AN INFORMATION-BASED APPROACH TO CREDIT-RISK MODELLING

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Agenda

- Credit Risk
- The Information-based Approach
- Defaultable Discount Bond Dynamics
- Derivatives and Coupon Bond
- Considerations on the Model
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Credit Risk

Definition

In financial markets **credit risk** is the risk associated to the possibility that a counterparty in a financial contract will not fulfill a contractual commitment to meet her/his obligation stated in the contract.

**EXAMPLES**

PARMALAT

LEHMAN BROTHERS
1. **Problem of modelling**: How is Credit Risk described?
   - Structural Models
   - Intensity Models
   - Information-based Models

2. **Problem of valuating**: Given the model, how is a financial contract valuated?
   - Zero-Coupon Bond
   - Coupon Bond
   - Options
   - Credit Default Swap
   - …
Credit Risk

Basic Assumptions

1. Default-free interest rate system is deterministic.

$$\{P_{tT}\}_{0 \leq t \leq T < \infty}$$

2. Financial market is modelled through the specification of a probability space (the probability measure $Q$ is the risk-neutral measure).

$$\{\Omega, F, Q\}$$

3. All processes are adapted to the market filtration.

$$\{F_t\}_{0 \leq t < \infty}$$

The existence of a unique risk-neutral measure is ensured, even if the market may be incomplete.
Credit Risk
General settings (1/2)

Under these hypothesis, if $H_T$ represents a cash-flow at time $T > 0$, then its value $H_t$ at time $t < T$ is given by:

$$H_t = P_{tT}E[H_T | F_t]$$

**EXAMPLE:** Binary bond.

- $Q(H_T=h_1)=p_1$ (no default)
- $Q(H_T=h_0)=p_0=1-p_1$ (default)
Credit Risk
General settings (2/2)

A defaultable bond is a financial contract that, at a pre-specified instant of time (maturity), delivers to the owner a certain amount of money, if the default never occurs.

• The random variable $H_T$ represents the final value of the defaultable bond.
• $H_T$ takes value $h_i$ with a priori probability $p_i$ ($i=1,\ldots,n$): $Q(H_T=h_i)=p_i$.
• At time $t$, the price $B_{tT}$ of a defaultable bond with maturity $T>0$, is given by:

$$B_{tT} = P_{tT} E[H_T | F_t]$$

The purpose is to obtain the bond price process:

$$\{B_{tT}\}_{0\leq t \leq T}$$
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The Information-based Approach

The information-process (1/2)

There exist an $F_t$-adapted process accessible to market agents, modelling the flow of information concerning future cash-flow of the defaultable bond:

\[ \xi_t = \sigma H_T t + \beta_{tT} \]

- $\sigma$ is a constant (information parameter).
- $H_T$ is an $F_T$-measurable random variable.
- $\beta_{tT}$ is a standard Brownian bridge on $[0, T]$ independent from $H_T$ (it is $F_T$-measurable).

**Theorem:** $\xi_t$ satisfies the Markov property.
The Information-based Approach

The information-process (2/2)

\[ \xi_t = \sigma H_T t + \beta_{tT} \]

$t$ in $(0, T)$: news, rumors, stories and speculation are mixed together, building the information about $H_T$ arriving on the market.

$t = 0$: all the information is in the \textit{a priori} probability distributions

$t = T$: the moment of truth.
The Information-based Approach

Bond Price Process

Simplifying assumption: the subalgebra generated by the information process $\xi_t$ is the market filtration:

$$\{F^\xi_t\} = \{F_t\}$$

$$B_{tT} = P_{tT}H_{tT} = P_{tT}E\left[H_T | F^\xi_t\right] = P_{tT}E\left[H_T | \xi_t\right]$$
The Information-based Approach

Bayes formula

\[
E \left[ H_T | \xi_t \right] = \sum_{i=0}^{n} h_i \pi_{it}
\]

\[
\pi_{it} = Q(H_T = h_i | \xi_t) = \frac{p_i \rho(\xi_t | H_T = h_i)}{\sum_i p_i \rho(\xi_t | H_T = h_i)}
\]

\[
Q(\xi_t \leq x | H_T = h_i) = \int_{-\infty}^{x} \rho(\xi | H_T = h_i) d\xi
\]

\[
\rho(\xi | H_T = h_i) = \frac{1}{\sqrt{2\pi t(T - t)/T}} \exp \left( -\frac{1}{2} \frac{(\xi - \sigma h_i t)^2}{t(T - t)/T} \right)
\]
The Information-based Approach

Bond price process

\[ H_{tT} = \frac{\sum_i p_i h_i \exp \left[ \frac{T}{T-t} (\sigma h_i \xi_t - \frac{1}{2} \sigma^2 h_i^2 t) \right]}{\sum_i p_i \exp \left[ \frac{T}{T-t} (\sigma h_i \xi_t - \frac{1}{2} \sigma^2 h_i^2 t) \right]} \]

Next step: obtain the defaultable bond dynamics

\[ dB_{tT} = ? \]
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Defaultable Discount Bond Dynamics

The Brownian motion

\[ W_t = \xi_t + \int_0^t \frac{1}{T-s} \xi_s ds - \sigma T \int_0^t \frac{1}{T-s} H_{sT} ds \]

**Theorem:** \( W_t \) is an \( F_t \)-Brownian motion.

The conditional probability:

\[ d\pi_{it} = \frac{\sigma T}{T-t}(h_i - H_{tT})\pi_{it} dW_t \]
**Defaultable Discount Bond Dynamics**

**Dynamics**

**Bond price dynamics:**

\[ dB_{tT} = r_t B_{tT} dt + \Sigma_{tT} dW_t \]

- **The short rate:**
  \[ r_t = -\partial \ln P_{0t} / \partial t \]

- **Absolute bond volatility:**
  \[ \Sigma_{tT} = \frac{\sigma T}{T - t} P_{tT} V_{tT} \]

- **Conditional variance:**
  \[ V_{tT} = \sum_i (h_i - H_{tT})^2 \pi_{it} \]
Defaultable Discount Bond Dynamics

Simulations of a digital bond.

\( \sigma = 35\% \)

\( \sigma = 55\% \)

\( \sigma = 75\% \)

\( \sigma = 95\% \)
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Derivatives and Coupon Bond

European call option (1/3)

An European call option is a financial contract that gives the owner the right to buy a pre-specified asset (the underlying) at a pre-specified price (the strike price) at a given instant of time.

\[ C_0 = P_{0t} E[(B_{tT} - K)^+] \]

- \( T \) is the maturity of the defaultable bond.
- \( t \) is the maturity of the option.
- \( K \) is the strike price.
Derivatives and Coupon Bond

European call option (2/3)

\[
C_0 = P_{0t}E \left[ (B_{tT} - K)^+ \right] = \\
= P_{0t}E \left[ \frac{1}{\Phi_t} \left( \sum_{i=0}^{n} \left( P_{tT} h_i - K \right) p_{it} \right)^+ \right]
\]

\[
p_{it} = p_i \exp \left[ \frac{T}{t(T-t)} \left( \sigma h_i \xi_t - \frac{1}{2} \sigma^2 h_i^2 t \right) \right]
\]

\[
\Phi_t = \sum_{i=0}^{n} p_{it}
\]
Derivatives and Coupon Bond

European call option (3/3)

Change of measure by using factor $\Phi_t$: from measure $Q$ to measure $B$ (the bridge measure).

Binary case ($i=1$):

$$C_0 = P_{0t} E^B \left[ \left( \sum_{i=0}^{n} (P_{iT} h_i - K) p_{it} \right)^+ \right]$$

$$C_0 = P_{0t} \left[ p_1 (P_{iT} h_1 - K) N(d^+) - p_0 (K - P_{iT} h_0) N(d^-) \right]$$

$$d^\pm = \ln \left[ \frac{(P_{iT} h_1 - K) p_1}{(K - P_{iT} h_0) p_0} \right] \pm \frac{1}{2} \sigma^2 (h_1^2 - h_0^2) \tau$$

$$\sigma \sqrt{\tau} (h_1 - h_0)$$
Derivatives and Coupon Bond

Numerical results

Call option: $C_0 = f(B_0)$

Put option: $P_0 = f(B_0)$

Call option: $\Delta = \frac{\partial C_0}{\partial B_0}$

Call option: Vega $= \frac{\partial C_0}{\partial \sigma}$
Derivatives and Coupon Bond

The X-factor Approach

Modeling more complex situations: how to describe multiple cash-flow?

Idea: if we have $n$ cash-flows, each at time $T_i$, we can build $n$ information processes $\xi^{(i)}$, $i=1,\ldots,n$, describing the information regarding the corresponding cash-flows.

\[ T_1, T_2, \ldots, T_n, \; n \in \mathbb{N}, \; X_{T_i} \sim Be(p_1^{(i)}), \; X_{T_i} \perp X_{T_j}, \; i \neq j \]

\[ \xi^{(i)} = \sigma_i X_{T_i} t + \beta_{iT_i}^{(i)}, \; t \in [0, T_i], \; \beta_{iT_i}^{(i)} \perp \beta_{iT_j}^{(j)}, \; i \neq j \]

\[
E \left[ X_{T_i} | \xi^{(i)}_t \right] = \frac{p_1^{(i)} \exp \left[ \frac{T_i}{T_i-t} \left( \sigma_i \xi^{(i)}_t - \frac{1}{2} \sigma_i^2 t \right) \right]}{p_0^{(i)} + p_1^{(i)} \exp \left[ \frac{T_i}{T_i-t} \left( \sigma_i \xi^{(i)}_t - \frac{1}{2} \sigma_i^2 t \right) \right]}
\]
A Credit Default Swap (CDS) is a credit derivative between two counterparties, whereby one makes periodic payments \( g \) to the other and receives the promise of a payoff \( h \) if a third party defaults. The former party receives credit protection and is said to be the buyer while the other party provides credit protection and is said to be the seller. The third party is known as the reference entity. It often happen that the coupon \( g \) and the payoff \( h \) are chosen in such way the value \( V_t \) of the CDS at time \( t=0 \) is \( V_0 = 0 \).

\[
V_t = g \left[ \sum_{i=1}^{n} \tilde{P}_{tT_i} \left( \prod_{j=1}^{i} X_{tT_j} \right) \right] - h \left[ \sum_{i=1}^{n} \tilde{P}_{tT_i} (1 - X_{tT_i}) \left( \prod_{j=0}^{i} X_{tT_j} \right) \right]
\]

\[
\tilde{P}_{tT_i} = \begin{cases} 
P_{tT_i} & \text{if } 0 \leq t \leq T_i \\
0 & \text{if } t > T_i
\end{cases}
\]

\[
X_{tT_j} = \begin{cases} 
E\left[ X_{T_j} | \xi_t^{(j)} \right] & \text{if } 0 \leq t < T_j \\
X_{T_j} & \text{if } t \geq T_j
\end{cases}
\]

\(^(*)\) In the first formula \( X_{tT0} = 1 \) for convenience
Derivatives and Coupon Bond

Coupon Bond

A Coupon Bond is a contract between a buyer and a seller in which at time \( t=0 \) the buyer gives to the seller \( p \) euro (principal). The seller will pay to the buyer at some pre-specified dates \( T_1, \ldots, T_n \) a pre-specified amount of money (coupon) \( c_i, \ i=1, \ldots, n \), and at time \( T_n \) the seller will pay even the principal \( p \).

\[
S_t = \sum_{i=1}^{n} c_i \tilde{P}_{tT_i} \left( \prod_{j=1}^{i} X_{tT_j} \right), \quad t \in [0, T_n]
\]

\[
\tilde{P}_{tT_i} = \begin{cases} 
P_{tT_i} & \text{if } 0 \leq t \leq T_i \\
0 & \text{if } t > T_i 
\end{cases}
\]

\[
X_{tT_j} = \begin{cases} 
E \left[ X_{T_j} | \xi_t^{(j)} \right] & \text{if } 0 \leq t < T_j \\
X_{T_j} & \text{if } t \geq T_j 
\end{cases}
\]
Simulation of the dynamics of a 5-years CDS. Earnings are positive for the seller of protection.

Simulation of the dynamics of a 5-years Coupon Bond.
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Consideration on the Model

Further development

- **Stochastic default-free interest rate system**
  \[ dP_{tT} = P_{tT} \left[ (r_t + b_1^2(t, T) + b_2^2(t, T)) dt + b(t, T) d\tilde{W}_t \right] \]

- **Final cash-flow** \((H_T)\) dependent from the “noise”
  \[ H_T = f(\theta, \beta.T) \]

- **Generalized noise process**
  \[ \xi_t = g(H_T, b.T) \]
Consideration on the Model

Conclusion

• A new class of models for Credit-risk has been analyzed.
• Central role of the information arriving on the market.
• It is possible to obtain bond price process (relating the \( a\ priori \) probability with the \( a\ posteriori \)).
• Explicit formula for bond price dynamics.
• Possibility of pricing derivatives (vanilla options, CDS, …).
Bibliography

THANK YOU VERY MUCH!

Grazie mille!