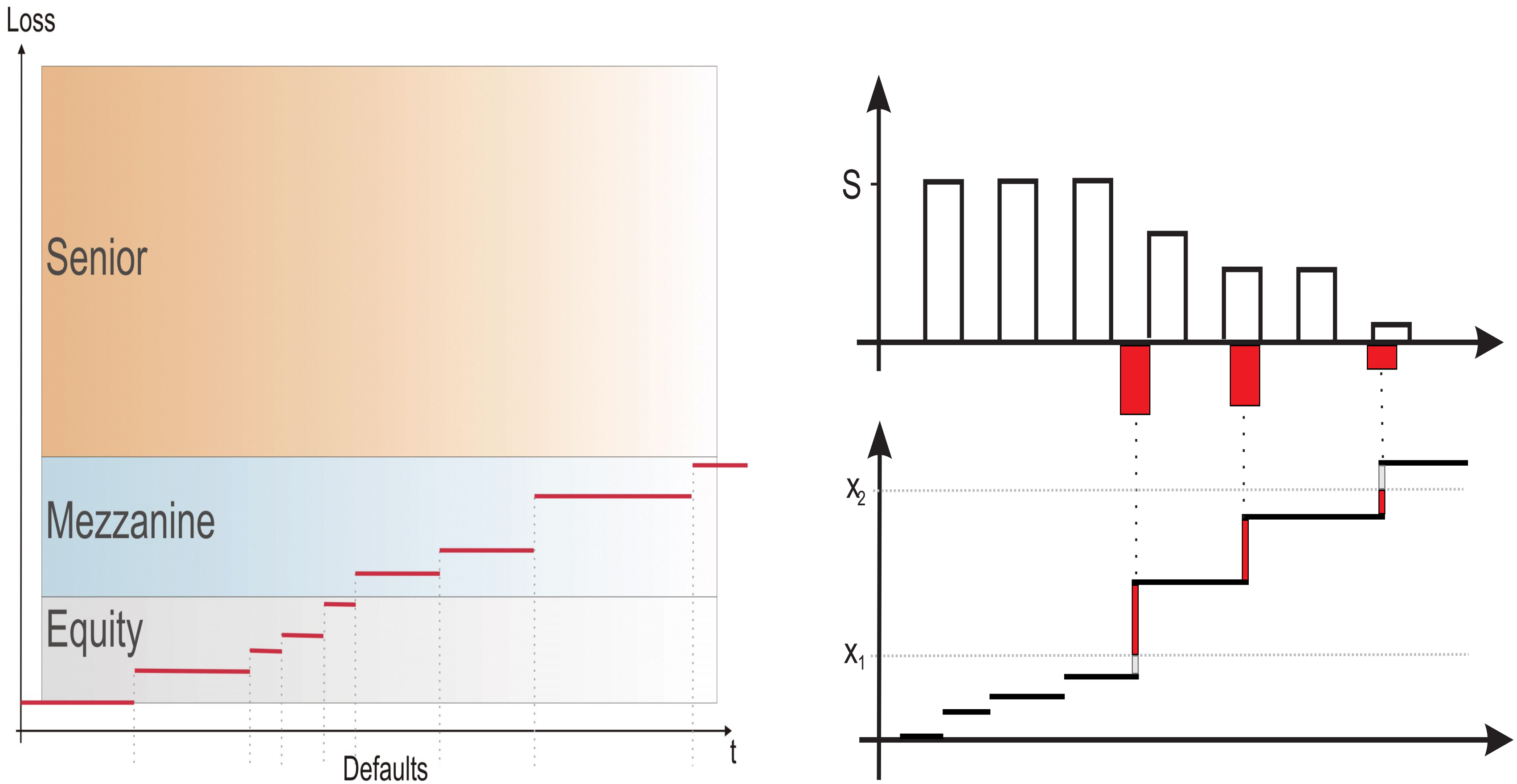
$$N_t^{[l,u]} := ((mu) - L_t)^+ - ((ml) - L_t)^+;$$

the cumulative loss of tranche $[l,u]$ is $L_t^{[l,u]} := N_0^{[l,u]} - N_t^{[l,u]}$.

- Default payments. At kth default time $T_k < T$ protection-seller makes default payment $\Delta L_{T_k}^{[l,u]}$ (the part of the portfolio loss which falls in the layer $[l,u]$).

- Premium payments. Protection-seller receives periodic premium payments at $0 < t_1 < \cdots < t_N = T$ of size $s^{[l,u]}(t_n - t_{n-1})N_{t_n}^{[l,u]}$, $s^{[l,u]}$ the tranche spread. At default-date $T_k \in [t_{n-1}, t_n]$ he receives an accrued premium of size $s^{[l,u]}(T_k - t_{n-1})\Delta L_{T_k}^{[l,u]}$.

Graphical illustration



Loss process for a given portfolio and corresponding tranche losses (left); premium and default payments for a tranche with lower attachment point X_1 (right)

Pricing Premium Payments.

Notation Q represents a risk-neutral measure used for pricing; risk-free interest rate r and LGD δ_i are deterministic; $D(t) = \exp(-\int_0^t r(s))ds$ is default-free discount factor.

Premium-payment leg. The value V^{prem} can be expressed in terms of the distribution of L_t. Given a generic tranche-spread x we have

$$V_0^{\text{prem},[l,u]} = xE^Q\Big(\sum_{n=1}^{N}(t_n - t_{n-1})D(t_n)N_{T_n}^{[l,u]}\Big),$$

and $N_{t_n}^{[l,u]} = ((mu) - L_t)^+ - ((ml) - L_t)^+$ is a function of L_{t_n}.

Pricing Default-Payment Leg

The cumulative default payments of a tranche with attachment points $[l, u]$ are given by $L_t^{[lu]}$. We therefore obtain

$$V_0^{\text{def},[l,u]} = E^Q\Big(\int_0^T D(t)dL_t^{[lu]}\Big) \approx \sum_{n=1}^{N} D(t_{n-1})E^Q\big(L_{t_n}^{[l,u]} - L_{t_{n-1}}^{[l,u]}\big).$$

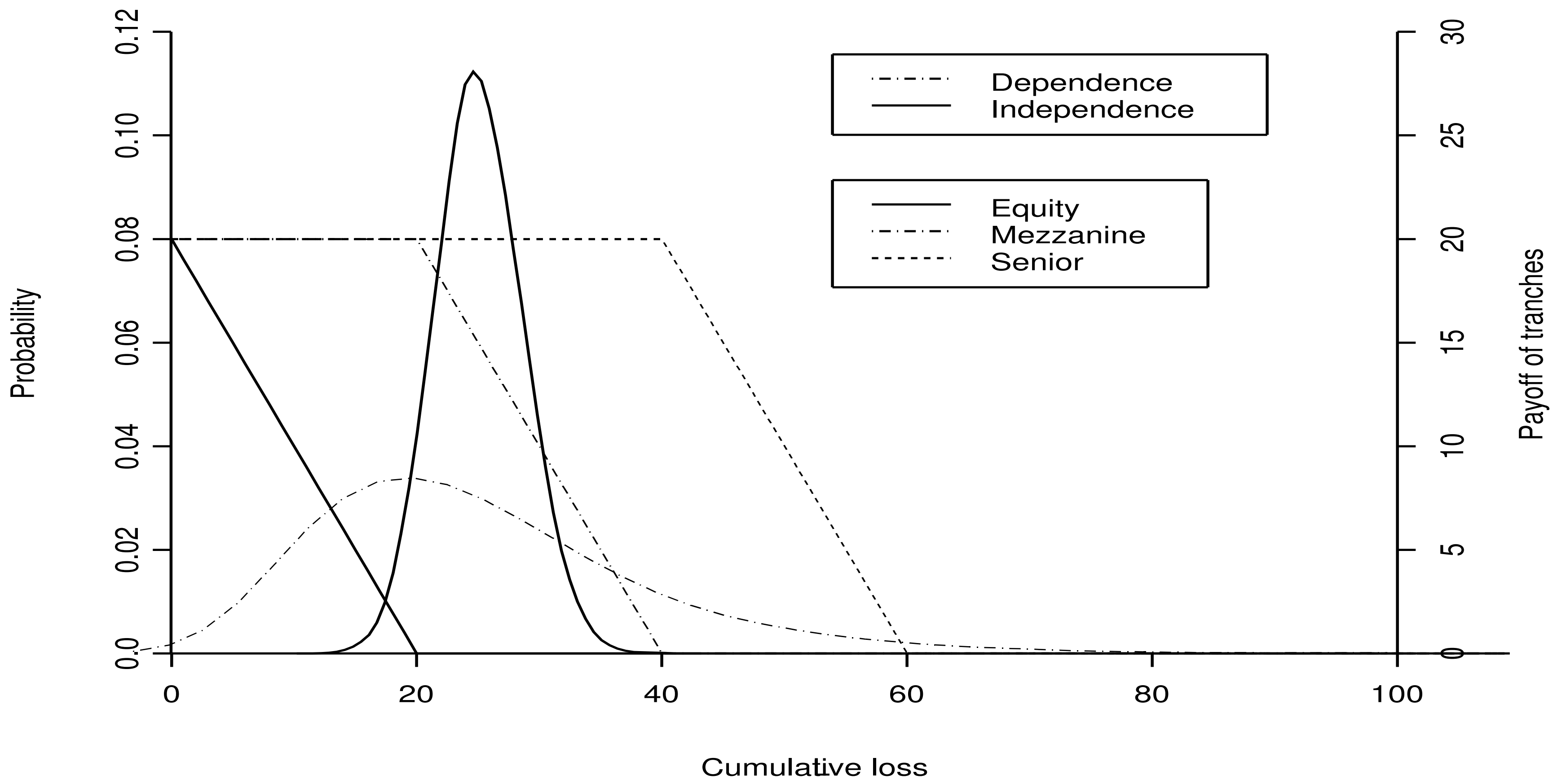
- $L_t^{[lu]}$ is a function of portfolio loss $L_t \Rightarrow$ For pricing purposes we need again distribution of L_{t_n}, $n = 1, \ldots, N$.
- No initial payment $\Rightarrow$ (fair) tranche spread $s^{[l,u]}$ determined from $V_0^{\text{prem},[l,u]} \overset{!}{=} V_0^{\text{def},[l,u]}$
- CDO-spreads depend only on the family of one-dimensional (marginal) distributions of L.

Default Correlation and CDO Tranches

More dependence (higher default correlation), same marginal default probabilities $\Rightarrow$

- Senior tranche suffers more frequent losses on average, hence fair spread increases.
- Equity tranche is wiped out less frequently on average, hence fair spread decreases.
- Impact on mezzanine tranches unclear.

Default Correlation and CDO Tranches ctd



Notional of a three CDO-tranches with attachment points at 20, 40 and 60 with two different loss distributions overlayed.

A2. Factor Copula Models

Copulas

- A copula is a df C on$[0,1]^m$ with uniform margins.
- Copulas and dependence structure. If a multivariate df F has continuous margins $F_1, \ldots, F_m$ and $\mathbf{X} \sim F$, the copula C of F is the df of $(F_1(X_1), \ldots F_m(X_m))$, and we have Sklars identity

$$F(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m)),$$

- Survival copulas. Similarly, the survival function of $\mathbf{X}$ can be written as $\overline{F}(x_1, \ldots, x_m) = \hat{C}\left(\overline{F}_1(x_1), \ldots, \overline{F}_m(x_m)\right)$, where the survival copula $\hat{C}$ is given by $\hat{C}(u_1, \ldots, u_m) = \bar{C}(1-u_1, \ldots, 1-u_m)$.

Sklar's Theorem

Let F be a joint distribution function with margins $F_1, \ldots, F_d$. There exists a copula C such that for all $x_1, \ldots, x_d$ in $[-\infty, \infty]$

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

If the margins are continuous then C is unique; otherwise C is uniquely determined on Ran$F_1 \times$ Ran$F_2 \ldots \times$ RanF_d.

And conversely, if C is a copula and $F_1, \ldots, F_d$ are univariate distribution functions, then F defined above is a multivariate df with margins $F_1, \ldots, F_d$.

Sklar's Theorem: Proof in Continuous Case

Henceforth, unless explicitly stated, vectors $\mathbf{X}$ will be assumed to have continuous marginal distributions. In this case:

$$\begin{aligned} F(x_1, \ldots, x_d) &= P\left(X_1 \le x_1, \ldots, X_d \le x_d\right) \\ &= P\left(F_1(X_1) \le F_1(x_1), \ldots, F_d(X_d) \le F_d(x_d)\right) \\ &= C(F_1(x_1), \ldots, F_d(x_d)). \end{aligned}$$

The unique copula C can be calculated from $F, F_1, \ldots, F_d$ using

$$C(u_1, \ldots, u_d) = F\left(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d)\right).$$

Copulas and Dependence Structures

Sklar's theorem shows how a unique copula C describes the dependence structure of the multivariate df of a random vector $\mathbf{X}$. This motivates a further definition.

Definition. The copula of $\mathbf{X} = (X_1, \ldots, X_d)$ is the df C of $(F_1(X_1), \ldots, F_d(X_d))$.

Example. Gauss copula $C_{\mathsf{P}}^{\mathsf{Ga}}$ is the copula of $\mathbf{X} \sim N_m(\mathbf{0}, \mathsf{P})$, P a correlation matrix. Symmetry of $N_m(\mathbf{0}, \mathsf{P}) \Rightarrow C_{\mathsf{P}}^{\mathsf{Ga}} = \hat{C}_{\mathsf{P}}^{\mathsf{Ga}}$.

Copula Models

In copula models marginal distribution and survival copula of default times $(\tau_1, \dots, \tau_m)$ are specified separately. Hence survival function of default times is given by

$$\overline{F}(t_1, \dots, t_m) = \hat{C}\left(\overline{F}_1(t_1), \dots, \overline{F}_m(t_m)\right), \qquad (1)$$

Specifying dependence structure $\hat{C}$ and marginal distribution $\overline{F}_i$ separately is useful for calibration: the model is calibrated to given term structure of (single-name) CDS spreads by specifying $\overline{F}_i$; calibration of dependence structure (i.e. $\hat{C}$) can then be done independently.

One-factor Gauss-Copula Model.

Model. Put $X_i = \sqrt{\rho_i} V + \sqrt{1-\rho_i}\epsilon_i$ for 'asset correlation' $\rho_i \in (0,1)$ and $V, (\epsilon_i)_{1 \le i \le m}$ iid standard normal rvs. Set $U_i = \Phi(X_i)$, so that $\mathbf{U} \sim C_P^{\mathsf{Ga}}$.

Survival function. We have with $d_i(t) := \Phi^{-1}\left(\overline{F}_i(t)\right)$

$$\overline{F}(t_1, \ldots, t_m) = P\left(U_1 \le \overline{F}_1(t_1), \ldots, U_m \le \overline{F}_m(t_m)\right) \qquad (2)$$

$$= P\left(X_1 \le d_1(t_1), \ldots, X_m \le d_m(t_m)\right) \qquad (3)$$

Conditioning on the systematic factor V we thus get from the factor structure of X

$$\overline{F}(t_1, \ldots, t_m) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \prod_{i=1}^{m} \underbrace{P\left(X_i \le d_i(t_i) \mid V = v\right)}_{= \Phi\left(\frac{d_i(t_i) - \sqrt{\rho_i} v}{\sqrt{1-\rho_i}}\right)} e^{-v^2/2} dv \, .$$

Mixture Representation of Survival Function

A similar mixture representation holds in all factor copula models. Denote by $g_{\mathbf{V}}$ the density of the factor vector $\mathbf{V}$. Then

$$\overline{F}(t_1, \ldots, t_m) = \int_{\mathbb{R}^p} \prod_{i=1}^{m} \overline{F}_{\tau_i|\mathbf{V}}(t_i \mid \mathbf{v})\, g_V(\mathbf{v}) d\mathbf{v}.$$

Mixture representation very useful for simulation and pricing. We have the following 'algorithm'

1. Simulate a realization of $\mathbf{V}$.

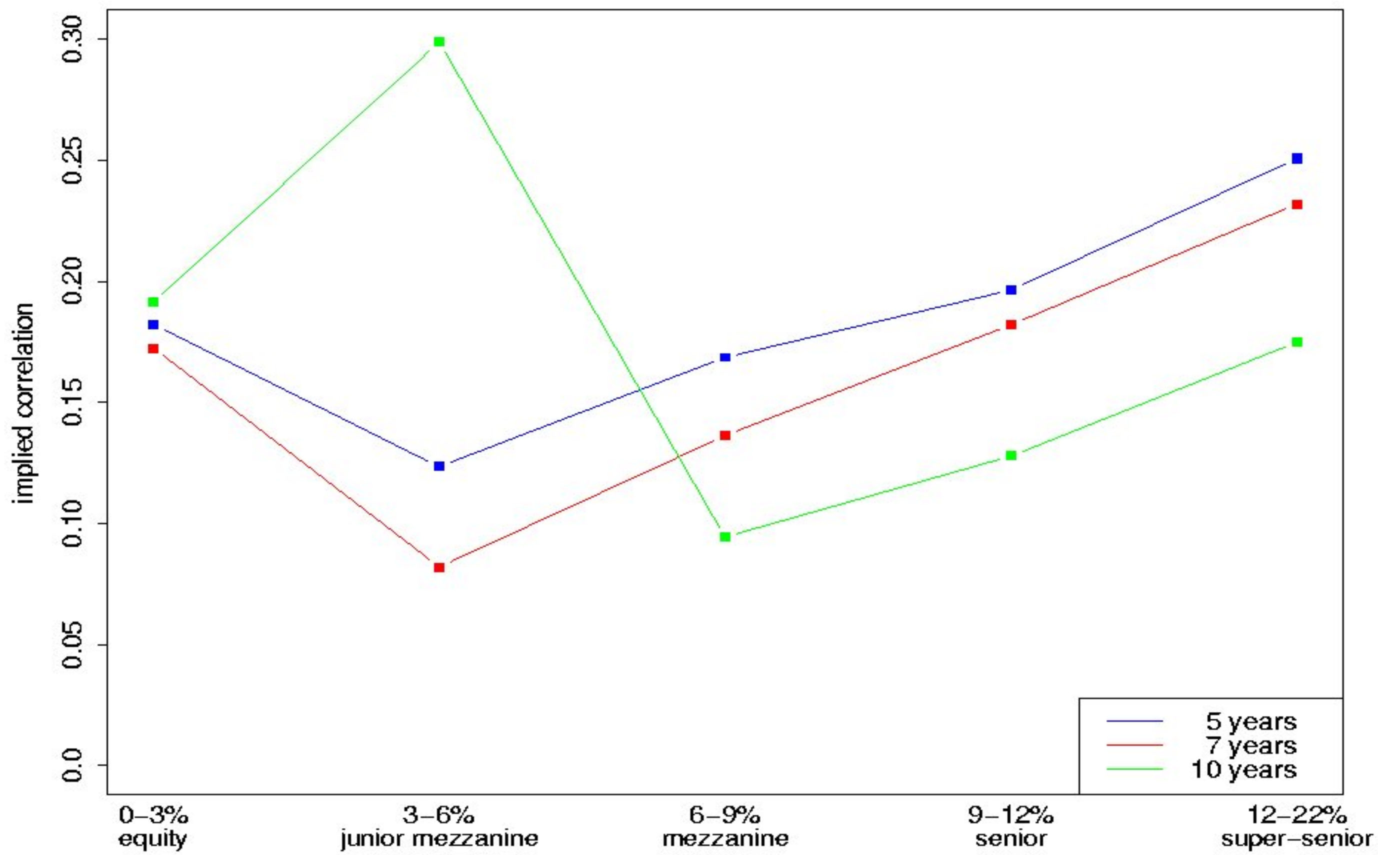
2. Simulate independent rvs τ_i with df $1 - \overline{F}_{\tau_i|\mathbf{V}}(t \mid \mathbf{V})$, $1 \leq i \leq m$.

Importance sampling techniques can be employed to speed up simulations. Particularly useful for rare event simulation as in the pricing of CDO tranches with high attachment points.

A3. Correlation Skews

One factor Gauss copula model has become important benchmark on markets for synthetic CDO-tranches. But the model is unable to explain all observed tranche prices simultaneously, largely because the only free parameter is asset correlation ρ_i

- Implied tranche correlation is the value of ρ in a homogeneous Gaussian copula model leading to the observed tranche price. (generally not uniquely defined).
- Implied base correlation is the value of ρ explaining the price (spread) of an equity tranche with the corresponding attachment point ([0,3], [0,6], [0,9], . . .). Base correlation is unique. Moreover, (hypothetical) prices of equity tranches can be computed recursively from observed prices of CDO tranches.



Typical implied correlation skew (Data from Nov,13, 2006).

Modelling Correlation Skews

Goal. Find a single model that explains (approximately) prices of all tranches. Existing approaches include

- Alternative copulas. See eg. [Burtschell et al., 2005] for a survey; a successful approach can be based on GH family (Eberlein-Frey-Hammerstein-08)
- The implied copula model of [Hull and White, 2006]
- Random default correlations and recovery rates, positively correlated with default probabilities (see eg. [Andersen and Sidenius, 2004]).

Dynamic Models

Alternatively one may look at dynamic models which specify not only the distribution of $\tau_1, \ldots, \tau_m$ at a given point in time but also the random evolution of this distribution over time.

- Dynamic Markov-chain models for L or Y (both bottom up or top-down); see Part 2
- Common shock models based on Marshall-Olkin copula
- HJM-type models for the whole loss distribution such as [Schönbucher, 2006] or [Sidenius et al., 2005] or [Filipovic et al., 2009]. These models are automatically calibrated to correlation skew (but mathematically difficult).
- Models with incomplete information and nonlinear filtering [Frey and Schmidt, 2009] or [Frey and Runggaldier, 2008].

A4. Risk Management: market practice

The holder of a CDO tranche is exposed to the following risks

- Changes in credit spreads or credit quality of underlying names (spread risk)
- Default- or event risk
- Changes in (market perceived) correlation structure
- Change in value due to time decay

In order to control and manage risk of CDO-position market uses sensitivity-based approach (model-free)

Here we give an account of current market practice using 'Fitch-terminology' ([Neugebauer, 2006].

a) Credit-Spread Sensitivity

The following quantities are commonly used

DV01. ("Dollar Value of a Basis Point"). $\mathrm{DV01}_i$ measures the change in the value of a synthetic CDO tranche given that the CDS-spread of name i changes by 1 bp.

Properties (in standard copula models).

- $\mathrm{DV01}_i$ increases with decreasing subordination and is highest for equity tranche.
- $\mathrm{DV01}_i$ increases with increasing spread on name i.

Systemic DV01. Measures the change in value of a synthetic CDO tranche given that the CDS-spread of all names in the portfolio (index spread) changes by 1 bp. Again, highest for the equity tranche.

Delta (Hedging with single-name CDS)

Denote by $\mathrm{SwapDV01}_i$ the change in the value of a CDS on name i with exposure e_i given a 1bp-change in the spread. Spread Delta wrt credit i given by $\Delta_i = \frac{\mathrm{DV01}_i}{\mathrm{SwapDV01}_i}$.

- Δ_i measures percentage of notional amount e_i that needs to be sold/bought to hedge a protection buyer/seller position in CDO against small fluctuations in spread of credit i. Often the notional amount $\Delta_i e_i$ is quoted instead of Δ_i

- As with $\mathrm{DV01}_i$, Δ_i is highest for the equity tranche and decreasing with increasing subordination.

- Δ_i changes with changes in CDS spreads, correlation structure and with changing time to maturity so that hedges need to be adjusted (no static hedge) $\Rightarrow$ Problems with transaction cost and liquidity.

b) Jump-to-default risk

Jump-to default risk is often measured by Value on Default (VOD_i). VOD_i measures change in P&L of a position assuming that name i defaults and that all other spreads remain unchanged. (No default contagion!)

- VOD_i is highest for the equity tranche which actually incurs the default payment
- VOD is higher for low-spread (high-quality) names than for high-spread names (assuming identical LGD), since quality of remaining portfolio is higher after the default of a high-spread (low-quality) name.
- In order to be "spread- and default neutral", one could use long-maturity CDS to hedge spread-risk and short-maturity CDS to hedge event risk

A5. Discussion

Factor copula models are convenient for calibration. But presentation and standard usage of the models is static, as distribution of default times is imposed at the outset. Issues related to dynamics (properties of default intensities or behavior of credit spreads) play only minor role. This creates problems:

- Without explicit dynamics of default intensities and credit spreads there are no model-based hedging strategies.
  - ⋆ Ad hoc sensitivity-based hedging might induce unaccounted drift- and time-decay effects
  - ⋆ Standard hedging neglects default contagion
- No pricing of exotic products such as forward starting CDOs possible; for this a model for the dynamic evolution of the distribution of L is needed.

B. Markov-Chain Models for Credit Portfolios

Overview

- Introduction
- A General Markov-Chain Model
- Homogeneous Models
- Nonparametric Calibration Methodology

B1. Introduction

Basic modelling idea. Default intensities are model primitives. Default intensity of firm i is modelled as function $\lambda_i(\Psi_t, \mathbf{Y}_t)$ of some factor process Ψ and of default-state $\mathbf{Y}_t$ of portfolio at time t.

Advantages.

- Intuitive parametrization of dependence between defaults;
- spread risk (via Ψ) and contagion (via dependence of λ on $\mathbf{Y}_t$) can be modelled explicitly $\Rightarrow$ useful tool for studying dynamic hedging of credit derivatives;
- Markov process techniques available for analysis and simulation.

Disadvantage. Calibration of inhomogeneous models to term structure of defaultable bonds or CDSs more difficult than with copula models.

Related literature.

Bottom up models (Modelling default state of individual firms)

- [Frey and Backhaus, 2008], [Frey and Backhaus, 2007], [Bielecki and Vidozzi, 2008], [Herbertsson, 2007], [Laurent et al., 2007] (model construction, calibration, hedging)
- Early work by [Jarrow and Yu, 2001] [Davis and Lo, 2001]

Top-down models (Modelling directly the dynamics of aggregate portfolio loss)

- [Arnsdorf and Halperin, 2007], [Lopatin and Misirpashaev, 2007] or [Cont and Minca, 2008]
- Related ideas also in [Schönbucher, 2006]

B2. A General Markov-Chain Model

We start with the case of deterministic Ψ.

Model is conveniently defined as Markov chain with state space $S = \{0,1\}^m$ and transition rates (from $\mathbf{y}$ to $\mathbf{x}$)

$$\lambda(t, \mathbf{y}, \mathbf{x}) = \begin{cases} 1_{\{y_i=0\}} \lambda_i(t, \mathbf{y}), & \text{if } \mathbf{x} = \mathbf{y}^i \text{ for some } i \in \{1 \dots, m\}, \\ 0 & \text{else,} \end{cases}$$

where $\mathbf{y}^i \in S$ is obtained from $\mathbf{y} \in S$ by flipping ith coordinate.

Intuition. Chain can jump only to neighbouring states which differ from $\mathbf{Y}_t$ by exactly one default; if $Y_{t,i} = 0$, the probability of a jump in $[t, t+h)$ to state $\mathbf{Y}_t^i$ (default of firm i) is $\approx h\lambda_i(t, \mathbf{Y}_t)$.

Model properties

- The generator of $(\mathbf{Y}_t)$ equals

$$G_{[t]}f\,(\mathbf{y}) = \sum_{i=1}^{m} 1_{\{y_i=0\}}\lambda_i(t,\mathbf{y})\big(f(t,\mathbf{y}^i) - f(t,\mathbf{y})\big).$$

- Denote by $\mathcal{H}_t = \sigma(\{\mathbf{Y}_s \colon s \le t\})$ the default history of the portfolio.

$$M_{t,i} = Y_{t,i} - \int_0^{t\wedge\tau_i} \lambda_i(s,\mathbf{Y}_s)ds$$

  is an (H_t)-martingale by the Dynkin formula, so that $\lambda_i(s,\mathbf{Y}_s)$ is in fact the $(\mathcal{H}_t)$-default intensity.

- Denote by $p(t,s,\mathbf{x},\mathbf{y}) = P_{(t,\mathbf{x})}(\mathbf{Y}_s = \mathbf{y})$, $s \ge t$, the transition probabilities of the chain. They satisfy the Kolmogorov forward- and backward equation, here an ODE system. For time-independent λ transition probabilities can be computed as matrix exponentials.

Simulation

Step 1: Determine first default time T_1 given initial state $\mathbf{y}^{(\mathbf{0})}$. Use that T_1 has intensity $\lambda_t^{(1)} = \sum_{i=1}^m \big(1 - y^{(0)}(i)\big)\lambda_i(t, \mathbf{y}^{(0)})$. Hence simulate $\theta_1 \sim \mathrm{Exp}(1)$ and put

$$T_1 := \inf\left\{ t \geq 0 : \int_0^t \lambda_s^{(1)} ds \geq \theta_1 \right\}.$$

Step 2: Determine ξ_1. Use that

$$P(\xi_1 = i \mid T_1 = t) = \frac{\big(1 - y^{(0)}(i)\big)\lambda_i(t, \mathbf{y}^{(0)})}{\sum_{j=1}^m \big(1 - y^{(0)}(j)\big)\ \lambda_j(t, \mathbf{y}^{(0)})};$$

Step 3: Put $\mathbf{y}^{(1)} := (\mathbf{y}^{(0)})^{\xi_1}$ (flip coordinate ξ_1 from 0 to 1). Repeat Step 1 and 2 with $\mathbf{y}^{(1)}$ instead of $\mathbf{y}^{(0)}$ to determine T_2 and ξ_2.

Step 4: Proceed this way until maturity.

Adding a random factor process Ψ

Motivation. Without additional factors default intensities and credit spreads evolve deterministically between defaults. This is unrealistic, hence we introduce an additional factor process Ψ and model default intensities as $\lambda_i(t, \Psi_t, Y_t)$.

Two ways for working with the model

- Two-step approach: Generate first a trajectory of Ψ and consider the model as conditional Markov chain with transition intensities depending on simulated trajectory.
- If Ψ is also modelled as finite-state Markov chain we may work directly with the Markov chain $\Gamma = (\mathbf{Y}_t, \Psi_t)_{t \geq 0}$. We mainly use this approach.

B3. Homogeneous Models

In homogeneous models (all firms are exchangeable) default intensities depend only on total number of number of defaults. Formally,

$$\lambda_i(t, \Psi_t, \mathbf{Y}_t) = h\left(t, \Psi_t, M_t\right), \; 1 \le i \le m, \quad \text{where } M_t = \sum_{i=1}^{m} Y_{t,i}. \quad (4)$$

- Natural economic interpretation: more defaults $\Rightarrow$ riskier environment for surviving firms.
- Homogeneity facilitates analytic treatment.
- Natural interpretation as a total loss model as used in top-down approach.

Examples for homogeneous models

- Linear counterparty risk model.
  $h(t, \psi, l) = \lambda_0 \psi + \lambda_1 l$, $\lambda_0 > 0, \lambda_1 \geq 0$. λ_0 is a level-parameter; λ_1 measures increase in default intensity of surviving firms at a default event.

- Convex counterparty risk model.

$$h(t, \psi, l) = \lambda_0 \psi + \frac{\lambda_1}{\lambda_2} \left(e^{\lambda_2 \left(\frac{l}{m} - \bar{\mu}(t) \right)^+} - 1 \right), \ \lambda_0 > 0, \lambda_1 \geq 0, \lambda_2 \geq 0.$$

  Here $\bar{\mu}(t)$ measures expected proportion of defaulted firms until t; convexity parameter λ_2 controls tendency of the model to generate default cascades.

Interpretation as total loss model (Ψ deterministic)

With homogeneous default intensities (4), $M_t = \sum_{i=1}^{m} 1_{\{\tau_i \le t\}}$ is itself a Markov chain. Intensity for transition of M_t from l to $l+1$ is

$$a_l(t) := (m-l)h(t,l);$$

no other transitions possible.

- State space of (M_t) is $S^M := \{0, 1, \dots m\}$ so that $|S^M| = m+1$ (instead of 2^m for the full model).
- With deterministic LGD δ the portfolio loss satisfies $L_t = \delta M_t$, so that we can view the model as dynamic portfolio-loss model.
- Extension to stochastic intensities possible; in that case we consider $\tilde{\Gamma} = (\Psi, M)$.

Forward equation

Forward equations will be important tools for analysis and calibration.

Assume $M_0 = 0$ and put $p_l(t) := P(M_t = l)$, $l = 0, \ldots, m$. Then $p_0, \ldots, p_m$ satisfy the following ODE system

$$\begin{aligned} \frac{d}{dt} p_0(t) &= -a_0(t) p_0(t) \\ \frac{d}{dt} p_l(t) &= \underbrace{a_{l-1}(t) p_{l-1}(t)}_{\text{probability inflow}} - \underbrace{a_l(t) p_l(t)}_{\text{probability outflow}}, \quad l = 1, \ldots, m \end{aligned} \tag{5}$$

Probability inflow due to jumps from $l-1$ to l; probability outflow due to jumps from l to $l+1$.

B4. Nonparametric Calibration Methodology

- Most models use parametric calibration methodology: a parametric form of the default intensities $h(t, \psi, l)$ was calibrated to observed index- and tranche spreads by minimizing some distance between market and model prices.

- Alternatively a nonparametric approach can be used. Here the function h is determined from observed CDO-spreads via forward induction, using the forward equation. The approach parallels the implied-volatility model of [Dupire, 1994]. The method is also used in HJM-type model of [Schönbucher, 2006].

- Approach presumes that we know put-option prices $E^Q\big((K - L_t)^+\big)$ for all $t > 0$, $K = 1, \ldots, m$, which can be obtained from CDO-tranches. Not all attachment points and maturities traded $\Rightarrow$ in practice one has to use interpolation.

The approach

W.l.o.g we assume $\delta = 1$ so that $M \equiv L$. Moreover, we concentrate on deterministic default intensities.

Step 1. Here one derives the implied probabilities $p_l^*(t)$ from the put-prices using the following 'butterfly-relationship':

$$Q(M_t = l) = E^Q\Big(\big((l-1) - L_t\big)^+ - 2\big(l - L_t\big)^+ + \big((l+1) - L_t\big)^+\Big),$$

$l = 0, \ldots, m-1$;

Step 2. Here one recursively derives the default intensities $h(t,l)$ from $p(t,l)$ using the Kolmogorov-forward equation. Details in [Schönbucher, 2006].

C. Dynamic hedging of credit derivatives

Overview

- Gains processes of CDSs and CDOs
- Dynamic risk-minimizing hedging strategies
- (Further) Examples

C1. Gains process of CDS- and CDO positions

The gains process of a position is the sum of the current market value and of the cumulative cash-flows associated with the position

CDSs. Consider CDS on name k with spread s, default payment δ and spread payment dates $0 < z_1 < \cdots < z_N = T$. Denote pricing measure by Q. Then the market value in t of a protection-buyer position equals (for $\tau_k > t$)

$$V_t^{\mathsf{CDS}} = E^Q\Big(\delta p_0(t,\tau_k)1_{\{\tau_k \le T\}} - s \sum_{n=n(t)}^{N} \Big\{(z_n - z_{n-1})p_0(t,z_n)1_{\{\tau_k > z_n\}}$$

$$+ (\tau_k - z_{n-1})p_0(t,\tau_k)1_{\{z_{n-1} < \tau_k \le z_n\}}\Big\} \mid \mathcal{F}_t\Big).$$

The corresponding gains process G^{CDS} satisfies $G_0^{\mathsf{CDS}} = 0$,

$$dG_t^{\mathsf{CDS}} = -s(1 - Y_{t,k})dt + \delta dY_{t,k} + dV_t^{\mathsf{CDS}}.$$

Market value and gains process for CDOs

Consider CDO with attachment points $[l, u]$. Market value of protection-seller position given by

$$V_t^{[l,u]} = E^Q\Big(- \int_t^T dL_s^{[l,u]} + s^{[l,u]} \sum_{n=n(t)}^{N} \Big\{ p_0(t, z_n)(z_n - z_{n-1}) N_{z_n}^{[l,u]}$$

$$+ \sum_{k=1}^{m} p_0(t, T_k)(T_k - z_{n-1}) \Delta L_{T_k}^{[l,u]} 1_{\{z_{n-1} < T_k \le z_n\}} \Big\} \mid \mathcal{F}_t \Big)$$

Corresponding gains process $G^{[l,u]}$ satisfy $G_0^{[l,u]} = s^{\mathsf{upf}}(u - l)$,

$$dG_t^{[l,u]} = s^{[l,u]} N_t^{[l,u]} dt - dL_t^{[l,u]} + dV_t^{[l,u]},$$

(again for spread payments as continuous payment stream).

Hedging with an index

In homogeneous portfolio hedge ratios $\theta_{t,k}$ for a CDO-tranche wrt the individual CDS in the portfolio will be identical, $\theta_{t,k} \equiv \theta_t$ for all k.

$\Rightarrow$ A hedging strategy can be implemented by taking a protection-buyer position of size $m\theta_t$ in the index; much easier than running a dynamic portfolio strategy in, say, $m = 125$ single-name CDS.

C2. Risk-minimizing dynamic hedging strategies

Overview. We determine dynamic hedging strategies using one CDS per underlying name as hedging instrument. With spread- and event risk market is incomplete (unless more than one hedging instrument per name available) $\Rightarrow$ use concept of risk minimization.

Risk-minimizing strategies [Föllmer and Sondermann, 1986]. W.l.o.g. let $r \equiv 0$. We seek a representation of the form

$$G_t^{[l,u]} - G_0^{[l,u]} = \sum_{j=1}^{m} \int_0^t \theta_{s,i} dG_{s,i}^{\text{CDS}} + L_t^{\perp}, \ \ 0 \le t \le T, \qquad (6)$$

- $L^{\perp}$ represents the hedge error - such that for each t the remaining risk (conditional error variance) $E_Q\big((L_T^{\perp} - L_t^{\perp})^2 \mid \mathcal{F}_t\big)$ is minimized.

Risk-minimizing strategies

It is well-known that $L^\perp$ in (6) must be orthogonal to the hedging instruments, $\langle L^\perp, G^{\text{CDS}}_{\cdot,i}\rangle_t \equiv 0$, $1 \le i \le m$. Hence $\boldsymbol{\theta}_t = (\theta_{t,1}, \ldots, \theta_{t,m})$ can be determined from the equations

$$d\langle G^{[l,u]}, G^{\text{CDS}}_j\rangle_t = \sum_{i=1}^m \theta_{t,i} d\langle G^{\text{CDS}}_i, G^{\text{CDS}}_j\rangle_t, \quad j = 1, \ldots, m. \tag{7}$$

Strategy $\boldsymbol{\theta}_t$ is computed from (7) in a two-step procedure.

- represent $G^{[l,u]}$ and G^{CDS}_j as stochastic integrals with respect to $N_{t,i} := Y_{t,i} - \int_0^{t\wedge\tau_i} \lambda_i(\Psi_s, \mathbf{Y}_s)ds$ and the compensated jump-measure of Ψ and compute the quadratic covariations in (7)
- Solve the linear system (7).

Qualitative Results

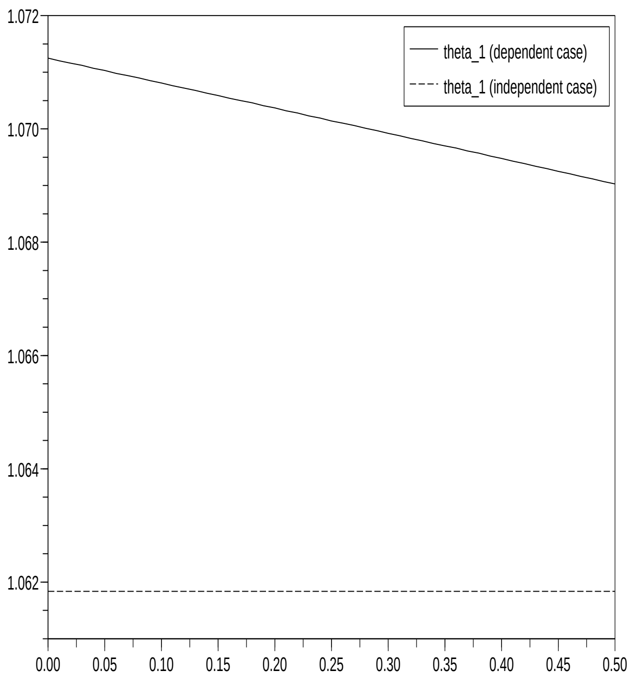
- If Ψ is constant (no spread risk) the market is typically complete ($L^{\perp} \equiv 0$)and the hedging strategy is given by the Δ^{def}.
- With both spread- and event risk (stochastic Ψ) market is incomplete.
- For homogeneous portfolios one can also use the underlying index as single hedging instrument; with inhomogeneous models this is no longer true.
- For low spread volatility $\boldsymbol{\theta}$ is close to Δ^{def}; with high spread volatility spread risk becomes more important and $\boldsymbol{\theta}$ is relatively close to the 'Spread-Delta'.

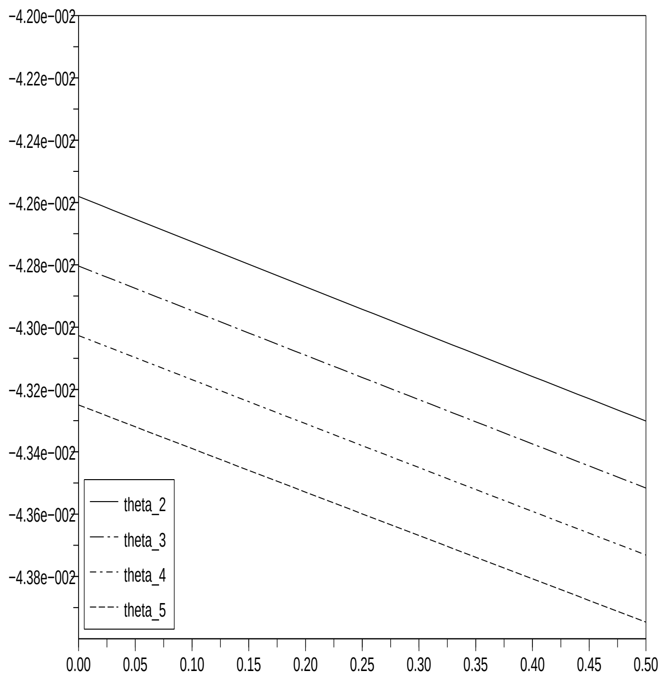
Numerical Examples

We start with examples for deterministic Ψ. Consider a portfolio of five firms; $\lambda(t, \mathbf{Y}_t)$ given by linear counterparty-risk model; hedging instruments zero coupon bonds with maturity $T = 5$ (years).

Example: Survival claim. payoff $H = (1 - Y_{T_0,1})$, $T_0 = 0.5 < T$. Independent defaults ⇒ $\boldsymbol{\theta}_t \equiv (0, {p_1(0,T_0)}/{p_1(0,T)}, 0, \dots, 0)$. Dependent defaults ⇒ short-position in bonds 2, . . . , 5.
Note that with contagion, bonds issued by firms not directly underlying the transaction are needed for hedging.

Numerical Illustration for survival claim





Hedge ratio for survival claim assuming $t < T_1$; left position in underlying risky bond θ_1, right position in other risky bonds, i.e. $\theta_2, \ldots, \theta_5$.

Hedging CDOs with CDSs for stochastic Ψ

	[0,3]-tranche			[3,6]-tranche			[6,9]-tranche		
	θ	Δ^{def}	Δ^{spread}	θ	Δ^{def}	Δ^{spread}	θ	Δ^{def}	Δ^{spread}
S_0^Ψ	0.344	0.344	-	0.138	0.138	-	0.058	0.058	-
S_1^Ψ	0.348	0.345	0.472	0.138	0.138	0.143	0.057	0.058	0.049
S_5^Ψ	0.414	0.366	0.473	0.136	0.134	0.139	0.050	0.053	0.046
S_{10}^Ψ	0.483	0.432	0.507	0.126	0.123	0.129	0.039	0.043	0.038

Note that For low spread volatility $\boldsymbol{\theta}$ is close to the Δ^{def}; with high spread volatility (state space S_3^Ψ) spread risk becomes more important and $\boldsymbol{\theta}$ is relatively close to the 'Spread-Delta'.

Conclusion

- Model-based hedging of credit derivatives possible, but requires new modelling framework.
- Empirical testing needed to assess performance of both approaches in real situations.
- True challenge: appropriate modelling of credit-derivative prices/dynamic evolution of default state
  - ⋆ Rich dynamics of credit spreads
  - ⋆ Realistic dependence structure
  - ⋆ Tractability

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