Strong Invariance Principle for Randomly Stopped Stochastic Processes

Andrii Andrusiv, Nadiia Zinchenko

Introduction

Let $(X_i : i \geq 1)$ be sequence of non-negative i.i.d.r.v. with d.f. F and ch.f. φ, $EX_i = m < \infty$

Denote

$$S(n) = \sum_{i=1}^{n} X_i, \ S(0) = 0, \ S(z) = S([z]),$$

where $[a]$ is entire of a>0.

Let $(Z_i : i \geq 1)$ be sequence of non-negative i.i.d.r.v. independent of $(X_i : i \geq 1)$ with d.f. F_1 and ch.f. φ_1, $EZ_i = 1/\lambda < \infty$

Denote

$$Z(n) = \sum_{i=1}^{n} Z_i, \ Z(0) = 0, \ Z(a) = Z([a])$$

and define the renewal counting process as

$$N(t) = \inf\{x \geq 0 : Z(x) > t\}$$

Main aim

The main aim of this talk is to study the asymptotic behavior of the random processes $S(N(t))$ and $N(t)$ when F and F_1 are heavy tailed. This problem has a deep relation with investigations of risk process $U(t)$ and approximation of ruin probabilities in Sparre Anderssen collective risk model

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k$$

$$U(t) = u + \sum_{k=1}^{\tilde{N}(t)} V_k - \sum_{i=1}^{N(t)} X_i$$

Weak invariance principle

Limit theorems for risk process such as (weak) invariance principle which constitute the weak convergence of $U(t)$ to the Wiener process $W(t)$ with drift (when $EX_i^2 < \infty$, $EZ_i^2 < \infty$) or to α-stable Lévy process $Y_\alpha(t)$ (when $EX_i^2 = \infty$, $EZ_i^2 < \infty$) lead to useful approximation of the ruin probability as a distribution of infimum of the Wiener process (Iglehard (1969), Grandell (1991), Embrechts, Klüppelberg and Mikosch (1997)) or infimum of the corresponding α-stable process (Furrer, Michna and Weron (1997), Furrer (1998)).

Strong invariance principle

Strong invariance principle (almost sure approximation) is a general name for the class of limit theorems which ensure the possibility to construct $(X_i : i \geq 1)$ and Lévy process $Y(t)$, $t \geq 0$ on the same probability space in such a way that with probability 1

$$|S(t) - mt - Y(t)| = o(r(t)) \quad \text{as} \quad t \to \infty$$

$$|S(t) - mt - Y(t)| = O(r(t)) \quad \text{as} \quad t \to \infty$$

were approximation error (rate) $r(\cdot)$ is non-random function depending only on assumption posed on X_i.

Strong invariance principle for partial sums

Based on Skorokhod embeded scheme Strassen (1964) proved the first variant of the strong invariance principle. In 1970-95 the further investigations were carried out by a number of authors, so firstly I will summarize their results.

Th.A1. It is possible to construct partial sum process $S(t),\ t \geq 0$ and a standard Wiener process $W(t), t \geq 0$ in such a way that a.s.

$$|S(t) - mt - W(t)| = o(r(t)),$$

with:

(i) $r(t) = t^{1/p}$ iff $E|X_i|^p < \infty, p > 2$,

(ii) $r(t) = (t \log\log(t))^{1/2}$ iff $E|X_i|^2 < \infty$,

(iii) $o(r(t))$ can be changed on $O(r(t)) = O(\log t)$ iff $Ee^{uX_i} < \infty$

for some $u > 0$

Domain of attraction of stable law

Suppose that $EX_i^2 = \infty$; more precisely we assume that $(X_i : i \geq 1)$ belongs to the domain of attraction of the stable law $G_{\alpha,\beta}$.

Here $G_{\alpha,\beta}$ is a d.f. of the stable law with parameters $1 < \alpha \leq 2$, $|\beta| \leq 1$ and ch.f. $g_{\alpha,\beta}(u) = \exp(-K(u))$

where $K(u) = K_{\alpha,\beta}(u) = |u|^{\alpha} (1 - i\beta \text{sign}(u) \tan(\pi\alpha/2))$.

$(X_i : i \geq 1) \in DNA(G_{\alpha,\beta})$ if for normalized and centered sums S_n^* there is a weak convergence

$$S_n^* = n^{-1/\alpha}(S(n) - mn) \Rightarrow G_{\alpha,\beta}$$

Domain of attraction of stable law

Denote by $Y(t) = Y_\alpha(t) = Y_{\alpha,\beta}(t)$, $t \geq 0$, the α-stable Lévy process with ch.f.

$$g_\alpha(t;u) = g_{\alpha,\beta}(t;u) = \exp(-tK_{\alpha,\beta}(u))$$

We omit index β if it is not essential.

The fact that $(X_i : i \geq 1) \in DNA(G_{\alpha,\beta})$ is not enough to obtain "good" error term above, thus, certain additional assumptions are needed. We formulate them in terms of ch.f.

Additional assumption

Assumption (C): there are $a_1 > 0$, $a_2 > 0$ and $l > \alpha$ such that for $|u| < a_1$

$$| f(u) - g_{\alpha,\beta}(u) | < a_2 |u|^l$$

where $f(u) = e^{-ium}\varphi(u)$ is a ch.f. of $(X_i - EX_i)$.

Put $A = [\max\{\alpha(\alpha+1), 2\alpha(2\alpha+1)/(l-\alpha)\}+1]$

Strong invariance principle for partial sums

Th.A2. (Zinchenko) For $1 < \alpha < 2$ and under assumption **(C)** it is possible to construct α-stable process $Y_{\alpha,\beta}(t)$, $t \geq 0$ such that a.s.

$$\sup_{0 \leq t \leq T} |S(t) - mt - Y_{\alpha,\beta}(t)| = o(T^{1/\alpha - \rho}),$$

for any $\rho = \rho(\alpha, l) \in (0, 1/4\alpha(A+1))$

Counting renewal processes

Order of magnitude of $N(t)$ is described by following theorem which includes strong law of large numbers (SLLN), Marcinkiewich-Zygmund SLLN and law of iterated logarithm for renewal process.

Counting renewal processes

Th.A3.

(i) If $0 < EZ_i = 1/\lambda < \infty$, then a.s.

$$N(t)/t \to \lambda$$

(ii) if $E|Z_i|^p < \infty$ for some $p \in (1, 2)$ then a.s.

$$t^{-1/p}(N(t) - \lambda t) \to 0$$

(iii) if $Var(Z_i) = \tau^2 < \infty$ then

$$\limsup_{t\to\infty}(2t \log\log t)^{-1/2} |N(t) - \lambda t| = \tau\lambda^{3/2}$$

while for the moments we have

$$EN(t) \sim \lambda t, \quad Var(N(t)) \sim \tau\lambda^{3/2}$$

α-stable Lévy process

L.A1. If $Y_\alpha(t)$ is an α-stable Lévy process with $0 < \alpha < 2$, then a.s. $\forall \varepsilon > 0$

$$Y_\alpha(t) = o(t^{1/\alpha + \varepsilon})$$

Keeping in mind these and equivalence in weak convergence for $Z(n)$ and associated $N(t)$ it is natural to ask about a.s. approximation of $N(t)$.

Strong invariance principle for counting renewal processes

Under assumptions $EZ^2 < \infty$ and $0 < EZ_i = 1/\lambda < \infty$ strong approximation of the counting process $N(t)$ associated with partial sum process

$$Z(x) = \sum_{i=1}^{[x]} Z_i$$

was investigated by a number of author. For instance, Csörgő, Horváth and Steinebach (1986) obtained that for non-negative r.v. Z_i the same error function $r(t)$ (see T.A1) provide a.s. approximation

$$|\lambda t - N(t) - \lambda W(\lambda t)| = o(r(t)) \vee O(r(t))$$

Strong invariance principle for counting renewal processes

Let consider the case $\{Z_i\} \in NDA(G_{\alpha,\beta})$ with $1 < \alpha < 2$ and $0 < EZ_i = 1/\lambda < \infty$

Th.1. Let Z_i satisfy (**C**) with $1 < \alpha < 2$ and $0 < EZ_i = 1/\lambda < \infty$ then a.s.

$$|t\lambda - N(t) - \lambda^{1+1/\alpha} Y_{\alpha,\beta}(t)| = o(r(t))$$

where $r(t)$ is any upper function for Lévy process.

Strong invariance principle

Let recall

$$D(t) = S(N(t))$$

Strong invariance principle for $D(t)$ was studied by Csörgő, Horváth, Steinbach, Deheuvels and other authors.

In the following we focus on the case $E|X_i|^2 = \infty$ when $(X_i : i \geq 1)$ belong to $DNA(G_{\alpha_1,\beta})$, $1 < \alpha_1 < 2$ while $(Z_i : i \geq 1)$ can be attracted to the normal law ($\alpha = 2$, $Var(Z_i) = \tau^2 < \infty$) or to the α_2-stable law, $1 < \alpha_2 < 2$

Our approach is close to the methods presented in Csörgő and Horváth (1993).

Strong invariance principle

Th.2.(Zinchenko) Let $(X_i : i \geq 1)$ satisfy (**C**) with $1 < \alpha < 2$ and $EZ_i^2 < \infty$. Then a.s.

$$|D(t) - m\lambda t - Y_{\alpha,\beta}(\lambda t)| = o(t^{1/\alpha - \rho_1}),\ \rho_1 \in (0, \rho_0)$$

for some $\rho_0 = \rho_0(\alpha, l)$

In this case $D(t)$ can be interpreted as total claims until moment t in classic risk model.

Developing such approach we proved rather general result concerning a.s. approximation of the randomly stopped process (not obligatory connected with the partial sum processes).

Strong invariance principle

Let $Z^*(t)$, $S^*(t)$ be two real-valued positive increasing càdlàg random processes,

$N^*(t)$ – the inverse of $Z^*(t)$ is defined by

$$N^*(t) = \inf\{t > 0 : Z^*(x) > t\}, \quad 0 \le t < \infty$$

Strong invariance principle

Th.3. Suppose that for some constants m, $a > 0$, $\sigma > 0$ and functions $r(t)$, $q(t)$ meet the conditions

$$r(t) \uparrow \infty, \quad \frac{r(t)}{t} \downarrow 0, \quad q(t) \uparrow \infty, \quad \frac{q(t)}{t} \downarrow 0 \quad \text{as } t \to \infty$$

$$\sup_{0 \le t \le T} |\sigma^{-1}(Z^*(t) - at) - W_1(t)| = O(r(T))$$

$$\sup_{0 \le t \le T} |S^*(t) - mt - Y_\alpha(t)| = O(q(T))$$

where $W_1(t)$ is a Wiener process and $Y_\alpha(t)$ being α-stable Lévy process, independent of $W_1(t)$,

Then $\forall \varepsilon > 0$

$$\left| S^*(N^*(t)) - \frac{mt}{a} - \left(Y_\alpha\left(\frac{t}{a}\right) - \frac{m\sigma}{a} W_2\left(\frac{t}{a}\right)\right)\right| =$$

$$= O((t \log\log t)^{1/(2\alpha - \varepsilon)} + q(t) + r(t)(\log t)^{1/2})$$

Strong invariance principle

Th.4. Let $(X_i : i \geq 1)$ satisfy (**C**) with $1 < \alpha_1 < 2$ and $(Z_i : i \geq 1)$ satisfy (**C**) with $1 < \alpha_2 < 2$, $\alpha_1 < \alpha_2$. Then a.s.

$$\left| S(N(t)) - m\lambda t - Y_{\alpha_1,\beta}(\lambda t) \right| = o(t^{1/\alpha_1 - \rho_2})$$

for some $\rho_2 = \rho_2(\alpha_1, l)$

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Thank you for attention!