Fractional and Multifractional Models in Finance

Georgiy Shevchenko

Kiev National Taras Shevchenko University

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Outline

1. Motivation for multifractionality
2. Mixed models
3. Multifractional processes
4. Non-Gaussian processes
5. Hermite and Rosenblatt processes
6. Local times
7. Estimation of the Hurst parameter
L. Bachelier *Théorie de la spéculuation* (1900)

Brownian motion as a model for prices

- Independent increments
- Stationary increments
- Gaussian laws

P. Samuelson (1940), F. Black, R. Merton, M. Scholes (1970s)

Geometric Brownian motion

Independent stationary returns

Log-Gaussian laws
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The fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a centered Gaussian process $B^H = \{B^H_t, t \geq 0\}$ with stationary increments and the covariance function

$$EB_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$
Definition
The fractional Brownian motion (fBm) with Hurst index \( H \in (0, 1) \) is a centered Gaussian process \( B^H_t = \{B^H_t, t \geq 0\} \) with stationary increments and the covariance function

\[
EB_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).
\]

- Stationary increments
- Self-similarity
- Long memory for \( H > 1/2 \); dependent returns in general
- Gaussian distribution
1 year vs 1 day (ca. 360 ticks)
2 years vs 2 days (ca. 720 ticks)
5 years vs 5 days (ca. 1800 ticks)
Properties of price process

- Stationary increments
- Self-similarity
- Long memory
Properties of price process

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Properties of price process

- Stationary increments
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## Google Scholar Query “Multifractional Brownian”

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Two kinds of multifractionality: *variability under time change* and *variability under scale change*. The first one is often modeled with a changing Hurst parameter; the second with a mixtures of processes.
Contents

1 Motivation for multifractionality

2 Mixed models

3 Multifractional processes

4 Non-Gaussian processes

5 Hermite and Rosenblatt processes

6 Local times

7 Estimation of the Hurst parameter
Setting

\((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P})\) filtered probability space.

\(\mathcal{W} = (\mathcal{W}_t, t \geq 0) \mathcal{F}_t\)-Wiener process.

\(B^H = (B^H_t, t \geq 0) \mathcal{F}_t\)-adapted fBm with \(H > 1/2\).
Setting

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P})$ filtered probability space.

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$B^H = (B^H_t, t \geq 0)$ $\mathcal{F}_t$-adapted fBm with $H > 1/2$.

For financial applications, it is most natural that both integrals are forward, i.e. they are limits of forward integral sums.

Integral w.r.t. $\mathcal{W}$ is Itô.

Integral w.r.t. $B^H$ is pathwise Young integral.
Pathwise integral w.r.t. fBm

For continuous $f, g$, $\alpha \in (0, 1)$, $x \in (a, b)$

$$(D^\alpha_{a+} f)(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(x - a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(u)}{(x - u)^{1+\alpha}} du \right),$$

$$(D^\beta_{b-} g)(x) = \frac{(-1)^\beta}{\Gamma(1 - \beta)} \left( \frac{g(x)}{(b - x)^\beta} + \beta \int_x^b \frac{g(x) - g(u)}{(x - u)^{1+\beta}} du \right).$$
Pathwise integral w.r.t. fBm

For continuous \( f, g, \alpha \in (0, 1), x \in (a, b) \)

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(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(x - a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(u)}{(x - u)^{1+\alpha}} du \right),
\]

\[
(D_{b-}^\beta g)(x) = \frac{(-1)^\beta}{\Gamma(1 - \beta)} \left( \frac{g(x)}{(b - x)^\beta} + \beta \int_x^b \frac{g(x) - g(u)}{(x - u)^{1+\beta}} du \right).
\]

The fractional (Young) integral is

\[
\int_a^b f(x) dg(x) = (-1)^\alpha \int_a^b (D_{a+}^{\alpha} f)(x) (D_{b+}^{1-\alpha} g_b)(x),
\]

where \( g_b(x) = g(x) - g(b) \).
Pathwise integral w.r.t. fBm

For continuous $f, g, \alpha \in (0, 1), x \in (a, b)$

$$(D_{a+}^\alpha f)(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(x - a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(u)}{(x - u)^{1+\alpha}} \, du \right),$$

$$(D_{b-}^\beta g)(x) = \frac{(-1)^\beta}{\Gamma(1 - \beta)} \left( \frac{g(x)}{(b - x)^\beta} + \beta \int_x^b \frac{g(x) - g(u)}{(x - u)^{1+\beta}} \, du \right).$$

The fractional (Young) integral is

$$\int_a^b f(x)dg(x) = (-1)^\alpha \int_a^b (D_{a+}^\alpha f)(x)(D_{b+}^{1-\alpha} g_{b})(x),$$

where $g_{b}(x) = g(x) - g(b)$.

In good cases (e.g. in ours)

$$\int_a^b f(x)dg(x) = \lim \sum_k f(x_k)(g(x_{k+1}) - g(x_k)).$$
Pathwise integral w.r.t. fBm

For $\alpha \in (1 - H, 1)$: $\Lambda_\alpha(B^H) := \sup_{s \in (0, T)}(D_{b-}^{1-\alpha} B^H_b)(s) < \infty$ a.s. Hence we can define for $f$ with $D_{a+}^{\alpha} f \in L_1[0, T]$ we can define the integral w.r.t. fBm:

$$\int_0^T f_s dB^H_s = (-1)^\alpha \int_0^T (D_{0+}^\alpha f)(x)(D_{b-}^{1-\alpha} B^H_b)(x)dx$$

and

$$\left| \int_0^T f_s dB^H_s \right| \leq \| f \|_{1, \alpha, T} \Lambda_\alpha(B^H),$$

where

$$\| f \|_{1, \alpha, T} = \int_0^T \left( \frac{|f(s)|}{s^\alpha} + \int_0^s \frac{|f(s) - f(z)|}{(s - z)^{1+\alpha}}dz \right) ds.$$
Black–Scholes model:

\[ B_t = e^{rt} \]
\[ S_t = S_0 e^{(b-\sigma^2/2)t + \sigma W_t}. \]
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Fractional Black–Scholes model:

\[ B_t = e^{rt} \]

\[ S_t = S_0 e^{bt + \sigma B_t^H} . \]
Mixed Black–Scholes model:

\[ B_t = e^{rt} \]
\[ S_t = S_0 e^{(b-\sigma^2/2)t + \sigma W_t + \mu B_t^H} \]

\( W \) and \( B^H \) are frequently assumed to be independent.
A *strategy* is a pair of adapted processes \((\beta_t, \gamma_t)\), its capital at time \(t\)

\[ V_t = \beta_t B_t + \gamma_t S_t. \]
A strategy is a pair of adapted processes \((\beta_t, \gamma_t)\), its capital at time \(t\)

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Strategy is self-financing if

\[ dV_t = \beta_t dB_t + \gamma_t dS_t. \]

Strategy is arbitrage if \(V_0 = 0\), \(P(V_T \geq 0) = 1\), \(P(V_T > 0) > 0\).

- No arbitrage in the strong sense;
A strategy is a pair of adapted processes \((\beta_t, \gamma_t)\), its capital at time \(t\)

\[ V_t = \beta_t B_t + \gamma_t S_t. \]

Strategy is \textit{self-financing} if

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Strategy is \textit{arbitrage} if \(V_0 = 0, P(V_T \geq 0) = 1, P(V_T > 0) > 0\).

- No arbitrage in the strong sense;
- No arbitrage in a weak sense (NFLVR) for \(H > 3/4\);
- Arbitrage in a weak sense for \(H \in (1/2, 3/4)\).
Theorem (Hitsuda (1968), Cheridito (2001))

There is a unique Volterra kernel \( h = h_\alpha \in L^2[0, T]^2 \) s.t

\[
B_t = M^{H,\alpha}_t - \int_0^t \int_0^s h(s, u) dM^{H,\alpha}_u ds
\]

is a Brownian motion. Furthermore,

\[
M^{H,\alpha}_t = B_t - \int_0^t \int_0^s r(s, u) dB_u ds
\]

with a kernel \( r = r_\alpha \in L^2([0, T]^2) \).
Theorem (Hitsuda (1968), Cheridito (2001))

There is a unique Volterra kernel \( h = h_\alpha \in L_2[0, T]^2 \) s.t.

\[
B_t = M_t^{H, \alpha} - \int_0^t \int_0^s h(s, u) dM_u^{H, \alpha} \, ds
\]

is a Brownian motion. Furthermore,

\[
M_t^{H, \alpha} = B_t - \int_0^t \int_0^s r(s, u) dB_u \, ds
\]

with a kernel \( r = r_\alpha \in L^2([0, T]^2) \).

This kernel solves

\[
\alpha^2 H(2H-1)(t-s)^{2H-2} = r_\alpha(t, s) + \int_0^s r_\alpha(t, x)r_\alpha(s, x) \, dx, \quad 0 \leq s < t \leq T,
\]
By Girsanov, $M_{t}^{H,\alpha}$ is a Brownian motion w.r.t. the measure with the density

$$
\frac{d\tilde{P}}{dP} = \exp \left( \int_{0}^{T} \int_{0}^{s} r(s, u) dB_u dB_s - \frac{1}{2} \int_{0}^{T} \left( \int_{0}^{s} r(s, u) dB_u \right)^2 ds \right).
$$
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\]

- No arbitrage
- Completeness
- Problems sometimes can be reduced to classical BS model
Mixed stochastic differential equations

Consider the equation

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t c(s, X_s)dB^H_s.$$  

This was studied by many authors: by Kubilius (2002) for time-independent coefficients and zero drift, by Mishura (2008) for $H \in (3/4, 1)$ and bounded coefficients, and by Guerra and Nualart (2008) for independent $B^H$ and $W$.

Main obstacle: the integral w.r.t. $W$ is well estimated in a mean (square) sense, while $B^H$ in pathwise sense with a random constant, hence no reasonable contractivity.
Assumptions

Will assume that for some $K > 0$, $t, s \in [0, t]$, $x, y \in \mathbb{R}$

$$
|a(s, x)| + |c(s, x)| \leq K(1 + |x|), \quad |b(t, x)| + |\partial_x c(t, x)| \leq K,
$$

$$
|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)|
+ |\partial_x c(t, x) - \partial_x c(t, y)| \leq K|x - y|,
$$

$$
|a(s, x) - a(t, x)| + |b(s, x) - b(t, x)| + |c(s, x) - c(t, x)|
+ |\partial_x c(s, x) - \partial_x c(t, x)| \leq K|s - t|^{1/2}.
$$
Consider the equation

\[ X_t = X_0 + \int_0^t \left( a(X_s)ds + b(X_s)dW_s + c(X_s)dB_s^H \right). \]
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Euler approximations: equidistant \( n \)-point partition
\( \{ \nu_k = k\delta, k = 0, 1, \ldots, n \} \), \( \delta = T/n \).

\[
X_{\nu_{k+1}}^\delta = X_{\nu_k}^\delta + a(X_{\nu_k}^\delta)\delta + b(X_{\nu_k}^\delta)(W_{\nu_{k+1}} - W_{\nu_k}) + c(X_{\nu_k}^\delta)(B_{\nu_{k+1}}^H - B_{\nu_k}^H),
\]

with \( X_0^\delta = X_0 \).
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Euler approximations: equidistant \( n \)-point partition
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\[ X_{\nu_{k+1}}^\delta = X_{\nu_k}^\delta + a(X_{\nu_k}^\delta)\delta \\
\quad + b(X_{\nu_k}^\delta)(W_{\nu_{k+1}} - W_{\nu_k}) + c(X_{\nu_k}^\delta)(B_{\nu_{k+1}}^H - B_{\nu_k}^H), \]

with \( X_0^\delta = X_0. \)

Integral form (also a continuous interpolation): setting
\[ t_u^\delta = \max\{ \nu_n : \nu_n \leq u \}, \]

\[ X_u^\delta = X_0 + \int_0^u \left( a(X_{t_s}^\delta)ds + \int_0^u b(X_{t_s}^\delta)dW_s + \int_0^u c(X_{t_s}^\delta)dB_s^H \right). \]
Consider the equation

\[ X_t = X_0 + \int_0^t \left( a(X_s)ds + b(X_s)dW_s + c(X_s)dB^H_s \right). \]

Define a norm

\[ \|f\|_{\infty, \alpha, T} = \sup_{s \in [0, T]} \left( |f(s)| + \int_0^s \frac{|f(s) - f(z)|}{(s - z)^{1+\alpha}} dz \right). \]
Consider the equation
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**Theorem (Mishura, Sh (2011))**

*Equation has a solution such that for any \( \alpha \in (1 - H, \kappa) \)

\[ \| X \|_{\infty, \alpha, T}^2 < \infty \quad \text{a.s.} \]

*This solution is unique in the class of processes satisfying this condition for some \( \alpha > 1 - H \).*

Can be proved in multidimensional case. Instead of fractional Brownian motion one can take any process, almost surely Hölder continuous with Hölder exponent \( \gamma > 1/2 \).
For $N \geq 1$ define a stopping time $\tau_N = \inf \{ t : K_\varepsilon^W(t) + K_\varepsilon^H(t) \geq N \} \wedge T$ and a stopped process $X_{t}^{\delta,N} = X_{t \wedge \tau_N}^{\delta}$.

Lemma

For $\alpha \in (1 - H, 1/2)$, $p > 0$, $N \geq 1$

$$\sup_\delta E \left[ \left\| X_{\cdot}^{\delta,N} \right\|_{\infty,\alpha,T}^p \right] < \infty.$$
Garsia–Rodemich–Rumsey inequality

For $f \in C([0, T])$ and $p > 0$, $\theta > 1/p$

$$\sup_{0 \leq v < u \leq T} \frac{|f(u) - f(v)|}{(u - v)^{\theta - 1/p}} \leq C_{p, \theta, T} \left( \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^\theta p + 1} \, dx \, dy \right)^{1/p}.$$
For $f \in C([0, T])$ and $p > 0$, $\theta > 1/p$

$$\sup_{0 \leq v < u \leq T} \frac{|f(u) - f(v)|}{(u - v)^{\theta - 1/p}} \leq C_{p, \theta, T} \left( \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{\theta p + 1}} \, dx \, dy \right)^{1/p}.$$ 

It follows e.g. that for $\varepsilon > 0$, $t, s \in [0, T]$

$$|W_t - W_s| \leq K_{\varepsilon}^W(T)|t - s|^{1/2 - \varepsilon}, \quad \left| B^H_t - B^H_s \right| \leq K_{\varepsilon}^H(T)|t - s|^{H - \varepsilon},$$

where $K_{\varepsilon}^W, K_{\varepsilon}^H$ are random, but all their moments (even exponential) are finite.
Other ingredients

Maximal inequalities for Itô integrals.
Other ingredients

Maximal inequalities for Itô integrals.
Estimates for stochastic integrals w.r.t. fBm.
Other ingredients

Maximal inequalities for Itô integrals.
Estimates for stochastic integrals w.r.t. fBm.

Generalized Gronwall lemma (Nualart and Rășcanu (2002)): if $f \in C([0, T])$ satisfies

$$f(t) \leq a + bt^\alpha \int_0^t f(s)s^{-\alpha}(t - s)^{-\alpha}ds$$

for all $t \in [0, T]$, then

$$f(t) \leq aC_\alpha e^{c_\alpha t^{1/(1-\alpha)}}.$$
Fundamental property

Take a subpartition \( \{ \theta_j = j\mu, 0 \leq j \leq nm \} \), where \( \mu = \delta/m \). Denote \( t^\mu_u = \max\{\theta_n : \theta_n \leq u\} \)

\[
X^\mu_u = X^\mu_0 + \int_0^u \left( a(X^\mu_{t^\mu_s})ds + \int_0^u b(X^\mu_{t^\mu_s})dW_s + \int_0^u c(X^\mu_{t^\mu_s})dB^H_s \right).
\]
Fundamental property

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\]

Auxiliary norm:

\[
\| f \|_{2,\alpha,t}^2 = \int_0^t \left( |f(s)| + \int_0^s |f(s) - f(z)| (s - z)^{-1-\alpha} dz \right)^2 \times (s^{-\alpha} + (t-s)^{-\alpha-1/2}) ds,
\]
Again, \( \tau_N = \inf \{ t : K_\epsilon^W(t) + K_\epsilon^H(t) \geq N \} \land T \) and a stopped process \( X_{t}^{\delta,N} = X_{t \wedge \tau_N}^{\delta} , \ X_{t}^{\mu,N} = X_{t \wedge \tau_N}^{\mu} \).

Also for \( R \geq 1 \) define \( B_t = B_{t}^{R,\alpha,\delta,\mu} = \{ \| X_{\delta}^{\delta} \|_{\infty,\alpha,t} + \| X_{\mu}^{\mu} \|_{\infty,\alpha,t} \leq R \} \).

**Theorem**

Let \( \alpha \in (1 - H, 1/2) \). Then for any \( 0 < \eta < \kappa - \alpha \) and \( N, R \geq 1 \) the following estimate holds

\[
E \left[ \| X_{\delta,N}^{\delta} - X_{\mu,N}^{\mu} \|_{2,\alpha,T}^2 \mathbb{1}_{B_T} \right] \leq M_{R,N} \delta^{1-2(\alpha+\varepsilon)},
\]

where the constant \( M_{R,N} \) is independent of \( \delta, \mu \).

Moreover,

\[
E \left[ \sup_{t \in [0,T]} \| X_{t}^{\delta,N} - X_{t}^{\mu,N} \|_{2,\alpha,T}^2 \mathbb{1}_{B_T} \right] \leq M_{R,N} \delta^{1-2(\alpha+\varepsilon)}.
\]
Proof ingredients

The same +

Gikhman–Skorokhod trick:

$A_s$ adapted decreasing family of sets.

$\gamma_s$ adapted process.

Then

$$\left| \int_0^t \gamma_s dW_s \right| \mathbb{1}_{A_t} \leq \left| \int_0^t \gamma_s \mathbb{1}_{A_s} dW_s \right|.$$
Take a partition \( \{ t_0 = 0 < t_1 < \cdots < t_k = t \} \) and a simple process

\[
\gamma_s = \sum_{j=0}^{k-1} \alpha_j \mathbb{1}_{[t_j, t_{j+1})}(s),
\]

where \( \alpha_j \) is \( \mathcal{F}_{t_j} \)-measurable.

\[
\left| \sum_{j=0}^{k-1} \alpha_j (W_{t_{j+1}} - W_{t_j}) \mathbb{1}_{A_t} \right| = \left| \sum_{j=0}^{k-1} \alpha_j (W_{t_{j+1}} - W_{t_j}) \mathbb{1}_{A_t} \mathbb{1}_{A_{t_j}} \right| = \left| \sum_{j=0}^{k-1} \alpha_j \mathbb{1}_{A_{t_j}} (W_{t_{j+1}} - W_{t_j}) \mathbb{1}_{A_t} \right| \leq \left| \sum_{j=0}^{k-1} \alpha_j \mathbb{1}_{A_{t_j}} (W_{t_{j+1}} - W_{t_j}) \right|. 
\]
Proof of main theorem

*Existence*

Take a sequence of dyadic partitions.

Step 1. Fundamental property of Euler approximations stopped at $\tau_N$ in both $E[\|\cdot\|_{2,\alpha,T}]$ and the supremum norm $\Rightarrow$ existence of the limit $X^N$. 

Step 2. Consistency (in $N$) of the limits: for $M < N$, $t \leq \tau_M X^N_t = X^M_t$.

Therefore, existence of $X$ s.t. $X^N_t = X_t \wedge \tau_N$.

Step 3. Proof that $X$ solves the equation: first up to moment $\tau_N$ and then passing to the limit.

Uniqueness

Let $X_1, X_2$ be two solutions. They coincide up to the moment $\sigma_N = \inf\{t: K_W \epsilon(t) + K_H \epsilon(t) \geq N\}$. Pass to the limit.
Proof of main theorem

Existence

Take a sequence of dyadic partitions.

Step 1. Fundamental property of Euler approximations stopped at $\tau_N$ in both $E[\|\cdot\|_{2,\alpha,T}]$ and the supremum norm $\Rightarrow$ existence of the limit $X_N^N$.

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Therefore, existence of $X$ s.t. $X_t^N = X_{t\wedge \tau_N}$. 
Proof of main theorem

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Therefore, existence of $X$ s.t. $X^N_t = X_{t \wedge \tau_N}$.
Step 3. Proof that $X$ solves the equation: first up to moment $\tau_N$ and then passing to the limit.

Uniqueness
Let $X^1, X^2$ be two solutions. They coincide up to the moment $\sigma_N = \inf \{ t : K^W_\varepsilon (t) + K^H_\varepsilon (t) \geq N \}$. Pass to the limit.
Challenges

Integrability

It is only possible to prove that $X \in L^p(\Omega, Q)$ with
$$dQ = C \exp\{-(K^W_T + K^H_T)\kappa\} dP.$$  
Moments under $P$. 

Challenges

Integrability

It is only possible to prove that $X \in L^p(\Omega, Q)$ with $dQ = C \exp\{-K^W_T + K^H_T\kappa\} dP$.

Moments under $P$.

Almost sure convergence

A subsequence is convergent almost surely, but which one?
Challenges

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$$dQ = C \exp\{- (K^W_T + K^H_T)\kappa\} dP.$$ Moments under $P$.

Almost sure convergence

A subsequence is convergent almost surely, but which one?

Rate of convergence

The pathwise rate of convergence is $\delta^{1/2-}$ for SDE with Brownian motion, $\delta^{2H-1-}$ for SDE with fBm.

$$X_t = X_0 + \int_0^t \left( a(X_s) ds + (b(X_s) - c(X_s)) dW_s + c(X_s) d(B^H_s + W_s) \right).$$ Milstein scheme?
The Mandelbrot-van Ness (moving average) representation:

\[ B_t^H = \frac{1}{\Gamma(H + 1/2)} \int_{\mathbb{R}} \left[ (t - s)^{H-1/2} - (-s)^{H-1/2} \right] dW_s. \]
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Volterra kernel representation:

\[ B_t^H = \int_0^t K_H(t, s) \, dW_s, \]
\[ K_H(t, s) = C_H s^{1/2-H} \int_s^t (v - s)^{H-3/2} v^{H-1/2} \, dv. \]
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Harmonizable representation:

\[ B_t^H = K_H^1 \int_{\mathbb{R}} \frac{e^{itx} - 1}{x^{1/2+H}} d\tilde{W}_x. \]
Multifractional processes

Start with a representation

\[ B_t^H = \int_{\mathbb{R}} f_H(t, s) dW_s \]

of fBm and replace constant parameter \( H \) by a Hurst function \( H_t \):

\[ M_t^H = \int_{\mathbb{R}} f_{H_t}(t, s) dW_s. \]
The moving average multifractional Brownian motion:

\[ M_{t}^{MA} = \int_{\mathbb{R}} \left[ (t - s)^{H_{t} - 1/2} - (-s)^{H_{t} - 1/2} \right] dW_s. \]
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The Volterra type multifractional Brownian motion:

\[ M_t^V = \int_0^t K_{H_t}(t, s) dW_s. \]
The moving average multifractional Brownian motion:

\[ M_t^{MA} = \int_{\mathbb{R}} \left[ (t - s)^{H_t - 1/2} - (-s)^{H_t - 1/2} \right] dW_s. \]

The Volterra type multifractional Brownian motion:

\[ M_t^V = \int_0^t K_{H_t}(t, s) dW_s. \]

Real harmonizable multifractional Brownian motion:

\[ M_t^{RH} = \int_{\mathbb{R}} \frac{e^{itx} - 1}{\left| x \right|^{1/2 + H_t}} d\tilde{W}_x. \]
The moving average multifractional Brownian motion:

$$M_t^{MA} = \int_{\mathbb{R}} \left[(t - s)^{H_t - 1/2} - (-s)^{H_t - 1/2}\right] dW_s.$$ 

The Volterra type multifractional Brownian motion:

$$M_t^V = \int_0^t K_{H_t}(t, s) dW_s.$$ 

Real harmonizable multifractional Brownian motion:

$$M_t^{RH} = \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{1/2 + H_t}} d\tilde{W}_x.$$ 

Usual assumption:

$$|H_t - H_s| \leq C |t - s|^{\gamma}$$ 

for $\gamma > \sup_t H_t$. 

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Properties

For $t, s$ close enough, $u \in [t, s]$

$$E \left[ (M_t - M_s)^2 \right] \leq C |t - s|^{2H_u}.$$  

Global Hölder continuity:

$$|M_t - M_s| \leq C |t - s|^\kappa$$

a.s. for any $\kappa < \inf_t H_t.$
Properties

For $t, s$ close enough, $u \in [t, s]$

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Global Hölder continuity:

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a.s. for any $\kappa < \inf_t H_t$.

Pointwise Hölder exponent at $t$ is $H_t$. 
Localisability

**Definition**

Process $X$ is $H$-localisable at $t_0$ with the local version $Y$ if

$$\left\{ \frac{1}{s^H}(X_{t_0+st} - X_{t_0}), \ t \geq 0 \right\} \to \{Y_t, \ t \geq 0\}, \ s \to 0^+.$$
Localisability

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Each of mBm’s is $H_{t_0}$-localisable at any point $t_0$ with the local version $B^{H_{t_0}}$ (up to a constant).
Integration

Most well understood for \( \inf H_t > 1/2 \) (we will assume this).

For \( f \) with \( D_{a+}^\alpha f \in L_1[a, b] \) the integral w.r.t. \( \text{mBm} \):

\[
\int_a^b f_s dM_s = (-1)^\alpha \int_a^b (D_{a+}^\alpha f)(s)(D_{b-}^{1-\alpha} M_{b-})(s)ds.
\]
Integration

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SDEs are solvable under the same conditions as those with fBm (e.g. Nualart and Rășcanu (2002)).
Integration

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$$

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When $H$ is differentiable, it is also possible to define Itô–Skorokhod integral as usually for Gaussian processes, see Alòs et al (2001). (And it is done in Boufoussi et al. (2010).)
(Multi)fractional Brownian motion is not a semimartingale. Possible to approximate it by semimartingales. Smoothness is worsened! Another idea: to approximate by smooth processes $\Rightarrow$ random ODEs or Itô SDEs with random drift.
The simplest idea (averaged):

\[ M_t^\varepsilon = \frac{t}{\varepsilon} \int_t^{t+\varepsilon} M_s ds. \]

More involved (but also more helpful in many cases):
Take a Volterra representation of fBm (or Volterra mBm)

\[ B_H^t = \int_0^t K_H(t, s) dW_s, \]

\[ K_H(t, s) = C_H s^{1/2 - H} \int_s^t (v - s)^{H - 3/2} v^{H - 1/2} dv. \]

It is not possible to change the order of integration here!
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\[ B^H,\varepsilon_t = C_H \int_0^t s^{1/2-H} \int_0^{v-\varepsilon} (v - s)^{H-3/2} v^{H-1/2} dW_s, dv \]
The simplest idea (averaged):

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It is not possible to change the order of integration here!

\[ M_t^\varepsilon = C_H t \int_0^t s^{1/2-H_t} \int_0^{v-\varepsilon} (v - s)^{H_t-3/2} v^{H_t-1/2} dW_s, dv \]

Similarly for moving average and harmonizable version.
The seminorm

\[ \| f \|_{1,\beta} := \sup_{0 \leq s < t \leq T} \left( \frac{|f(t) - f(s)|}{(t - s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u - s)^{1+\beta}} \, du \right) < \infty, \]

is useful for controlling integrals w.r.t. fBm.
The seminorm

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is useful for controlling integrals w.r.t. fBm.

**Theorem (Ralchenko, Sh (2010))**

*For any* \( \beta < \inf_t H_t \)

\[ \| M_t - M_t^\varepsilon \|_{1,\beta} \to 0 \]

*in probability. Moreover, the averaged approximations converge almost surely. Yet moreover, the subsequence with* \( \varepsilon = 2^{-k} \) *is convergent almost surely for the approximations by kernel regularization.*
Consider the equation

\[ X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t c(X_s)dM_s. \]

and its smooth approximations:

\[ X_t^\varepsilon = X_0 + \int_0^t a(X_s^\varepsilon)ds + \int_0^t c(X_s^\varepsilon)dM_s^\varepsilon. \]

These are just random ODEs:

\[ dX_t^\varepsilon = (a(X_s^\varepsilon) + c(X_s^\varepsilon)\dot{M}_s^\varepsilon)ds. \]

**Theorem (Ralchenko and Sh (2010))**

It holds

\[ \sup_{t\in[0,T]} |X_t - X_t^\varepsilon| \to 0 \]

in probability (and almost surely as above).
Similarly, for a mixed equation

\[ X_t = X_0 + \int_0^t \left( a(X_s) ds + b(X_s) dW_s + c(X_s) dM_s \right). \]

its approximations

\[ X_t^\varepsilon = X_0 + \int_0^t \left( a(X_s^\varepsilon) ds + b(X_s^\varepsilon) dW_s + c(X_s^\varepsilon) dM_s^\varepsilon \right) \]

are in fact Itô SDEs with a random drift:

\[ dX_t^\varepsilon = \left( a(X_s^\varepsilon) + c(X_s^\varepsilon) \dot{M}_s^\varepsilon \right) ds + b(X_s^\varepsilon) dW_s. \]

**Theorem (Mishura, Posashkova, Sh, working paper)**

\[ \sup_{t \in [0, T]} |X_t - X_t^\varepsilon| \to 0 \]

in probability.
Main tools of the proof:

Garsia–Rodemich–Rumsey inequality: for \( f \in C([0, T]) \) and \( p > 0 \), \( \theta > 1/p \)

\[
\sup_{0 \leq v < u \leq T} \frac{|f(u) - f(v)|}{(u - v)^{\theta - 1/p}} \leq C_{p,\theta,T} \left( \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{\theta p+1}} \, dx \, dy \right)^{1/p}.
\]

These are two-parameter generalizations of the spoken result.
Motivation: image processing; fractional SPDEs.

Let \( t = (T_1, T_2) \in \mathbb{R}^2_+, [0, T] = [0, T_1] \times [0, T_2] \).

For \( s = (s_1, s_2), t = (t_1, t_2) \) we write \( s < t \) whenever \( s_1 < s_2 \) and \( t_1 < t_2 \)
and denote \( \Delta_s f(t) := f(t) - f(s_1, t_2) - f(t_1, s_2) + f(s) \).

One of the main tools: the two-parameter generalization of
Garsia–Rodemich–Rumsey inequality (Ralchenko (2007)): for
\( f \in C([0, T], p > 0, \theta_1 > 1/p, \theta_2 > 1/p \)

\[
\frac{|\Delta_s f(t)|}{|t - s|^{\theta p - 1}} \leq C_{p, \theta, T} \left( \int_{[0, T]^2} \frac{|\Delta_x f(y)|^p}{|x - y|^{\theta p + 1}} \, dx \, dy \right)^{1/p}.
\]

where \( |t - s|^{\theta p \pm 1} = |t_1 - s_1|^{\theta_1 p \pm 1} |t_2 - s_2|^{\theta_2 p \pm 1} \).
Assumptions

\{B_t, t \in [0, T]\} Gaussian random field.

- For all \( s < t \)

\[ E \left[ (\Delta_s B_t)^2 \right] \leq C(|t_1 - s_1| |t_2 - s_2|)^{2\lambda}; \]

- \( B_t \) is almost surely jointly continuous.
Examples

\{ W_t, t \in \mathbb{R}^2 \} \text{ standard Wiener sheet: continuous centered Gaussian field with independent increments over rectangles and } E[(\Delta_s W_t)^2] = (t_1 - s_1)(t_2 - s_2) \text{ for } t > s.

Anisotropic fractional Brownian sheet with Hurst parameter \( H = (H_1, H_2) \) is

\[ B^H_t := C_H \int_{\mathbb{R}^2} \prod_{i=1,2} \left[ (t_i - u_i)^{H_i-1/2} - (-u_i)^{H_i-1/2} \right] dW_u. \]

Centered with covariance

\[ E[B^H_t B^H_s] = \frac{1}{4} \prod_{i=1,2} \left( t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right). \]

Increments satisfy

\[ E[(\Delta_s B^H_t)^2] = (t_1 - s_1)^{2H_1}(t_2 - s_2)^{2H_2}. \]
Anisotropic multifractional Brownian sheet with Hurst function $H(t) = (H_1(t), H_2(t))$ is

$$M_t := \int_{\mathbb{R}^2} \prod_{i=1,2} \left[ (t_i - u_i)^{H_i(t)-1/2} - (-u_i)^{H_i(t)-1/2} \right] dW_u.$$ 

Assumptions on the Hurst function: $H_i(t) \in (1/2, 1)$,

$$|H_i(t) - H_i(s)| \leq c_1 (|t_1 - s_1|^\nu + |t_2 - s_2|^\nu)$$

$$|\Delta_s H_i(t)| \leq c_2 (|t_1 - s_1| |t_2 - s_2|)^\nu.$$
\[ \{ B_t, t \in [0, T]\} \text{ Gaussian random field.} \]

- for all \( s < t \)

\[
E \left[ (\Delta_s B_t)^2 \right] \leq C (|t_1 - s_1| |t_2 - s_2|)^{2\lambda};
\]

- \( B_t \) is almost surely jointly continuously.

Smooth approximations:

\[
B_t^\varepsilon = \frac{1}{\varepsilon^2} \int_{t_1}^{t_1 + \varepsilon} \int_{t_2}^{t_2 + \varepsilon} B_s ds_2 ds_1.
\]

**Theorem (Ralchenko, Sh (2011))**

For any \( \beta_1 < \lambda, \beta_2 < \lambda \)

\[
\| B_t - B_t^\varepsilon \|_{1, \beta_1, \beta_2, T} \to 0
\]

almost surely.
Contents

1 Motivation for multifractionality
2 Mixed models
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7 Estimation of the Hurst parameter
Stable laws

Symmetric $\alpha$-stable ($S_\alpha S$) random variable $\alpha \in (1, 2)$ with scale parameter $\sigma^\alpha$

$$E \left[ e^{i\lambda \xi} \right] = e^{-|\sigma \lambda|^\alpha}.$$ 

Crucial property

$$S_\alpha S(\sigma_1^\alpha) + S_\alpha S(\sigma_2^\alpha) = S_\alpha S(\sigma_1^\alpha + \sigma_2^\alpha).$$
Stable laws

Symmetric $\alpha$-stable ($S\alpha S$) random variable $\alpha \in (1, 2)$ with scale parameter $\sigma^\alpha$

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Crucial property

$$S\alpha S(\sigma_1^\alpha) + S\alpha S(\sigma_2^\alpha) = S\alpha S(\sigma_1^\alpha + \sigma_2^\alpha).$$

$S\alpha S$ Lévy motion

- $L_0^\alpha = 0$;
- $L^\alpha$ has independent increments;
- $L_t^\alpha - L_s^\alpha$ is $S\alpha S(|t - s|)$. 

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Fractional stable processes

Fractional (moving average) stable process

\[ Z_{t}^{\alpha,H,MA} = \frac{1}{\Gamma(H + 1/2)} \int_{\mathbb{R}} \left[ (t - s)^{H-1/\alpha} - (-s)^{H-1/\alpha} \right] dL_{s}^{\alpha}. \]

Volterra type fractional stable process

\[ Z_{t}^{\alpha,H,V} = \int_{0}^{t} K_{H}(t, s) dL_{s}^{\alpha}. \]
Independently scattered rotationally invariant complex $S_\alpha S$ random measure

1. For $A \subset \mathbb{R}$ and $\theta \in \mathbb{R}$: $e^{i\theta} M(A) \overset{d}{=} M(A)$; $\text{Re } M(A)$ is $S_\alpha S$ with the scale parameter $\lambda(A)$.

2. For disjoint $A_1, \ldots, A_n \subset [0, \infty)$ the values $M(A_1), \ldots, M(A_n)$ are independent.

3. For any Borel set $A \subset \mathbb{R}$ $M(-A) = \overline{M(A)}$. 
For \( f : \mathbb{R} \to \mathbb{C} \) s.t.
\[
f(-x) = \overline{f(x)} \text{ for all } x \in \mathbb{R}
\]
and
\[
\|f\|_{L^\alpha(\mathbb{R})} = \int_{\mathbb{R}} |f(x)|^\alpha \, dx < \infty
\]
it is possible to define a stochastic integral
\[
\int_{\mathbb{R}} f(x) M(dx),
\]
which is a S\(\alpha\)S r.v. with the scale parameter \( \|f\|_{L^\alpha(\mathbb{R})} \).

Stochastic integral gives an isometry between the space of S\(\alpha\)S real random variables spanned by the measure \( M \) with the norm
\[
\|\xi\|_{\alpha} = -\log \mathbb{E} \left[ e^{i\xi} \right]
\]
and the subspace of \( L^\alpha(\mathbb{R}) \) consisting of functions having adjoint values at symmetric points.
Real harmonisable fractional stable process:

\[ Z_t^H = \int_\mathbb{R} \frac{e^{itx} - 1}{|x|^{1/\alpha + H}} M(dx). \]

Multifractional counterpart (rhmsp):

\[ X_t = \int_\mathbb{R} \frac{e^{itx} - 1}{|x|^{1/\alpha + H_t}} M(dx). \]

As before,

\[ |H_t - H_s| \leq C|t - s|^{\gamma}, \]

\[ \gamma > \sup_t H_t. \]
For $s < t$ close enough, $u \in [s, t]$

$$C_1 |t - s|^{H_u} \leq \|X(t) - X(s)\|_{\alpha} \leq C_2 |t - s|^{H_u}.$$ 

This does not imply continuity (in contrast to Gaussian case).
Properties

For $s < t$ close enough, $u \in [s, t]$

$$C_1 |t - s|^{H_u} \leq \|X(t) - X(s)\|_\alpha \leq C_2 |t - s|^{H_u}.$$  

This does not imply continuity (in contrast to Gaussian case).
But it is possible to prove that $\text{rhm}_{\alpha} X$ is Hölder continuous of any order $\kappa < \inf H_t$. 
LePage representation

\{v_j, j \geq 1\} symmetric iid with \(E[|v_1|^\alpha] < \infty\)
\{\Gamma_j, j \geq 1\} independent if \(v\) sequence of Poissonian arrivals with unit intensity.

Then

\[ S_\alpha = \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} v_j \]

is \(S_\alpha S\) with the scale parameter \(C_\alpha E[|v_1|^\alpha]\), \(C_\alpha = (\int_0^\infty x^{-\alpha} \sin x \, dx)^{-1}\).
\( f(t, x) : [0, \infty) \times \mathbb{R} \) good kernel, \( f(t, x) = \overline{f(t, -x)} \), \( Y_t = \int_{\mathbb{R}} f(t, x) M(dx) \).

**Theorem**

\( \varphi \) any probability density on \( \mathbb{R} \) equivalent to the Lebesgue measure.
\( \{ \Gamma_k, k \geq 1 \}, \{ \xi_k, k \geq 1 \}, \{ g_k, k \geq 1 \} \) three independent sets of random variables s.t.

- \( \{ \Gamma_k, k \geq 1 \} \) Poisson process arrivals with unit intensity;
- \( \{ \xi_k, k \geq 1 \} \) iid with density \( \varphi \);
- \( \{ g_k, k \geq 1 \} \) iid rotationally invariant complex Gaussian with \( \mathbb{E} [ |\text{Re} g_k|^\alpha ] = 1 \).

Then \( Y \) has the same distribution as

\[
Y'_t = (C_\alpha)^{-1/\alpha} \text{Re} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} f(t, \xi_k) g_k.
\]
Definition

Process $X$ is $H$-localisable at $t_0$ with the local version $Y$ if

$$\left\{ \frac{1}{s^H}(X_{t_0+st} - X_{t_0}), t \geq 0 \right\} \to \{Y_t, t \geq 0\}, s \to 0^+.$$
Definition

Process $X$ is $H$-localisable at $t_0$ with the local version $Y$ if

$$\left\{ \frac{1}{s^H}(X_{t_0+st} - X_{t_0}), \ t \geq 0 \right\} \rightarrow \{ Y_t, \ t \geq 0 \}, \ s \rightarrow 0^+. $$

Rhmsp $X$ is $H_{t_0}$-localisable at any point $t_0$ with the local version $Z^{H_{t_0}}$. 
Double stochastic integrals

$W$ two-sided standard Wiener process (Wiener measure on $\mathbb{R}$).

$\hat{L}^2(\mathbb{R}^2)$ the space of symmetric functions with the norm $\| \cdot \|$.

Take simple $f \in \hat{L}^2(\mathbb{R}^2)$ of the form

$$f(x, y) = \sum_{k,j=1}^{n} a_{kj} \mathbb{1}_{A_k}(x) \mathbb{1}_{A_j}(y),$$

with $A_1, \ldots, A_n$ being finite disjoint intervals of $\mathbb{R}$, $a_{kj} = a_{jk}$, $a_{kk} = 0$. 

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with $A_1, \ldots, A_n$ being finite disjoint intervals of $\mathbb{R}$, $a_{kj} = a_{jk}$, $a_{kk} = 0$.

Define

$$I_2(f) = \int\int_{\mathbb{R}^2} f(x, y) W(dx) W(dy) = \sum_{k,j=1}^{n} a_{kj} W(A_k) W(A_j),$$

where $W(A_k)$ is the increment of $W$ over $A_k$.

$I_2/\sqrt{2}$ is a linear isometry to $L^2(\Omega)$ which can be extended to $\hat{L}^2(\mathbb{R}^2)$. 
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$I_2/\sqrt{2}$ is a linear isometry to $L^2(\Omega)$ which can be extended to $\hat{L}^2(\mathbb{R}^2)$.

For asymmetric $f$, $I_2(f) = I_2(\hat{f})$, $\hat{f}(x, y) = (f(x, y) + f(y, x))/2$. 
1. If \( a, b \in L^2(\mathbb{R}) \), \( f = a \otimes b \in \hat{L}^2(\mathbb{R}^2) \) symmetric tensor product, \( f(x, y) = (a(x)b(y) + a(y)b(x))/2 \), then \( l_2(h) = l_1(a)l_1(b) - (a, b) \). (Here \( l_1(a) = \int_{\mathbb{R}} a(x)W(dx) \).

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2. To \( f \in \hat{L}^2(\mathbb{R}^2) \) corresponds a Hilbert–Schmidt operator on \( L^2(\mathbb{R}): \)

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Let \( \lambda_{k,f} \) be its eigenvalues ordered by absolute value decrease, \( \varphi_{k,f} \) be correspondent (orthonormal) eigenfunction. Then 
\[
 f(x, y) = \sum_{k \geq 1} \lambda_{k,f} \varphi_{k,f} \otimes \varphi_{k,f},
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whence \( l_2(f) = \sum_{k \geq 1} \lambda_{k,f}(\zeta_k^2 - 1) \), 
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This implies

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\left| E \left[ e^{i\alpha l_2(f)} \right] \right| \leq \left( 1 + 4\alpha^2 \sum_{k \geq 1} \lambda_{k,f}^2 + 16\alpha^4 \sum_{j<k} \lambda_{j,f}^2 \lambda_{k,f}^2 + 64\alpha^6 \sum_{j<k<l} \lambda_{j,f}^2 \lambda_{k,f}^2 \lambda_{l,f}^2 \right)^{-1/4}
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\{\xi_n, n \geq 1\} \text{ stationary standard Gaussian sequence with } \\
\mathbb{E}[\xi_0, \xi_n] = L(n)n^{(2H-2)/k}; \\
H_m \quad m^{th} \text{ Hermite polynomial; } \\
g : \mathbb{R} \rightarrow \mathbb{R} \text{ is s.t. } \mathbb{E}[g(\xi_0)] = 0, \mathbb{E}[g(\xi_0)^2] < \infty, g \text{ of Hermite rank } k: \\
\mathbb{E}[g(\xi_0)H_j(\xi_0)] = 0, j < k, \mathbb{E}[g(\xi_0)H_k(\xi_0)] \neq 0.
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Then 
\[ n^{-H} \sum_{i=1}^{[nt]} g(\xi_i) \to \]

\[ c(H, k) \int \cdots \int \int_0^t \prod_{j=1}^k (s - y_i)^{-\frac{1}{2} - \frac{1-H}{k}} \, ds \, W(dy_1) \cdots W(dy_k), \]

the Hermite process of rank \(k\) (with Hurst parameter \(H\)).

\(k = 1\): fractional Brownian motion.

\(k = 2\): Rosenblatt process:
\[ R^H_t = c(H, 2) \int \int_0^t (s - x)^{H/2-1}(s - y)^{H/2-1} \, ds \, W(dx) \, W(dy). \]
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Harmonizable representation:

\[ R_t^H = d(H, 2) \int \cdots \int_{\mathbb{R}^2} \frac{e^{i(\lambda_1+\lambda_2)t} - 1}{i(\lambda_1 + \lambda_2)} |\lambda_1\lambda_2|^{-H/2} \tilde{W}(d\lambda_1) \tilde{W}(d\lambda_2). \]

- \( R^H \) is \( H \)-self-similar: for any \( c > 0 \) \( \{ c^{-H} R_{ct}^H \} \overset{d}{=} \{ R_t^H \} \);
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\[ \mathbb{E} \left[ R_t^H R_s^H \right] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right). \]
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- Rosenblatt process is Hölder continuous of order \( \kappa < H \).
Multifractional Rosenblatt process with the Hurst function $H$

$$X_t = \int \int \int_{\mathbb{R}^2} \int_0^t (s-x)^{H_t/2-1} (s-y)^{H_t/2-1} ds \ W(dx) W(dy).$$
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$H$ satisfies

$$|H_t - H_s| \leq C |t - s|^{\gamma}$$

with $\gamma > \sup_{t} H_t$. 
Properties of mRp

For $s < t$ close enough, $u \in [s, t]$

$$\mathbb{E} \left[ (X_t - X_s)^2 \right] \leq C(t - s)^{H_u}.$$ 

On any interval $[a, b] \subset [0, \infty)$ trajectories of mRp are a.s. Hölder continuous of any order $\kappa < \min_{t \in [a, b]} H_t$.

MRp $\{X_t, t \geq 0\}$ is $H_{t_0}$-localisable at point $t_0$ with the local version $\left\{ R_t^{H_{t_0}} \right\}$ up to a constant.
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Financial motivation

- Valuation of plain vanilla options
- Stop-loss start-gain strategies
- First-passage problems
- Optimal stopping problems
- Passport options...
$\{Z_t, t \geq 0\}$ separable random process.

Occupation measure for $Z$: for Borel $A \subset [0, \infty), B \subset \mathbb{R}$

$$\mu(A, B) = \lambda(\{s \in A, Z_s \in B\}),$$
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Analytic method: for a given path, write $L(t, x)$ as the inverse Fourier transform

$$L(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ivx} \int_{\mathbb{R}} e^{ivy} L(t, y) dy \, dv = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{iv(Z_s-x)} ds \, dv.$$
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\]

For any \( x, y \in \mathbb{R}, t, h \in \mathbb{R}_+ \) and \( m \geq 2 \)

\[
E [L([t, t+h], x)^m] = \frac{1}{(2\pi)^m} \int_{[t,t+h]^m} \int_{\mathbb{R}^m} E\left[ e^{i \sum_{j=1}^{m} v_j(Z_{s_j}-x_j)} \right] \prod_{j=1}^{m} d\nu_j \prod_{j=1}^{m} ds_j.
\]

Similarly (but more involved) for increments in \( x \).
Local non-determinism

A random process $X$ is $\| \cdot \|$ \textit{locally non-deterministic (LND)} on $T$ if

1. $\|X_t\| > 0$ for all $t \in T$;
2. $\|X_t - X_s\| > 0$ for all sufficiently close distinct $s, t \in T$;
3. for any $n > 1$ there exists $C_n > 0$ s.t. for any $t_1 < t_2 < \cdots < t_n \in T$ sufficiently close together one has

$$\|X_{t_n} - \text{span} \{X_{t_1}, \ldots, X_{t_n}\}\| \geq C_n \|X_{t_n} - X_{t_{n-1}}\|.$$
Local non-determinism

A random process $X$ is $\| \cdot \|$ *locally non-deterministic* (LND) on $\mathbf{T}$ if

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$$\| X_{t_n} - \text{span} \{ X_{t_1}, \ldots, X_{t_n} \} \| \geq C_n \| X_{t_n} - X_{t_{n-1}} \| .$$

Another form of the third condition:

3. for any $t_1 < t_2 < \cdots < t_n \in \mathbf{T}$ sufficiently close together and any $a_1, \ldots, a_n \in \mathbb{R}$ one has

$$\| a_1 X_{t_1} + \sum_{k=1}^{n-1} a_k (X_{t_{k+1}} - X_{t_k}) \| \geq C_n \left( \| a_1 X_{t_1} \| + \sum_{k=1}^{n-1} \| a_k (X_{t_{k+1}} - X_{t_k}) \| \right).$$
Local times: results

Theorem (Dozzi and Sh (2011), Sh (2011))

Both rhmsp and mRp have square integrable local times.

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Theorem (Dozzi and Sh (2011))

The local time of the rhmsp is jointly continuous in \((t, x)\) for \(t > 0\), moreover, for any \(\kappa < (1/ \max_t H_t - 1)/2\) it is \(\kappa\)-Hölder continuous in \(x\).
Proof contents

Existence of local time

mRp: First the existence of local time for Rosenblatt process, using an estimate for characteristic function.

rhmsp: lower bounds for increments: for \( s < t \) close enough, \( u \in [s, t] \)

\[
\|X(t) - X(s)\|_\alpha \geq C_1 |t - s|^{H_t}.
\]
Regularity of local time for rhmsp
Implied by LND property.
One of main ingredients to prove it is the Hausdorff–Young inequality
\( f \in L^\alpha(\mathbb{R}) \cap L^1(\mathbb{R}), \alpha \in (1, 2), \beta = \alpha/(\alpha - 1) \) exponent adjoint to \( \alpha \).
\( \mathcal{F}_\alpha f \) Fourier transform.
\[
\| \mathcal{F}_\alpha f \|_{L^\beta(\mathbb{R})} \leq C_\alpha \| f \|_{L^\alpha(\mathbb{R})}.
\]
Allows to extend Fourier transform to \( L^\alpha(\mathbb{R}) \).
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Localizability $\Rightarrow M$ looks (up to a constant) as fBm with Hurst parameter in a neighborhood of $t_0$.

We need an estimator for Hurst parameter of fBm with good rate of convergence!
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We need an estimator for Hurst parameter of fBm with good rate of convergence!
Quadratic variation

\{B^H, H \in [0, T]\} fBm with \(H > 1/2\).

\(N \geq 1\), equidistant partition \(\{\nu_k = k\delta, k = 0, \ldots, N\}\), \(\delta = T/N\).

Quadratic variation:

\[
V_N = \sum_{k=1}^{N} \left( B^H_{\tau_k} - B^H_{\tau_{k-1}} \right)^2 \sim c_H N^{1-2H}.
\]

A variation-based estimator:

\[
\hat{H}^{QV}_N = \frac{1}{2} \left( 1 + \frac{\log V_N}{\log N} \right).
\]

Poor speed of convergence; non-central asymptotics for large \(H\).
Generalized quadratic variation

Istas and Lang (1996, fBm); Benassi et al (1998, mBm)

\[ G^N = \sum_{k=1}^{N-1} \left( B_{\tau_{k+1}}^H - 2B_{\tau_k}^H + B_{\tau_{k-1}}^H \right)^2 \sim c_H N^{1-2H}. \]

Correspondent estimator:

\[ \hat{H}_{N}^{GQV} = \frac{1}{2} \left( 1 + \frac{\log G_N}{\log N} \right). \]
How to go from an estimator for fBm to the one for mBm?

\[ \mathcal{N}_{N,\varepsilon}(t) = \{ k \leq N : |\tau_k - t| < \varepsilon \}. \]

Take \( \varepsilon = N^{-\eta} \), so that \( |\mathcal{N}_{N,\varepsilon}(t)| \approx N^{1-\eta} \).

By localizability, in \( N_{N,\varepsilon}(t)(t) \),

\[ M_s \approx M_t + N^{-\eta H_t} B_{(s-t)/\varepsilon}^H. \]

Generalized quadratic variation in \( N_{N,\varepsilon}(t) \) is

\[ V_{N,\varepsilon}(t) \sim c_H N^{(1-\eta)(1-2H_t)} N^{-2\gamma H_t} = N^{1-\gamma-2H_t}. \]

Estimator:

\[ \hat{H}_N(t) = \frac{1}{2} \left( 1 - \gamma - \frac{\log V_{N,\varepsilon}(t)}{\log N} \right). \]

Consistent estimator for \( H_t \) (Benassi et al (1998)).

Asymptotic normality

To prove asymptotic normality, the following theorem is useful, which is a consequence of a stronger statement by Nualart and Pecatti (2005).

**Theorem**

Let \( f_n \in \hat{L}^2(\mathbb{R}^2) \) be such that \( E[I_2(f_n)^2] \to 1 \).

The following statements are equivalent:

- \( I_2(f_n) \xrightarrow{d} N(0, 1) \)
Asymptotic normality

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**Theorem**

Let $f_n \in \hat{L}^2(\mathbb{R}^2)$ be such that $E[l_2(f_n)^2] \to 1$. The following statements are equivalent:

- $l_2(f_n) \xrightarrow{d} N(0, 1)$
- $E[l_2(f_n)^4] \to 3$. 