Part II: Topics in fractional Brownian motion

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- On stochastic integrals with respect to fBm.
- Change of variables formula.
- Mixed Brownian - fractional Brownian models.
- Arbitrage.
On stochastics integrals with respect to fBm

For the rest of the lectures, the Hust index $H$ is assumed to be $H > \frac{1}{2}$.

We have shown that fractional Brownian motion is not a semimartingale. We will have at least the following problems:

- The methods of classical stochastic analysis are not directly applicable.
- The change of limit and integral must be carefully justified.

Fractional Brownian motion is a Gaussian process, so one can apply the isometry of the covariance kernel to define Wiener integrals, or one can use the Malliavin approach to develop stochastic integration theory with respect to fractional Brownian motion; and there are many other possibilities.
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We will use later so-called forward integrals. These are pathwise integrals, and they are applied to the case when the integrator is a sum of standard Brownian motion and fractional Brownian motion, for example. When integrating with respect to fractional Brownian motion, a useful integral is based on an extension of Riemann-Stieltjes integrals. It turns out that so-called fractional Besov spaces are useful here. We recall with some definitions.
On stochastics integrals with respect to fBm
Fractional Besov spaces

Fix $0 < \beta < 1$.

(i) Let $W_1^\beta = W_1^\beta[0, T]$ be the space of real-valued measurable functions $f : [0, T] \rightarrow \mathbb{R}$ such that

$$
\|f\|_{1,\beta} := \sup_{0 \leq s < t \leq T} \left( \frac{|f(t) - f(s)|}{(t - s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u - s)^{1+\beta}} du \right) < \infty.
$$

(ii) Let $W_2^\beta = W_2^\beta[0, T]$ be the space of real-valued measurable functions $f : [0, T] \rightarrow \mathbb{R}$ such that

$$
\|f\|_{2,\beta} := \int_0^T \frac{|f(s)|}{s^\beta} ds + \int_0^T \int_0^s \frac{|f(u) - f(s)|}{(u - s)^{1+\beta}} duds < \infty.
$$
On stochastics integrals with respect to fBm
Fractional Besov spaces

The Besov-spaces are closely related to the spaces of Hölder continuous functions. More precisely, for any $0 < \epsilon < \beta \wedge (1 - \beta)$,

$$C^{\beta+\epsilon}[0, T] \subset \mathcal{W}_1^\beta[0, T] \subset C^{\beta-\epsilon}[0, T] \quad \text{and} \quad C^{\beta+\epsilon}[0, T] \subset \mathcal{W}_2^\beta[0, T].$$

where $C^\gamma[0, T]$ stands for Hölder continuous functions of order $\gamma$. 
Recall that the trajectories of $B^H$ for a.s. $\omega \in \Omega$, any $T > 0$ and any $0 < \gamma < H$ belong to $C^\gamma[0, T]$. This follows from the Kolmogorov continuity theorem. We obtain that the trajectories of $B^H$ for a.s. $\omega \in \Omega$, any $T > 0$ and any $0 < \beta < H$ belong to $W^\beta_1[0, T]$.

Recall the left-sided Riemann-Liouville fractional integral operator $I^\beta_+$ of order $\beta > 0$:

$$(I^\beta_0+ f)(s) = \frac{1}{\Gamma(\beta)} \int_0^s f(u)(s-u)^{\beta-1} du.$$
The corresponding right-sided fractional integral operator $I^\beta_-$ was already defined by

$$(I^\beta_{t-}f)(s) = \frac{1}{\Gamma(\beta)} \int_s^t f(u)(u-s)^{\beta-1} \, du.$$
Let $f : [0, T] \to \mathbb{R}$ and $0 < \beta < 1$. If $f \in I^\beta_+(L_1[0, T])$ (resp. $f \in I^\beta_-(L_\infty[0, T])$) then the Weyl fractional derivatives are defined by

$$
(D^\beta_{0+} f)(x) = \frac{1}{\Gamma(1 - \beta)} \left( \frac{f(x)}{x^\beta} + \beta \int_0^x \frac{f(x) - f(y)}{(x - y)^{\beta+1}} \, dy \right) 1_{(0, T)}(x),
$$

resp.

$$
(D^\beta_{T-} f)(x) = \frac{1}{\Gamma(1 - \beta)} \left( \frac{f(x)}{(T - x)^\beta} + \beta \int_x^T \frac{f(x) - f(y)}{(y - x)^{\beta+1}} \, dy \right) 1_{(0, T)}(x).
$$
On stochastics integrals with respect to fBm

The following proposition clarifies the construction of the stochastic integrals. This approach is by Nualart and Răscanu (2002), and initiated by M. Zähle (1998).

**Proposition** Let \( f \in W_2^\beta [0, T], g \in W_1^{1-\beta} [0, T] \). Then for any \( t \in (0, T] \) the Lebesgue integral

\[
\int_0^t (D_{0+}^\beta f)(x)(D_{t-}^{1-\beta} g_t)(x)dx
\]

exists, and we can define the **generalized Lebesgue-Stieltjes integral** by

\[
\int_0^t f dg := \int_0^t (D_{0+}^\beta f)(x)(D_{t-}^{1-\beta} g_t)(x)dx.
\]
On stochastics integrals with respect to fBm

The next theorem can be used to study the continuity of the integral.

**Theorem** [Nualart and Răşcanu] Let $f \in W_2^\beta[0, T]$ and $g \in W_1^{1-\beta}[0, T]$. Then we have the estimation

$$|\int_0^t f dg| \leq \frac{1}{\Gamma(\beta)} \|f\|_{2,\beta} \|g\|_{1,1-\beta}.$$  \hspace{1cm} (1)

**Corollary** Let $f, f^n \in W_2^\beta[0, T]$, $\|f^n - f\|_{2,\beta} \to 0$ as $n \to \infty$, and $g \in W_1^{1-\beta}[0, T]$. Then

$$\int f^n dg \to \int f dg.$$
Convex functions

We will prove a change of variables formula for convex functions. Therefore, we recall some results on convex functions. First, recall that every convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ has a left-derivative $f'_-$ and a right-derivative $f'_+$. The next theorem gives information about the left-derivative $f'_-$ and right-derivative $f'_+$.

**Theorem** The functions $f'_-$ and $f'_+$ are increasing, respectively left and right-continuous and the set $\{x : f'_-(x) \neq f'_+(x)\}$ is at most countable.
**Proposition**  The second derivative $f''$ of convex function $f$ exists in the sense of distributions, and it is a positive Radon measure; conversely, for any Radon measure $\mu$ on $\mathbb{R}$, there is a convex function $f$ such that $f'' = \mu$ and for any interval $I$ and $x \in \text{int}(I)$ we have the equality

$$f'_-(x) = \frac{1}{2} \int_I \text{sgn}(x - a) \mu(da) + \alpha_I,$$

where $\alpha_I$ is a constant and $\text{sgn} x = 1$ if $x > 0$ and $-1$ if $x \leq 0$. 
Convex functions

If the supp$(\mu)$ is compact, then one can globally state that

$$f'_-(x) = \frac{1}{2} \int \text{sgn}(x - a)\mu(da)$$

up to a constant term.
Let $S_t = \exp\{B^H_t\}$ be a geometric fractional Brownian motion, $H \in (\frac{1}{2}, 1)$, $t \in [0, T]$ and $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then the stochastic integral
\[
\int_0^T f'(S_t)S_t dB^H_t
\] (4)
can be understood in the sense of the generalized Lebesgue-Stieltjes integral a.s. $\omega \in \Omega$. 

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Recall that the second derivative of $f$ is a Radon measure $\mu$. We prove the result in the case that $\mathcal{K} := \text{supp}(\mu)$ is a compact set. We will show that for $\beta \in (1 - H, \frac{1}{2})$ we have that

$$\|f'(S_t)S_t\|_{2,\beta} < \infty \quad a.s.$$

Recall the definition of the fractional Besov norm:

$$\|f\|_{2,\beta} := \int_0^T \frac{|f(s)|}{s^\beta} ds + \int_0^T \int_0^s \frac{|f(u) - f(s)|}{(u - s)^{1+\beta}} duds < \infty.$$
$f'_-(S_t)$ is integrable

On the proof

Obviously,

$$\int_0^T \frac{|f'_-(S_t)||S_t|}{t^\beta} dt \leq \max_{t \in [0,T]} (S_t)|f'_-(\max_{t \in [0,T]} S_t)| \int_0^T \frac{1}{t^\beta} dt < \infty \quad \text{a.s.}$$

For the second term in the definition of fractional Besov norm we must work harder.
\(f'_{-}(S_t)\) is integrable

On the proof

By the triangular inequality,

\[
\int_0^T \int_0^t |f'_{-}(S_t)S_t - f'_{-}(S_s)S_s| \frac{1}{(t-s)^{\beta+1}} \, ds \, dt \leq I_1 + I_2,
\]

where,

\[
I_1 = \int_0^T \int_0^t |f'_{-}(S_t)||S_t - S_s| \frac{1}{(t-s)^{\beta+1}} \, ds \, dt,
\]

\[
I_2 = \int_0^T \int_0^t |S_s||f'_{-}(S_t) - f'_{-}(S_s)| \frac{1}{(t-s)^{\beta+1}} \, ds \, dt.
\]
Using the Hölder continuity property of geometric fractional Brownian motion trajectories one can bound from above $I_1$ as

$$|I_1| \leq |f'_-(\max_{t \in [0,T]} S_t)|C(\omega) \int_0^T \int_0^t \frac{(t - s)^{H-\delta}}{(t - s)^{\beta+1}} dsdt < \infty \quad \text{a.s.,}$$

where $\delta \in (0, H - \beta)$ and $C$ is a almost surely finite random variable such that

$$|S_t - S_s| \leq C(\omega)|t - s|^{H-\delta}.$$
$f'(S_t)$ is integrable

On the proof

We use the representation (3) to show $l_2$ is finite almost surely.

$$|l_2| \leq \max_{t \in [0,T]} (S_t) \int_0^T \int_0^t \frac{|f'(S_t) - f'(S_s)|}{(t-s)^{\beta+1}} ds dt \leq l_{2,1} + l_{2,2},$$

where,

$$
\begin{cases}
    l_{2,1} = \max_{t \in [0,T]} (S_t) \int_0^T \int_0^t \int_K \frac{1_{\{S_s<a<S_t\}}}{(t-s)^{\beta+1}} \mu(da) ds dt, \\
    l_{2,2} = \max_{t \in [0,T]} (S_t) \int_0^T \int_0^t \int_K \frac{1_{\{S_t<a<S_s\}}}{(t-s)^{\beta+1}} \mu(da) ds dt.
\end{cases}
$$
By Tonelli’s theorem we have

\[
E \int_0^T \int_0^t \int_\mathcal{K} \frac{1_{\{S_s < a < S_t\}}}{(t - s)^{\beta + 1}} \mu(da) ds dt = \int_\mathcal{K} E \left( \int_0^T \int_0^t \frac{1_{\{S_s < a < S_t\}}}{(t - s)^{\beta + 1}} ds dt \right) \mu(da)
\]

\[
\leq \int_\mathcal{K} M \mu(da) = M \mu(\mathcal{K}) < \infty,
\]

since \(\mu\) is a Radon measure and the upper bound \(M\) is independent of \(a\).
$f'(S_t)$ is integrable

End of the proof

This implies

$$
\int_0^T \int_0^t \int_K \frac{1_{\{S_s < a < S_t\}}}{(t-s)^{\beta+1}} \mu(da)dsdt < \infty \quad \text{a.s.}
$$

Therefore $|I_2| < \infty$ a.s., thus the integral (4) exists as a generalized Lebesgue–Stieltjes integral.
One can show that the process $S$ satisfies the following stochastic differential equation:

$$S_t = 1 + \int_0^t S_u dB^B_u;$$

using this we can write (4) as

$$\int_0^T f_-'(S_t) dS_t.$$
In the same lines one can show the pathwise stochastic integral
\[
\int_0^T f'(S_t) S_t dB_t^H
\]
is well-defined in the sense of the generalized Lebesgue-Stieltjes integral a.s. Moreover, we have
\[
\int_0^T f_-(S_t) S_t dB_t^H = \int_0^T f_+(S_t) S_t dB_t^H.
\]
Next we consider the Itô formula, which is more interesting for us.

**Theorem** Let \( S_t = \exp\{B^H_t \} \) be a geometric fractional Brownian motion with \( H \in (\frac{1}{2}, 1) \), \( t \in [0, T] \) and \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a convex function. Then

\[
 f(S_T) = f(S_0) + \int_0^T f'(S_t) S_t dB^H_t,
\]

where the stochastic integral is understood in the sense of generalized Lebesgue-Stieltjes integral.

We indicate the idea of the proof.
The claim is true for smooth functions, because of the zero quadratic variation of fBm.

We want to show that the equation (3) is valid for convex $f$, where $f'$ is replaced with the left derivative $f'_-$ and the integral is generalized Lebesgue - Stieltjes integral. This is possible.
An example

The above results are true also for fractional Brownian motion $B^H$, when $H > \frac{1}{2}$. If $f : \mathbb{R} \to \mathbb{R}$ is a convex function with left-derivative $f'_-$, then the integral

$$\int_0^t f'_-(B^H_u)dB^H_u$$

exists as a generalized Lebesgue-Stieltjes integral. Moreover, we have the change of variables formula

$$f(B^H_t) - f(0) = \int_0^t f'_-(B^H_u)dB^H_u.$$
Comments of chang of variables formula

Take $f(x) = |x|$ we have the following version of Tanaka’s formula

$$|B^H_T| = \int_0^T \text{sgn}(B^H_u) dB^H_u.$$  \hspace{1cm} (5)

So no local time appears in the formula.
We prove the approximation result for integrals under the assumption that the second derivative of the convex function $f$ has a finite support.

**Theorem**  Assume $T = 1$ and let $f$ be a convex function which positive Borel measure $\mu$ corresponding to its second derivative with finite support $\mathcal{K}$, i.e. $\mu(\mathcal{K}) < \infty$. Let $t_i = \frac{i}{n}; \quad i = 0, 1, \ldots, n$. Then,

$$\sum_{i=0}^{n} f'(S_{t_{i-1}})(S_t - S_{t_{i-1}}) \xrightarrow{\text{a.s.}} \int_0^1 f'(S_t)dS_t.$$
Consider the following problem:

- The process $g$ is an adapted process such that $\int_0^T g_u dB^H_u$ is defined as a Riemann-Stieltjes integral. We know that

\[
(\star) \quad \int_0^T g_u dB^H_u = 0 \text{ Leb } \otimes \mathbb{P} - \text{a.s.}
\]

When one can conclude that $g = 0 \text{ Leb } \otimes \mathbb{P} - \text{a.s.}$?
Uniqueness of the integral representations

- If $g$ is a deterministic process, then this is true.
Uniqueness of the integral representations

- If $g$ is a deterministic process, then this is true.
- Let $g$ be a simple predictable process,

$$g = \sum_{k=1}^{m} \gamma_i 1_{[t_{i-1}, t_i)},$$

where $\gamma_i \in L^2(F_{t_{i-1}}, \mathbb{P})$, $\bar{t} = \{t_j : 0 = t_0 < t_1 < \cdots < t_n = T\}$ is a partition of the interval, and assume that ($\star$) holds. Then one can show that $\gamma_i = 0$, for example using the representation of fractional Brownian motion with respect to standard Brownian motion.
Zero problem

The next is by G. Shevchenko. We construct a bounded adapted process $u_t$ such that $\int_0^t u_s dB^H_s \to 0$ a.s. as $t \to 1-$ and $u$ is a.s. bounded.

Put $b_n = n^3$, $a_n = n^{-2}$, $\Delta_n = n^{-\alpha}/\zeta(\alpha)$, where $\alpha \in (1, 2H)$, $\zeta(\alpha) = \sum_{n \geq 1} n^{-\alpha}$, and define $t_n = \sum_{k=1}^{n-1} \Delta_k$ so that $t_n \to 1-$, $n \to \infty$.

Further, let $\tau_n = \min\{t \geq t_n : \left| B^H_t - B^H_{t_n} \right| \geq a_n \wedge t_{n+1} \}$ and

$$\varphi(t) = \sum_{n=0}^{\infty} b_n (B^H_t - B^H_{t_n}) 1_{[t_n, \tau_n]}(t).$$
Observe that $\varphi(t) \leq b_na_n = n$ on $[t_n, t_{n+1})$.

Denote $v_t = \int_0^t \varphi(s) dB_s^H$. Then

$$v_{tn} = \frac{1}{2} \sum_{k=0}^{n-1} b_k (B_{\tau_k}^H - B_{t_k}^H)^2 \geq \frac{1}{2} \sum_{k=0}^{n-1} b_k a_k^2 1_{A_k} = \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{k} 1_{A_k},$$

where $A_n = \{\sup_{t\in[t_n, t_{n+1}]} |B_t^H - B_{tn}^H| \geq a_n\}$. 
The following fact (a small ball probability estimate) is known. Assume that \( \{X_t, t \in [0, 1]\} \) is a centered Gaussian process with \( X_0 = 0 \), whose increments satisfy
\[
(X(t + h) - X(t))^2 = \sigma^2(h),
\]
where \( \sigma \) is non-decreasing and convex on \([0, 1]\). Then
\[
P \left( \sup_{t \in [0,1]} |X_t| < \sigma(x) \right) \leq 2e^{-0.17/x}.
\]
Zero problem

By the self-similarity property of fractional Brownian motion,

\[ \{B_t^H - B_{t_n}^H, \ t \in [t_n, t_{n+1}]\} \overset{d}{=} \{\Delta_n^{1/H} B_t^H, \ t \in [0, 1]\}. \]

Hence, thanks to the above small ball probability estimate (with \( \sigma(h) = h^H \)), we can write

\[
P(\Omega \setminus A_n) = P\left( \sup_{t \in [0,1]} |B_t^H| < a_n \Delta_n^{-1/H} \right) \leq 2 \exp\{-0.17 (a_n \Delta_n^{-1/H})^{-1/H}\} = 2 \exp\{-0.17 \left( n^2 n^{-\alpha/H} \right)^{1/H}\} = 2 \exp\{-0.17 n^{(2-\alpha/H)}\}.\]
Thus it is clear that $\sum_{n \geq 1} P(\Omega \setminus A_n) < \infty$, so by the Borel–Cantelli lemma for almost all $\omega$ there exists $N(\omega)$ so that $\omega \in A_n$ for $n \geq N(\omega)$.

Hence two conclusions.

First, $\lim_{t \to 1^-} v_t = \infty$ a.s. Indeed, $v_{tn} \to \infty$, $n \to \infty$, but $v_t \geq v_{tn}$ for $t \geq t_n$, whence the assertion follows. Moreover, almost surely

$$v_{tn} \geq \sum_{k:N(\omega) \leq k < n} \frac{1}{k} \geq \log n - K(\omega).$$
Second, the function
\[ u_t = \varphi(t)f'(v_t), \]
where \( f(x) = xe^{-x^2} \), is almost surely bounded. Indeed, it is clear that \( |f'(x)| \leq Ce^{-x} \). Therefore, for \( t \in [t_n, t_{n+1}) \)
\[ |u_t| \leq Cne^{-vt} \leq Cne^{K(\omega)-\log n} = C(\omega). \]

Finally, by the Itô formula,
\[
\int_0^t u_s dB_s^H = \int_0^t \varphi(s)f'(v_s)dB_s^H = \int_0^t f'(v_s)dv_s = f(v_t) \to 0, \ t \to 1.
\]
Transaction costs

We have seen that for a convex function $f$ we have the following:

$$f(S_T) = f(S_0) + \int_0^T f'(S_u) dS_u,$$

where $S$ the geometric fractional Brownian motion

$$S_t = S_0 e^{B^H_t}; \quad t \in [0, 1], \quad \text{and} \quad H > \frac{1}{2}.$$  

The two-assets market model consists of :

(i) Riskless asset (bond), $B_t = 1; \quad t \in [0, 1]$ which corresponds to zero interest rate.

(ii) Risky asset (stock) whose price is modeled by geometric fractional Brownian motion
Transaction costs

Now consider the discretized version of hedging strategy, i.e.

\[
\theta^n_t = \sum_{i=1}^{n} f'_-(S^n_{t_{i-1}}) 1_{(t_{i-1}, t_i^n]}(t); \quad t \in (0, 1].
\]

In the presence of proportional transaction costs, the value of this portfolio at terminal date is

\[
V_1(\theta^n) = f(S_0) + \int_0^1 \theta^n_t dS_t - k \sum_{i=1}^{n} S^n_{t_{i-1}} |f'_-(S^n_{t_i}) - f'_-(S^n_{t_{i-1}})|.
\] (6)
The *occupation measure* related to fractional Brownian motion is defined by

$$
\Gamma_{B^H}(I \times U) = \lambda\{t \in I : B^H_t \in U\} = \int_I 1_{\{B^H_t \in U\}} dt
$$

where $I$ and $U$ are Borel sets on time interval $[0, T]$ and the real line respectively and $\lambda$ stands for Lebesgue measure. It is well-known that the occupation measure has a jointly continuous density (*local time*) which is denoted by $l^H(x, t) := l^H(x, [0, t])$ and is Hölder continuous in $t$ of any order $\alpha < 1 - H$ and in $x$ of any order $\beta < \frac{1-H}{2H}$. 
Transaction costs

The following result is by Azmoodeh (2010):

Theorem[Azmoodeh] Assume the level of transaction costs $k = k_n = k_0 n^{-(1-H)}$, where $k_0 > 0$. Then

$$\mathbb{P} \lim_{n \to \infty} V_1(\theta^n) = f(S_1) - J,$$

where

$$J = J(k_0) := \sqrt{\frac{2}{\pi}} k_0 \int \int_0^1 S_t l^H(\ln a, dt) \mu(da),$$

and the inner integral in the right hand side is understood as limit of Riemann-Stieltjes sums a.s.
Transaction costs
On the proof

We have the identity

\[ V_1(\theta^n) - f(S_1) = l_n^1 - k_0 l_n^2, \]

where

\[
\begin{align*}
    l_n^1 &= \sum_{i=1}^{n} f'_{-}(S_{t_i^n})(S_{t_i^n} - S_{t_{i-1}^n}) - \int_0^1 f'_{-}(S_t) dS_t, \\
    l_n^2 &= \Delta_n^{1-H} \sum_{i=1}^{n} S_{t_{i-1}^n} |f'_{-}(S_{t_i^n}) - f'_{-}(S_{t_{i-1}^n})|. 
\end{align*}
\]
Transaction costs
On the proof

We already know that

\[ I_n^1 = \sum_{i=1}^{\n} f'(S_{t_{i-1}}^n)(S_{t_i}^n - S_{t_{i-1}}^n) - \int_0^1 f'(S_t) dS_t \to 0. \]

The study limiting behaviour of the term \( I_n^2 \) is done as follows. We know that we have

\[ f'_-(x) = \frac{1}{2} \int \text{sgn}(x - a) \mu(da). \]
Transaction costs
On the proof

Assume that supp $\mu = \{a\}$. One can then use the fact that local time $l^H$ can be approximated with the level crossings of polygonal approximations to show that $l_n^2 \to 0$. 
Transaction costs
On the proof

- Assume that supp $\mu = \{a\}$. One can then use the fact that local time $l^H$ can be approximated with the level crossings of polygonal approximations to show that $l^2_n \to 0$.
- The next step is to assume that supp $\mu = \{a_1, \ldots, a_p\}$. 
Assume that $\text{supp} \, \mu = \{a\}$. One can then use the fact that local time $l^H$ can be approximated with the level crossings of polygonal approximations to show that $I_n^2 \to 0$.

The next step is to assume that $\text{supp} \, \mu = \{a_1, \ldots, a_p\}$.

The last step is based on the fact that one can approximate a convex function using convex linear approximations [see the next slide].
Transaction costs

On the proof

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. For each interval $[a, b]$, let

$\pi = \{a = a_0 < a_1 < ... < a_n = b\}$ be a partition of the interval and

$$\|\pi\| := \max_{1 \leq i \leq n} (a_i - a_{i-1}).$$

A piecewise linear function through points $(a_i, f(a_i))$ is called a convex linear approximation of convex function $f$ on the interval $[a, b]$ based on the partition $\pi$. Let $[a, b]$ be a closed interval and $\{\pi_m\}$ be a sequence of partitions of the interval $[a, b]$ such that

$$\|\pi_m\| \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty$$

where $\pi_m = \{a_1, a_2, ..., a_{n(m)}\}$. 
On the proof
Approximation

Let $P_m$ be a convex linear approximation of convex function $f$ on the interval $[a, b]$ based on partitions $\pi_m$. Then we have

(i) On the interval $(a, b)$ as $m \to \infty$

$$P_m \to f \quad \text{and} \quad (P_m)'_\cdot \to f'_\cdot \quad \text{pointwise}.$$  

(ii) For any bounded continuous function $g$ we have

$$\int_{[a,b]} gd\mu_m \to \int_{[a,b]} gd\mu \quad \text{as} \quad m \to \infty,$$

where $\mu_m$ stands for Radon measure corresponding to the second derivative of $P_m$. 
Transaction costs, local time and quadratic variation

Let $f$ be a convex function with positive Radon measure $\mu$ as its second derivative. Consider continuous semimartingale $X_t = X_0 e^{W_t}$ with local time $l_X$ and $X_0 \in \mathbb{R}_+$. By Itô-Tanaka formula we have

$$f(X_1) = f(X_0) + \int_0^1 f'(X_t) dX_t + \frac{1}{2} \int_{\mathbb{R}} l_X(a, [0, 1]) \mu(da)$$

$$= f(X_0) + \int_0^1 f'(X_t) dX_t + \frac{1}{2} \int_{\mathbb{R}} a l_{W}(\ln a, [0, 1]) \mu(da).$$

We have seen that the same structure appears in the limit with geometric fractional Brownian motion with proportional transaction costs.